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GLOBAL ASYMPTOTIC STABILITY OF PERIODIC SOLUTION FOR A COOPERATIVE SYSTEM WITH TIME DELAYS

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A periodic cooperative model with time delays is investigated. By using the continuation theorem of coincidence degree theory and by constructing a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the existence, uniqueness and global stability of positive periodic solutions of the system.

Key Words: **Cooperative System; Time Delay; Periodic Solution; Continuation Theorem of Coincidence Degree Theory; Lyapunov Functional; Global Stability**

1. INTRODUCTION

A rather characteristic behaviour of population dynamics is the often observed oscillatory phenomenon of the population densities. One mechanism to do this is to introduce time delays in the models, which is amore realistic approach to the understanding of the population dynamics. Another mechanism to produce oscillatory dynamics is to take into account the fluctuation in the environment. However, in many of the models considered so far, it has been assumed that all biological and environmental parameters are constant in time. Mathematically, this means that the models are autonomous with time t not appearing explicitly in the equations. Any biological or environmental parameters, however, are naturally subject to fluctuation in time. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment (such as seasonal effects of weather, food supplies, mating habits and so forth). We refer to Pianka[1] for a discussion of the relevance of periodic environments to evolutionary theory.

In 1976, May suggested the following model:

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t)\left[1 - \frac{x_1(t)}{a_1 + b_1 x_2(t)} - c_1 x_1(t)\right], \\ \dot{x}_2(t) &= r_2 x_2(t)\left[1 - \frac{x_2(t)}{a_2 + b_2 x_1(t)} - c_2 x_2(t)\right] \end{aligned} \qquad \ldots (1.1)$$

to describe the interaction between a pair of mutualists[2], where $x_1(t)$ and $x_2(t)$ are the densities of the species X_1 and X_2 at time t respectively and r_i, a_i, b_i and c_i $(i = 1, 2)$ are all positive constants. It was proved that system (1.1) has a globally asymptotically stable positive equilibrium.

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Cui and Chen[3] discussed the following two-species cooperative model with periodic coefficients

$$\begin{aligned} \dot{x}_1(t) &= r_1(t)\,x_1(t)\left[1 - \frac{x_1(t)}{a_1(t) + b_1(t)\,x_2(t)} - c_1(t)\,x_1(t)\right], \\ \dot{x}_2(t) &= r_2(t)\,x_2(t)\left[1 - \frac{x_2(t)}{a_2(t) + b_2(t)\,x_1(t)} - c_2(t)\,x_2(t)\right] \end{aligned} \qquad \ldots (1.2)$$

where $r_i(t), a_i(t), b_i(t)$ and $c_i(t)$ are assumed to be continuous and bounded positive functions on $[0, +\infty), i = 1, 2$. When the coefficients in (1.2) are ω-periodic, a sufficient condition has been obtained in[3] for the existence of a unique globally asymptotically stable and strictly positive periodic solution for system (1.2).

In view of the fact that in real-life interactions, instantaneous responses are rare or weak relative to delayed responses, more realistic models should consist of delay differential systems without instantaneous (negative) feedbacks. The effect of time delays on the asymptotic behaviour of populations has been studied by a number of authors (see, for example[4-8]). Most of the known convergence results for delayed systems require dominating or strong instantaneous negative feedbacks or some restrictions on initial conditions. In order to justify the common belief that "small delays are negligible in some modelling process as far as stabilities are concerned", it is thus important to show under some reasonable assumptions that the global stability of an ecological system persists when time delays are small enough. This is indeed a long standing question in the qualitative analysis of systems of delay differential equations. Kuang[9] presented a partial answer to this open question for Lotka-volterra type systems with a saturated equilibrium.

In the present paper, we incorporate periodicity and time delays into system (1.1). We wish to concentrate on the consequences of incorporating time delays that are not constants but are periodic functions of time. In some instances (but not all) there are good reasons for allowing the delay to vary with time and the use of a periodic delay is especially relevant since it can model seasonal fluctuations of the environment. To do so, we discuss the following delayed periodic cooperative system

$$\begin{aligned} \dot{x}_1(t) &= r_1(t)\,x_1(t)\left[1 - \frac{x_1(t-\tau_{11}(t))}{a_1(t) + b_1(t)\,x_2(t-\tau_{11}(t))} - c_1(t)\,x_1(t-\tau_{11}(t))\right], \\ \dot{x}_2(t) &= r_2(t)\,x_2(t)\left[1 - \frac{x_2(t-\tau_{22}(t))}{a_2(t) + b_2(t)\,x_1(t-\tau_{21}(t))} - c_2(t)\,x_2(t-\tau_{22}(t))\right] \end{aligned} \qquad \ldots (1.3)$$

where $x_i(t)$ represents the density of the species X_i at time t, $i = 1, 2$. $\tau_{ii}(t) \geq 0$ denotes the time delay due to gestation period of species $X_i, i = 1, 2$. $\tau_{12}(t)$ and $\tau_{21}(t)$ are the time delays relating to the maturation of larvae of species X_1 and X_2, respectively. It is assumed in system (1.3) that only adult individuals of species X_1 can be benefit to species X_2 and vice versa.

We have assumed in (1.3) that when the other species X_j is absent, the species X_i is governed by the well known delay logistic equation

$$\frac{dx_i(t)}{dt} = r_i(t)\, x_i(t)\left[1 - \left(\frac{1}{a_i(t)} + c_i(t)\right) x_i(t - \tau_{ii}(t))\right].$$

In this paper, for system (1.3), we always assume that for all i, $j = 1, 2$:

(H1) $r_i(t), a_i(t), b_i(t)$ and $c_i(t)$ are continuously positive periodic functions with period ω;

(H2) $\tau_{ij}(t)$ is a non-negative and continuously differentiable periodic function with period ω, and $\min_{t \in [0, \omega]} (1 - \dot{\tau}_{ij}(t)) > 0$, where $\dot{\tau}_{ij}(t) = d\tau_{ij}(t)/dt$, $\tau = \max\{\tau_{ij}(t), t \in [0, \omega], i, j = 1, 2\}$.

We note that under assumption (H2), $t - \tau_{ij}(t)$ is increasing and $\lim_{t \to +\infty} (t - \tau_{ij}(t)) = +\infty, i, j = 1, 2$. Mathematically, the assumption (H2) will be needed in the proof of the uniqueness and global stability of periodic solutions of system (1.3).

Motivated by the application of system (1.3) to population dynamics, we assume that solutions of system (1.3) satisfy the initial conditions

$$x_i(\theta) = \phi_i(\theta), \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0, i = 1, 2; \qquad \ldots (1.4)$$

where each ϕ_i is a given non-negative and bounded continuous function on $[-\tau, 0]$.

It is well known by the fundamental theory of functional differential equations[10] that system (1.3) has a unique solution $x(t) = (x_1(t), x_2(t))^T$ satisfying the initial conditions (1.4). It is easy to verify that solutions of system (1.3) corresponding to initial conditions (1.4) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$. In this paper, the solution of system (1.3) satisfying initial conditions (1.4) is said to be positive.

The organization of this paper is as follows. In the next section, we discuss the existence of a positive ω-periodic solution of system (1.3)-(1.4). In Section 3, by means of a suitable Lyapunov functional, sufficient conditions are derived for the uniqueness and global stability of the positive periodic solution of (1.3)-(1.4).

2. Existence of a Periodic Solution

In this section, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, we show the existence of a positive periodic solution of (1.3)-(1.4). To this end, we first introduce the following concepts and notations.

Let X, Y be real Banach spaces, let L: $\mathrm{Dom}L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$, then the restriction L_P of L to $\mathrm{Dom}L \cap \mathrm{Ker}P : (I - P)X \to \mathrm{Im}L$ is invertible. Denote the inverse of L_p by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact. Since ImQ is isomorphic to Ker$L$, there exists an isomorphism $J:\mathrm{Im}Q\to\mathrm{Ker}L$.

For convenience of use, we introduce the continuation theorem (see[11], p. 40) as follows:

Lemma 2.1 — Let $\Omega\subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume

(a) for each $\lambda\in(0,1), x\in\partial\Omega\cap\mathrm{Dom}L$, $Lx\neq\lambda Nx$;

(b) for each $x\in\partial\Omega\cap\mathrm{KerL}$, $QNx\neq 0$;

(c) $\deg\{JQN,\Omega\cap\mathrm{KerL},0\}\neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}\cap$ *Dom L*.

In what follows we shall use the following notations:

$$\bar{f}=\frac{1}{\omega}\int_0^{\omega} f(t)\,dt,\quad f^L=\min_{t\in[0,\omega]}|f(t)|,\quad f^M=\max_{[0,\omega]}|f(t)|,$$

where f is a continuous ω-periodic function.

We are now in a position to state our result on the existence of a periodic solution of system (1.3).

Theorem 2.1 — *System (1.3) has at least one positive ω-periodic solution.*

PROOF : Since solutions of (1.3)-(1.4) remain positive for all $t\geq 0$, we let

$$u_1(t)=\ln[x_1(t)],\quad u_2(t)=\ln[x_2(t)] \qquad \text{... (2.1)}$$

and derive that

$$\frac{du_1(t)}{dt}=r_1(t)\left[1-\frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t)+b_1(t)\,e^{u_2(t-\tau_{12}(t))}}-c_1(t)\,e^{u_1(t-\tau_{11}(t))}\right],$$
$$\frac{du_2(t)}{dt}=r_2(t)\left[1-\frac{e^{u_2(t-\tau_{22}(t))}}{a_2(t)+b_2(t)\,e^{u_1(t-\tau_{21}(t))}}-c_2(t)\,e^{u_2(t-\tau_{22}(t))}\right]. \qquad \text{... (2.2)}$$

It is easy to see that if system (2.2) has one ω-periodic solution $\left(\overset{*}{u_1}(t),\overset{*}{u_2}(t)\right)^T$, then $x^*(t)=\left(\overset{*}{x_1}(t),\overset{*}{x_2}(t)\right)^T=\left(\exp\left[\overset{*}{u_1}(t)\right],\exp\left[\overset{*}{u_2}(t)\right]\right)^T$ is a positive ω-periodic solution of system (1.3). Therefore, to complete the proof, it suffices to show that system (2.2) has one ω-periodic solution.

Take

$$X = Y = \{(u_1(t),u_2(t))^T\in C(R,R^2):u_1(t+\omega)=u_1(t),\,u_2(t+\omega)=u_2(t)\},$$

and

$$\left\| (u_1(t), u_2(t))^T \right\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)|,$$

here $|\cdot|$ denotes the Euclidean norm, X is a Banach space. Set

$$L : \mathrm{Dom}L \cap X \to X, \quad L(u_1(t), u_2(t))^T = \left(\frac{du_1(t)}{dt}, \frac{du_2(t)}{dt} \right)^T,$$

where $\mathrm{Dom}L = \{(u_1(t), u_2(t))^T \in C(R, R^2)\}$ and $N : X \to X$,

$$N \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} r_1(t) \left[1 - \dfrac{e^{u_1(t - \tau_{11}(t))}}{a_1(t) + b_1(t)\, e^{u_2(t - \tau_{12}(t))}} - c_1(t)\, e^{u_1(t - \tau_{11}(t)} \right] \\ r_2(t) \left[1 - \dfrac{e^{u_2(t - \tau_{22}(t))}}{a_2(t) + b_2(t)\, e^{u_1(t - \tau_{21}(t))}} - c_2(t)\, e^{u_2(t - \tau_{22}(t)} \right] \end{bmatrix}.$$

Define two projectors P and Q by

$$P \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Q \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega} \displaystyle\int_0^\omega u_1(t)\, dt \\ \dfrac{1}{\omega} \displaystyle\int_0^\omega u_2(t)\, dt \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in X = Y.$$

It is clear that

$$\mathrm{Ker}L = \{x \mid x \in X, \; x = h, \; h \in R^2\},$$

$$\mathrm{Im}L = \left\{ y \mid y \in Y, \int_0^\omega y(t)\, dt = 0 \right\} \text{ is closed in } Y,$$

and

$$\dim \mathrm{Ker}L = \mathrm{codim}\, \mathrm{Im}L = 2.$$

Therefore, L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\mathrm{Im}P = \mathrm{Ker}L, \qquad \mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}\,(I - Q).$$

Furthermore, the inverse K_P of the form K_P: $\mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ exists and is given by

$$K_P(y) = \int_0^t y(s)\, ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s)\, ds dt.$$

Then $QN: X \to Y$ and $K_P(I-Q)N: X \to X$ are given by

$$QNx = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega r_1(t)\left[1 - \frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t)+b_1(t)\,e^{u_2(t-\tau_{12}(t))}} - c_1(t)e^{u_1(t-\tau_{11}(t))}\right]dt \\ \frac{1}{\omega}\int_0^\omega r_2(t)\left[1 - \frac{e^{u_2(t-\tau_{11}(t))}}{a_2(t)+b_2(t)\,e^{u_1(t-\tau_{21}(t))}} - c_2(t)e^{u_2(t-\tau_{22}(t))}\right]dt \end{bmatrix},$$

$$K_P(I-Q)Nx = \int_0^t Nx(s)\,ds - \frac{1}{\omega}\int_0^\omega\int_0^t Nx(s)\,ds dt - \left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega Nx(s)\,ds.$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using the Arzela-Ascoli Theorem, we can easily verify that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\frac{du_1(t)}{dt} = \lambda r_1(t)\left[1 - \frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t)+b_1(t)\,e^{u_2(t-\tau_{12}(t))}} - c_1(t)e^{u_1(t-\tau_{11}(t))}\right],$$

$$\frac{du_2(t)}{dt} = \lambda r_2(t)\left[1 - \frac{e^{u_2(t-\tau_{22}(t))}}{a_2(t)+b_2(t)\,e^{u_1(t-\tau_{21}(t))}} - c_2(t)e^{u_2(t-\tau_{22}(t))}\right]. \quad \text{... (2.3)}$$

Suppose that $(u_1(t), u_2(t))^T \in X$ is a solution of (2.3) for a certain $\lambda \in (0, 1)$. Integrating (2.3) over the interval $[0, \omega]$ leads to

$$\int_0^\omega \frac{r_1(t)\,e^{u_1(t-\tau_{11}(t))}}{a_1(t)+b_1(t)\,e^{u_2(t-\tau_{12}(t))}}dt + \int_0^\omega c_1(t)\,r_1(t)\,e^{u_1(t-\tau_{11}(t))}\,dt = \bar{r}_1\,\omega, \quad \text{... (2.4)}$$

$$\int_0^\omega \frac{r_2(t)\,e^{u_2(t-\tau_{22}(t))}}{a_2(t)+b_2(t)\,e^{u_1(t-\tau_{21}(t))}}dt + \int_0^\omega c_2(t)\,r_2(t)\,e^{u_2(t-\tau_{22}(t))}\,dt = \bar{r}_2\,\omega. \quad \text{... (2.5)}$$

It follows from (2.3), (2.4) and (2.5) that

$$\int_0^\omega \left|u_1'(t)\right| dt = \lambda\int_0^\omega \left| r_1(t)\left[1 - \frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t)+b_1(t)\,e^{u_2(t-\tau_{12}(t))}} - c_1(t)e^{u_1(t-\tau_{11}(t))}\right]\right| dt$$

$$< \lambda \int_0^\omega \left| r_1(t)\left[1 + \frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t) + b_1(t)\, e^{u_2(t-\tau_{12}(t))}} + c_1(t) e^{u_1(t-\tau_{11}(t))} \right] \right| dt$$

$$= 2\overline{r}_1\, \omega,$$

$$\int_0^\omega \left| u_2'(t) \right| dt = \lambda \int_0^\omega \left| r_2(t)\left[1 - \frac{e^{u_2(t-\tau_{22}(t))}}{a_2(t) + b_2(t)\, e^{u_2(t-\tau_{21}(t))}} - c_2(t) e^{u_2(t-\tau_{22}(t))} \right] \right| dt$$

$$< \lambda \int_0^\omega \left| r_2(t)\left[1 + \frac{e^{u_2(t-\tau_{22}(t))}}{a_2(t) + b_2(t)\, e^{u_1(t-\tau_{21}(t))}} + c_2(t) e^{u_2(t-\tau_{22}(t))} \right] \right| dt$$

$$= 2\overline{r}_2\, \omega,$$

that is

$$\int_0^\omega \left| u_i'(t) \right| dt < 2\overline{r}_i\, \omega, \quad i = 1, 2. \qquad \text{... (2.6)}$$

Since $(u_1(t), u_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \min_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \max_{t \in [0, \omega]} u_i(t), \quad i = 1, 2. \qquad \text{... (2.7)}$$

It follows from (2.4), (2.5) and (2.7) that

$$\overline{c_i r_i}\, \omega\, e^{u_i(\xi_i)} \le \int_0^\omega c_i(t)\, r_i(t)\, e^{u_i(t-\tau_{ii}(t))} dt < \overline{r}_i\, \omega, \quad i = 1, 2,$$

which yields

$$u_i(\xi_i) < \ln \left\{ \frac{\overline{r}_i}{\overline{c_i r_i}} \right\}. \qquad \text{... (2.8)}$$

Then we have

$$u_i(t) \le u_i(\xi_i) + \int_0^\omega \left| u_i'(t) \right| dt < \ln \left\{ \frac{\overline{r}_i}{\overline{c_i r_i}} \right\} + 2\,\overline{r}_i\, \omega, \quad i = 1, 2. \qquad \text{... (2.9)}$$

On the other hand, it follows from (2.4) and (2.5) that

$$\overline{r_1}\, \omega = \int_0^\omega r_1(t) \frac{e^{u_1(t-\tau_{11}(t))}}{a_1(t) + b_1\, e^{u_2(t-\tau_{12}(t))}} dt + \int_0^\omega c_1(t)\, r_1(t)\, e^{u_1(t-\tau_{11}(t))} dt$$

$$\le \int_0^\omega \left[\frac{r_1(t)}{a_1(t)} + c_1(t)\, r_1(t) \right] e^{u_1(\eta_1)} dt$$

$$= \omega e^{u_1(\eta_1)} [(\overline{r_1/a_1}) + \overline{c_1 r_1}],$$

$$\overline{r_2}\,\omega = \int_0^{\omega} r_2(t) \frac{e^{u_2(t-\tau_{22}(t))}}{a_2(t) + b_2 e^{u_1(t-\tau_{21}(t))}} dt + \int_0^{\omega} c_2(t)\, r_2(t)\, e^{u_2(t-\tau_{22}(t))} dt$$

$$\leq \int_0^{\omega} \left[\frac{r_2(t)}{a_2(t)} + c_2(t)\, r_2(t) \right] e^{u_2(\eta_2)} dt$$

$$= \omega e^{u_2(\eta_2)} [(\overline{r_2/a_2}) + \overline{c_2 r_2}],$$

which yields

$$u_i(\eta_i) > \ln \left\{ \frac{\overline{r_i}}{(\overline{r_i/a_i}) + \overline{c_i r_i}} \right\}, \quad i = 1, 2. \qquad \text{... (2.10)}$$

It follows from (2.6) and (2.10) that

$$u_i(t) \geq u_i(\eta_i) - \int_0^{\omega} \left| u_i'(t) \right| dt > \ln \left\{ \frac{\overline{r_i}}{(\overline{r_i/a_i}) + \overline{c_i r_i}} \right\} - 2\,\overline{r_i}\,\omega, \quad i = 1, 2. \qquad \text{... (2.11)}$$

which, together with (2.9), implies

$$\max_{t \in [0,\omega]} |u_i(t)| < \max \left\{ \left| \ln \left\{ \frac{\overline{r_i}}{\overline{c_i r_i}} \right\} \right| + 2\overline{r_i}\,\omega,\ \left| \ln \left\{ \frac{\overline{r_i}}{(\overline{r_i/a_i}) + \overline{c_i r_i}} \right\} \right| + 2\,\overline{r_i}\,\omega \right\} := R_i, \quad i = 1, 2. \qquad \text{... (2.12)}$$

Clearly, R_1 and R_2 are independet of λ. Denote $M = R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that each solution $(\alpha^*, \beta^*)^T$ of the system of algebraic equations

$$\overline{r_1} - \frac{1}{\omega} e^{\alpha} \int_0^{\omega} \frac{r_1(t)}{a_1(t) + b_1(t)\, e^{\beta}} dt - \overline{c_1 r_1}\, e^{\alpha} = 0,$$

$$\overline{r_2} - \frac{1}{\omega} e^{\beta} \int_0^{\omega} \frac{r_2(t)}{a_2(t) + b_2(t)\, e^{\alpha}} dt - \overline{c_2 r_2}\, e^{\beta} = 0 \qquad \text{... (2.13)}$$

satisfies $\left\| \alpha^*, \beta^*)^T \right\| = |\alpha^*| + |\beta^*| < M$, provided that system (2.13) has at least one solution. We now take $\Omega = \{(u_1(t), u_2(t))^T \in X : \|(u_1, u_2)^T\| < M\}$. This satisfies condition (*a*) in Lemma 2.1.

When $(u_1(t), u_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(u_1, u_2)^T$ is a constant vector in R^2 with $|u_1| + |u_2| = M$. If system (2.13) has solutions, then we have

$$QN\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \overline{r_1} - \frac{1}{\omega} e^{u_1} \int_0^\omega \frac{r_1(t)}{a_1(t) + b_1(t)\, e^{u_2}} dt - \overline{c_1\, r_1}\, e^{u_1} \\ \overline{r_2} - \frac{1}{\omega} e^{u_2} \int_0^\omega \frac{r_2(t)}{a_2(t) + b_2(t)\, e^{u_1}} dt - \overline{c_2\, r_2}\, e^{u_2} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If system (2.13) does not have a solution, we can directly derive

$$QN\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This proves that condition (*b*) in Lemma 2.1 is satisfied.

Finally, we will prove that condition (*c*) in Lemma 2.1 holds. To this end, we define $\phi : \operatorname{Dom} L \times [0, 1] \to X$ by

$$\phi(u_1, u_2, \mu) = \begin{bmatrix} \overline{r_1} - \overline{c_1\, r_1}\, e^{u_1} \\ \overline{r_2} - \overline{c_2\, r_2}\, e^{u_2} \end{bmatrix}$$

$$+ \mu \begin{bmatrix} -\frac{1}{\omega} e^{u_1} \int_0^\omega \frac{r_1(t)}{a_1(t) + b_1(t)\, e^{u_2}} dt \\ -\frac{1}{\omega} e^{u_2} \int_0^\omega \frac{r_2(t)}{a_2(t) + b_2(t)\, e^{u_1}} dt \end{bmatrix}$$

where $\mu \in [0, 1]$ is a parameter. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(u_1, u_2)^T$ is a constant vector in R^2 with $|u_1| + |u_2| = M$. We will show that when $(u_1(t), u_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L$, $\phi(u_1, u_2, \mu) \neq 0$. Assume the conclusion is not true, i.e., there is a constant vector $(u_1, u_2)^T$ with $|u_1| + |u_2| = M$ satisfying $\phi(u_1, u_2, \mu) = 0$, that is

$$\overline{r_1} - \frac{\mu}{\omega} e^{u_1} \int_0^\omega \frac{r_1(t)}{a_1(t) + b_1(t)\, e^{u_2}} dt - \overline{c_1\, r_1}\, e^{u_1} = 0,$$

$$\overline{r_2} - \frac{\mu}{\omega} e^{u_2} \int_0^\omega \frac{r_2(t)}{a_2(t) + b_2(t)\, e^{u_1}} dt - \overline{c_2\, r_2}\, e^{u_2} = 0.$$

A similar argument in (2.8) and (2.10) shows that

$$|u_i| < \max\left\{\left|\ln\left\{\frac{r_i}{\overline{c_i r_i}}\right\}\right|, \left|\ln\left\{\frac{\overline{r_i}}{(\overline{r_i/a_i}) + \overline{c_i r_i}}\right\}\right|\right\}, \qquad i = 1, 2.$$

Thus, we have

$$|u_1| + |u_2| < M,$$

which leads to a contradiction. Using the property of topological degree and taking $J = I$: $\mathrm{Im}Q \to \mathrm{Ker}\,L$, $(u_1, u_2)^T \to (u_1, u_2)^T$, we have

$$\begin{aligned}
&\deg\,(JQN(u_1, u_2)^T, \Omega \cap \mathrm{Ker}\,L, (0,0)^T) \\
&= \deg\,((\phi(u_1, u_2, 1), \Omega \cap \mathrm{Ker}\,L, (0,0)^T) \\
&= \deg\,((\phi(u_1, u_2, 0), \Omega \cap \mathrm{Ker}\,L, (0,0)^T) \\
&= \deg\,((\overline{r_1} - \overline{c_1 r_1}\, e^{u_1}, \overline{r_2} - \overline{c_2 r_2}\, e^{u_2})^T, \Omega \cap \mathrm{Ker}\,L, (0,0)^T).
\end{aligned}$$

The system of algebraic equations

$$\overline{r_1} - \overline{c_1 r_1}\, e^{u_1} = 0,$$

$$\overline{r_2} - \overline{c_2 r_2}\, e^{u_2} = 0$$

has a unique solution $\left(\overset{*}{u_1}, \overset{*}{u_2}\right)$ which satisfies

$$\overset{*}{u_1} = \ln\left\{\frac{\overline{r_1}}{\overline{c_1 r_1}}\right\}, \quad \overset{*}{u_2} = \ln\left\{\frac{\overline{r_2}}{\overline{c_2 r_2}}\right\}.$$

Thus, we have

$$\begin{aligned}
&\deg\,(JQN(u_1, u_2)^T, \Omega \cap \mathrm{Ker}\,L, (0,0)^T) \\
&= \mathrm{sign}\begin{vmatrix} -\overline{c_1 r_1}\, e^{\overset{*}{u_1}} & 0 \\ 0 & -\overline{c_2 r_2}\, e^{\overset{*}{u_2}} \end{vmatrix} \\
&= \mathrm{sign}\left\{\overline{c_1 r_1}\,\overline{c_2 r_2}\, e^{\overset{*}{u_1} + \overset{*}{u_2}}\right\} = 1.
\end{aligned}$$

By now we have proved that Ω satisfies all the rquirements in Lemma 2.1. Hence, (2.2) has at least one ω-periodic solution. Accordingly, system (1.3) has at least one positive ω-periodic solution. The proof is complete.

3. Uniqueness and Global Stability

We now proceed to a discussion of the uniqueness and global stability of the ω-periodic solution $x^*(t)$ in Theorem 2.1. It is immediate that if $x^*(t)$ is globally asymptotically stable then $x^*(t)$ is in fact unique. We first derive certain upper and lower bound estimates for solutions of (1.3)-(1.4).

Theorem 3.1 — *Let* $x(t) = (x_1(t), x_2(t))^T$ *denote any positive solution of system (1.3) with initial conditions (1.4). If system (1.3) satisfies (H1)-(H2), then there exists a* $T > 0$ *such that*

$$m_i \le x_i(t) \le M_i \quad \text{for} \quad t \ge T, \ i = 1, 2, \qquad \dots (3.1)$$

i.e. system (1.3) is uniformly persistent, where

$$M_i = \frac{1}{c_i^L} e^{\gamma_i^M \tau},$$

$$m_i = \frac{a_i^L}{1 + a_i^L c_i^M} \exp\left\{ r_i^M \left[1 - \left(\frac{1}{a_i^L} + c_i^M \right) \right] M_i \right\}. \qquad \dots (3.2)$$

Proof : Suppose $x(t) = (x_1(t), x_2(t))^T$ is a solution of system (1.3) which satisfies (1.4). Firstly, we prove that there exists a $T_1^* > 0$ such that

$$x_i(t) \le M_i \quad \text{for} \quad t \ge T_1^*, \qquad \dots (3.3)$$

where M_i is defined by (3.2), $i = 1, 2$. It follows from the positivity of the solution of (1.3) that

$$\frac{dx_i(t)}{dt} \le r_i(t)\, x_i(t) \left(1 - c_i^L x_i (t - \tau_{ii}(t)) \right). \qquad \dots (3.4)$$

Set $M_i^* = (1 + k_i)/c_i^L$, where $0 < k_i < \exp\left\{ r_i^M \tau \right\} - 1$. Suppose $x_i(t)$ is not oscillatory about M_i^*; that is, there exists a $T_{i1} > 0$ such that

$$x_i(t) > M_i^* \quad \text{for} \quad t > T_{i1}, \qquad (3.5)$$

or

$$x_i(t) < M_i^* \quad \text{for} \quad t > T_{i1}, \qquad \dots (3.6)$$

If (3.6) holds, then (3.3) follows. Suppose (3.5) holds, (3.4) implies that

$$\frac{dx_i(t)}{dt} < -k_i \gamma_i(t)\, x_i(t) < -k_i r_i^L x_i(t) \quad \text{for all} \quad t \ge T_{i1} + \tau.$$

This will lead to a contradiction. Therefore, there must exist a $T_{i2} \geq T_{i1}$, such that $x_i(T_{i2}) < M_i^*$. If $x_i(t) \leq M_i^*$ for all $t \geq T_{i2}$, then (3.3) follows. If not, then there must exist a $T_{i3} > T_{i2}$ such that $x_i(T_{i3}) > M_i^*$. Therefore, from the above discussion, there exists T_{i4} and T_{i5} such that $x_i(T_{i4}) = x_i(T_{i5}) = M_i^*$ and $x_i(t) > M_i^*$ for $T_{i4} < t < T_{i5}$, where $T_{i2} < T_{i4} < T_{i3} < T_{i5}$. Suppose that $x_i(t)$ on the interval $[T_{i4}, T_{i5}]$ attains its maximum at T_{i6}, where $T_{i4} < T_{i6} < T_{i5}$. Then it is easy to see from (3.4) that

$$0 = \frac{dx_{it}(T_{i6})}{dt} \leq r_i(T_{i6})\, x_i(T_{i6}) \left(1 - c_i^L x_i (T_{i6} - \tau_{ii}(T_{i6})) \right). \qquad \text{... (3.7)}$$

This leads to

$$x_i(T_{i6} - \tau_{ii}(T_{i6})) \leq 1/c_i^L. \qquad \text{... (3.8)}$$

Integrating both sides of (3.4) on the interval $[T_{i6} - \tau_{ii}(T_{i6}), T_{i6}]$, we have

$$\ln \left[\frac{x_i(T_{i6})}{x_i(T_{i6} - \tau_{ii}(T_{i6}))} \right] \leq \int_{T_{i6} - \tau_{ii}(T_{i6})}^{T_{i6}} r_i(t) \left[1 - c_i^L x_i (t - \tau_{ii}(t)) \right] dt.$$

It follows that

$$\left[\frac{x_i(T_{i6})}{x_i(T_{i6} - \tau_{ii}(T_{i6}))} \right] \leq \int_{T_{i6} - \tau_{ii}(T_{i6})}^{T_{i6}} r_i^M \, dt = r_i^M \tau_{ii}(T_{i6}). \qquad \text{... (3.9)}$$

From (3.8) and (3.9), we have

$$x_i(T_{i6}) \leq \frac{1}{c_i^L} e^{r_i^M \tau_{ii}(T_{i6})} \leq \frac{1}{c_i^L} e^{r_i^M \tau} \equiv M_i.$$

Since $x_i(T_{i6})$ is an arbitrary local maximum of $x_i(t)$, therefore, we can conclude that there exists a $T_i > 0$ such that $x_i(t) \leq M_i$ for all $t \geq T_i$. We set $T_1^* = \max\{T_1, T_2\}$. If $t \geq T_1^*$, then $x_i(t) \leq M_i$ for all $i = 1, 2$.

Similarly, we can verify that these exists a $T \geq T_1^*$, such that if $t \geq T$,

$$x_i(t) \geq \frac{a_i^L}{1 + a_i^L c_i^M} \exp\left\{ r_i^M \left[1 - \left(\frac{1}{a_i^L} + c_i^M \right) \right] M_i \right\} \equiv m_i.$$

The proof is complete.

We now formulate the uniqueness and global asymptotic stability of the positive ω-periodic solutions of system (1.3).

Theorem 3.2 — *In addition of (H1)-(H2), assume further that there exist* $k_1 > 0$, $k_2 > 0$, *such that the following hold*:

(H3) $\liminf_{t \to \infty} A_i(t) > 0,$

where

$$A_i(t) = k_i c_i(t) r_i(t) - \frac{k_i r_i(\sigma_{ii}^{-1}(t))\,[a_i(\sigma_{ii}^{-1}(t)) + b_i(\sigma_{ii}^{-1}(t))\,M_j]}{a_i^2(\sigma_{ii}^{-1}(t))\,(1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(t)))}$$

$$- \frac{k_j b_j(\sigma_{ij}^{-1}(t))\,r_j(\sigma_{ji}^{-1}(t))\,M_j}{a_j^2(\sigma_{ji}^{-1}(t))\,(1 - \dot{\tau}_{ji}(\sigma_{ji}^{-1}(t)))} - k_i r_i(t) \int_t^{\sigma_{ii}^{-1}(t)} c_i(s)\,r_i(s)\,ds$$

$$\times \left\{ 1 + \frac{M_i}{a_i^2(t)} [a_i(t) + b_i(t)\,M_j] + c_i(t)\,M_i \right\}$$

$$- \frac{k_i M_i r_i(\sigma_{ji}^{-1}(t))}{1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(t))} \int_{\sigma_{ii}^{-1}(t)}^{\sigma_{ii}^{-1}(t)(\sigma_{ii}^{-1}(t))} c_i(s)\,r_i(s)\,ds \qquad \ldots (3.10)$$

$$\times \left\{ \frac{1}{a_i^2(\sigma_{ii}^{-1}(t))} [a_i(\sigma_{ii}^{-1}(t)) + b_i(\sigma_{ii}^{-1}(t))\,M_j] + c_i(\sigma_{ii}^{-1}(t)) \right\}$$

$$- \frac{k_j M_j^2\, r_j(\sigma_{ji}^{-1}(t))\, r_j(\sigma_{ji}^{-1}(t))\, b_j(\sigma_{ji}^{-1}(t))}{a_j^2(\sigma_{ji}^{-1}(t))\,(1 - \dot{\tau}_{ji}(\sigma_{ji}^{-1}(t)))} \int_{\sigma_{ji}^{-1}(t)}^{\sigma_{jj}^{-1}(\sigma_{ji}^{-1}(t))} c_j(s)\,r_j(s)\,ds,$$

in which $\sigma_{ij}^{-1}(t)$ *is the inverse function of* $\sigma_{ij}(t) = t - \tau_{ij}(t)$, $i, j = 1, 2$, $i \neq j$. *Then system (1.3)-(1.4) has a unique positive ω-periodic solution* $x(t) = \left(x_1^*(t), x_2^*(t) \right)^T$ *which is globally asymptotically stable.*

PROOF : Due to the conclusion of Theorem 2.1, we only need to show the global asymptotic stability of the positive periodic solution of (1.3)-(1.4). Let $x(t) = \left(x_1^*(t), x_2^*(t) \right)^T$ be a positive ω-periodic solution of system (1.3) and $y(t) = (y_1(t), y_2(t))^T$ be any positive solution of system (1.3). It follows from Theorem 3.1 that there exist positive constants T and M_i (defined by (3.2)), such that for all $t \geq T$,

$$0 < x_i^*(t) \le M_i, \quad 0 < y_i(t) \le M_i, \quad i = 1, 2. \qquad \text{... (3.11)}$$

For $i = 1, 2$, we define

$$V_{i1}(t) = \left| \ln x_i^*(t) - \ln y_i(t) \right|. \qquad \text{... (3.12)}$$

For $i = 1, j = 2$ or $i = 2, j = 1$, calculating the upper right derivative of $V_{i1}(t)$ along the solution of (1.3), it follows for $t \ge T$ that

$$D^+ V_{i1}(t) = \left(\frac{\dot{x}_i^*(t)}{x_i^*(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right) \text{sgn}\,(x_i^*(t) - y_i(t))$$

$$= r_i(t)\,\text{sgn}\,(x_i^*(t) - y_i(t))\,[-c_i(t)\,(x_i^*(t-\tau_{ii}(t)) - y_i(t-\tau_{ii}(t)))$$

$$+ \frac{y_i(t) - \tau_{ii}(t))}{a_i(t) + b_i(t)\,y_j(t-\tau_{ij}(t))} - \frac{x_i^*(t-\tau_{ii}(t))}{a_i(t) + b_i(t)\,x_j^*(t-\tau_{ij}(t))} \Bigg]$$

$$= r_i(t)\,\text{sgn}\,(x_i^*(t) - y_i(t)) \Big\{ -c_i(t)\,(x_i^*(t-\tau_{ii}(t)) - y_i(t-\tau_{ii}(t)) \Big\}$$

$$+ \frac{1}{[a_i(t) + b_i(t)\,y_j(t-\tau_{ij}(t))]\,[a_i(t) + b_i(t)\,x_j^*(t-\tau_{ij}(t))]}$$

$$\times \Big\{ -(a_i(t) + b_i(t)\,x_j^*(t-\tau_{ij}(t)))$$

$$\times \Big[x_i^*(t-\tau_{ii}(t)) - y_i(t-\tau_{ii}(t)) \Big]$$

$$+ b_i(t)\,x_i^*(t-\tau_{ii}(t)) \Big[x_j^*(t-\tau_{ij}(t)) - y_j(t-\tau_{ij}(t)) \Big] \}\}$$

$$= \text{sgn}\,(x_i^*(t) - y_i(t))\ \{-c_i(t)\,r_i(t)\,(x_i^*(t) - y_i(t))$$

$$\frac{r_i(t)}{[a_i(t) + b_i(t)\,y_j(t-\tau_{ij}(t))]\,[a_i(t) + b_i(t)\,x_j^*(t-\tau_{ij}(t))]}$$

$$\times \Big\{ -(a_i(t) + b_i(t)\,x_j^*(t-\tau_{ij}(t)))$$

$$\times \Big[x_i^*(t-\tau_{ii}(t)) - y_i(t-\tau_{ii}(t)) \Big]$$

$$+ b_i(t)\,x_i^*(t-\tau_{ii}(t)) \Big[x_j^*(t-\tau_{ij}(t)) - y_j(t-\tau_{ij}(t)) \Big] \}$$

$$+ c_i(t)\,r_i(t) \int_{t-\tau_{ii}(t)}^{t} (\dot{x}_i^*(u) - \dot{y}_i(u)\,du\}$$

$$\leq - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right|$$

$$+ \frac{r_i(t)}{a_i^2(t)} \{(a_i(t) + b_i(t)\, x_j^*(t - \tau_{ij}(t)))$$

$$\times \left| x_i^*(t - \tau_{ii}(t)) - y_i^*(t)(t - \tau_{ii}(t)) \right|$$

$$+ b_i(t)\, x_i^*(t - \tau_{ij}(t)) \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \}$$

$$+ c_i(t)\, r_i(t) \left| \int_{t-\tau_{ii}(t)}^{t} (\dot{x}_i^*(u) - \dot{y}_i(u))\, du \right|$$

$$= - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right| + \frac{r_i(t)}{a_i^2(t)}$$

$$\{(a_i(t) + b_i(t)\, x_j^*(t - \tau_{ij}(t))) \times \left| x_i^*(t - \tau_{ii}(t)) - y_i(t - \tau_{ii}(t)) \right|$$

$$+ b_i(t)\, x_i^*(t - \tau_{ii}(t)) \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \}$$

$$+ c_i(t)\, r_i(t) \left| \int_{t-\tau_{ii}(t)}^{t} r_i(u) \; \{x_i^*(u) \right.$$

$$\times \left[1 - \frac{x_i^*(u - \tau_{ii}(u))}{a_i(u) + b_i(u)\, x_j^*(u - \tau_{ij}(u))} - c_i(u)\, x_i^*(u - \tau_{ii}(u)) \right]$$

$$- y_i(u) \left[1 - \frac{y_i(u - \tau_{ii}(u)}{a_i(u) + b_i(u)\, y_j(u - \tau_{ij}(u))} \right. \qquad \ldots (3.13)$$

$$\left. - c_i(u)\, y_i(u - \tau_{ii}(u))] \} \; du \right|$$

$$= - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right|$$

$$+ \frac{r_i(t)}{a_i^2(t)} \{(a_i(t) + b_i(t)\, x_j^*(t - \tau_{ij}(t)))$$

$$\times \left| x_i^*(t - \tau_{ii}(t)) - y_i(t - \tau_{ii}(t)) \right|$$

$$+ b_i(t)\, x_i^*(t - \tau_{ii}(t)) \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \}$$

$$+ c_i(t)\, r_i(t) \left| \int_{t-\tau_{ii}(t)}^{t} r_i(u) \;\{[x_i^*(u) - y_i(u)] \right.$$

$$+ \frac{1}{a_i(u) + b_i(u)\, y_j(u - \tau_{ij}(u))}$$

$$\times \frac{1}{a_i(u) + b_i(u)\, x_j^*(u - \tau_{ij}(u))}$$

$$\times \{(a_i(u) + b_i(u)\, y_j(u - \tau_{ij}(u)))$$

$$\times \{x_i^*(u)\, [y_i(u - \tau_{ii}(u)) - x_i^*(u - \tau_{ii}(u))]$$

$$+ y_i(u - \tau_{ii}(u))\, [y_i(u) - x_i^*(u)]\}$$

$$+ b_i(u)\, y_i(u)\, y_i(u - \tau_{ii}(u))$$

$$\times [x_j^*(u - \tau_{ij}(u)) - y_j(u - \tau_{ij}(u))]\}$$

$$+ c_i(u) \;\{x_i^*(u)\, [y_i(u - \tau_{ii}(u)) - x_i^*(u - \tau_{ij}(u))]$$

$$\left. + y_i(u - \tau_{ii}(u))\, [y_i(u) - x_i^*(u)]\}\} \; du \right|.$$

By Theorem 3.1, $0 < x_i^*(t) \le M_i$, $0 < y_i(t) \le M_i$ $(i = 1, 2)$ for $t \ge T$. This, together with (3.13), for $t \ge T + \tau$, leads to

$$D^+ V_{i1}(t) \le \; - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right| + \frac{r_i(t)}{a_i^2(t)} \;\{(a_i(t) + b_i(t)\, M_j)$$

$$\times \left| x_i^*(t - \tau_{ii}(t)) - y_i(t - \tau_{ii}(t)) \right|$$

$$+ b_i(t)\, M_i \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \}$$

$$+ c_i(t)\, r_i(t) \int_{t-\tau_{ii}(t)}^{t} r_i(u) \left\{ \left| x_i^*(u) - y_i(u) \right| \right.$$

$$+ \frac{1}{a_i^2(u)} \{(a_i(u) + b_i(u)\, M_j)$$

$$\times \left\{ M_i \left| y_i(u - \tau_{ii}(u)) - x_i^*(u - \tau_{ii}(u)) \right| \right.$$

$$+ M_i \mid y_i(u) - x_i^*(u) \mid$$

$$+ b_i(u) M_i^2 \mid x_j^* (u - \tau_{ij}(u)) - y_j(u - \tau_{ij}(u))\}$$

$$+ c_i(u) \left\{ M_i \left| y_i(u - \tau_{ii}(u)) - x_i^*(u - \tau_{ii}(u)) \right| \right.$$

$$+ M_i \left| y_i(u) - x_i^*(u) \right| \}\} \, du \qquad \text{... (3.14)}$$

$$\leq - c_i(t) r_i(t) \left| x_i^*(t) - y_i(t) \right| + \frac{r_i(t)}{a_i^2(t)} \{(a_i(t) + b_i(t) M_j)$$

$$\times \mid x_i^*(t - \tau_{ii}(t)) - y_i(t - \tau_{ii}(t)) \mid$$

$$+ b_i(t) M_i \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \}$$

$$+ c_i(t) r_i(t) \int_{t - \tau_{ii}(t)}^{t} \gamma_i(u) \left\{ + \frac{M_i}{a_i^2(u)} (a_i(u) + b_i(u) M_j) \right.$$

$$+ c_i(u) M_i] \left| x_i^*(u) - y_i(u) \right| + \left[\frac{M_i}{a_i^2(u)} (a_i(u) + b_i(u) M_j) \right.$$

$$+ c_i(u) M_i] \left| x_i^*(u - \tau_{ii}(u)) - y_i(u - \tau_{ii}(u)) \right|$$

$$\left. + \frac{b_i(u)}{a_i^2(u)} M_i^2 \left| x_j^*(u - \tau_{ij}(u)) - y_i(u - \tau_{ij}(u)) \right| \right\} du.$$

Define

$$V_{i2}(t) = \int_{t - \tau_{ii}(t)}^{t} \frac{r_i(\sigma_{ii}^{-1}(s)) [a_i(\sigma_{ii}^{-1}(s)) + b_i(\sigma_{ii}^{-1}(s)) M_j] \mid x_i^*(s) - y_i(s) \mid}{a_i^2(\sigma_{ii}^{-1}(s)) (1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(s)))} ds$$

$$+ M_i \int_{t - \tau_{ij}(t)}^{t} \frac{b_i(\sigma_{ij}^{-1}(s)) r_i(\sigma_{ij}^{-1}(s)) \mid x_j^*(s) - y_j(s) \mid}{a_i^2(\sigma_{ij}^{-1}(s)) (1 - \dot{\tau}_{ij}(\sigma_{ij}^{-1}(s)))} ds$$

$$+ \int_{t}^{\sigma_{ii}^{-1}(t)} \int_{\sigma_{ii}(s)}^{t} c_i(s) \gamma_i(s) r_i(u) \left\{ \left[1 + \frac{M_i}{a_i^2(u)} (a_i(u) \right. \right.$$

$$+ b_i(u) M_j) + c_i(u) M_i] \left| x_i^*(u) - y_i(u) \right| \qquad \text{... (3.15)}$$

$$+ \left[\frac{M_i}{a_i^2(u)} (a_i(u) + b_i(u) M_j) + c_i(u) M_i \right]$$

$$\times \left| x_i^* (u - \tau_{ii}(u)) - y_i (u - \tau_{ii}(u)) \right|$$

$$\left. + \frac{b_i(u)}{a_i^2(u)} M_i^2 \left| x_j^* (u - \tau_{ij}(u)) - y_j (u - \tau_{ij}(u)) \right| \right\} du\, ds.$$

It follows from (3.14) and (3.15) that for $t \geq T + \tau$

$$D^+ V_{i1}(t) + \dot{V}_{i2}(t) \leq - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right|$$

$$+ \frac{[r_i(\sigma_{ii}^{-1}(t)) \, [a_i(\sigma_{ii}^{-1}(t)) + b_i(\sigma_{ii}^{-1}(t))\, M_j\,] \, | x_i^*(t) - y_i(t) |}{a_i^2(\sigma_{ii}^{-1}(t))\, (1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(t)))}$$

$$+ \frac{b_i(\sigma_{ij}^{-1}(t))\, r_i(\sigma_{ij}^{-1}(t))\, M_i \, | x_j^*(t) - y_j(t) |}{a_i^2(\sigma_{ij}^{-1}(t))\, (1 - \dot{\tau}_{ij}(\sigma_{ij}^{-1}(t)))}$$

$$+ r_i(t) \int_t^{\sigma_{ii}^{-1}(t)} c_i(s)\, r_i(s)\, ds \left\{ \left[+ \frac{M_i}{a_i^2(t)} (a_i(t) \right.\right. \qquad \text{... (3.16)}$$

$$\left. + b_i(t) M_j) + c_i(t) M_i \,\right] \left| x_i^*(t) - y_i(t) \right|$$

$$+ \left[\frac{M_i}{a_i^2(t)} (a_i(t) + b_i(t) M_j) + c_i(t) M_i \right]$$

$$\times \left| x_i^*(t - \tau_{ii}(t)) - y_i(t - \tau_{ii}(t)) \right|$$

$$\left. + \frac{b_i(t)}{a_i^2(t)} M_i^2 \left| x_j^*(t - \tau_{ij}(t)) - y_j(t - \tau_{ij}(t)) \right| \right\}.$$

We now define

$$V_i(t) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t) \qquad \text{... (3.17)}$$

in which

$$V_{i3}(t) = \int_{t - \tau_{ii}(t)}^{t} \int_{\sigma_{ii}^{-1}(l)}^{\sigma_{ii}^{-1}(\sigma_{ii}^{-1}(l))} \frac{c_i(s)\, r_i(s)\, \gamma_i(\sigma_{ii}^{-1}(l))}{1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(l))}$$

$$\times \left[\frac{M_i}{a_i^2(\sigma_{ii}^{-1}(l))} (a_i(\sigma_{ii}^{-1}(l)) + b_i(\sigma_{ii}^{-1}(l))\, M_j) \right.$$

$$+ c_i(\sigma_{ii}^{-1}(l))\, M_i] \left| x_i^*(l) - y_i(l) \right| dsdl \qquad \dots (3.18)$$

$$+ M_i^2 \int_{t-\tau_{ij}(t)}^{t} \int_{\sigma_{ij}^{-1}(l)}^{\sigma_{ii}^{-1}(\sigma_{ij}^{-1}(l))} \frac{c_i(s)\, r_i(s)\, \gamma_i(\sigma_{ij}^{-1}(l))}{1 - \dot{\tau}_{ij}(\sigma_{ij}^{-1}(l))}$$

$$\times \frac{b_i(\sigma_{ij}^{-1}(l))}{a_i^2(\sigma_{ij}^{-1}(l))} \left| x_j^*(l) - y_j(l) \right| \Bigg\} dsdl.$$

It then follows from (3.16)-(3.18) that for $t \geq T + \tau$

$$D^+ V_i(t) \leq - c_i(t)\, r_i(t) \left| x_i^*(t) - y_i(t) \right|$$

$$+ \frac{[r_i(\sigma_{ii}^{-1}(t))\, [a_i(\sigma_{ii}^{-1}(t)) + b_i(\sigma_{ii}^{-1}(t))\, M_j] \mid x_i^*(t) - y_i(t) \mid}{a_i^2(\sigma_{ii}^{-1}(t))\, (1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(t)))}$$

$$+ \frac{b_i(\sigma_{ij}^{-1}(t))\, r_i(\sigma_{ij}^{-1}(t))\, M_i \mid x_j^*(t) - y_j(t) \mid}{a_i^2(\sigma_{ij}^{-1}(t))\, (1 - \dot{\tau}_{ij}(\sigma_{ij}^{-1}(t)))}$$

$$+ r_i(t) \int_t^{\sigma_{ii}^{-1}(t)} c_i(s)\, r_i(s)\, ds \left\{ \left[+ \frac{M_i}{a_i^2(t)} (a_i(t) \right.\right.$$

$$\left.\left. + b_i(t)\, M_j) + c_i(t)\, M_i \right] \left| x_i^*(t) - y_i(t) \right| \right\}$$

$$+ \frac{M_i\, r_i(\sigma_{ii}^{-1}(t))}{1 - \dot{\tau}_{ii}(\sigma_{ii}^{-1}(t))} \int_{\sigma_{ii}^{-1}(t)}^{\sigma_{ii}^{-1}(\sigma_{ii}^{-1}(t))} c_i(s)\, r_i(s)\, ds$$

$$\times \left[\frac{1}{a_i^2(\sigma_{ii}^{-1}(t))} (a_i(\sigma_{ii}^{-1}(t)) + b_i(\sigma_{ii}^{-1}(t))\, M_j) \right.$$

$$\left. + c_i(\sigma_{ii}^{-1}(t)) \right] \left| x_i^*(t) - y_i(t) \right|$$

$$+ \frac{M_i^2\, r_i(\sigma_{ij}^{-1}(t))}{1 - \dot{\tau}_{ij}(\sigma_{ij}^{-1}(t))} \int_{\sigma_{ii}^{-1}(t)}^{\sigma_{ij}^{-1}(\sigma_{ij}^{-1}(t))} c_i(s)\, r_i(s)\, ds \qquad \dots (3.19)$$

$$\times \frac{b_i(\sigma_{ij}^{-1}(t))}{a_i^2(\sigma_{ij}^{-1}(t))} \left| x_j^*(t) - y_j(t) \right|$$

Now we define a Lyapunov function $V(t)$ as

$$V(t) = k_1 V_1(t) + k_2 V_2(t). \qquad \text{... (3.20)}$$

Then it follows from (3.19), (3.20) that for $t \geq T + \tau$

$$D^+ V(t) \leq -A_1(t) \left| \overset{*}{x}_1(t) - y_1(t) \right| - A_2(t) \left| \overset{*}{x}_2(t) - y_2(t) \right| \qquad \text{... (3.21)}$$

where $A_1(t)$ and $A_2(t)$ are defined in (3.10).

By hypothesis (H3), there exist constants $\alpha_i > 0$ $(i = 1, 2)$ and $T^* \geq T + \tau$, such that

$$A_i(t) \geq \alpha_i > 0 \quad \text{for} \quad t \geq T^*. \qquad \text{... (3.22)}$$

Integrating both sides of (3.21) on interval $[T^*, t]$,

$$V(t) + \sum_{i=1}^{2} \int_{T^*}^{t} A_i(s) \left| \overset{*}{x}_i(s) - y_i(s) \right| ds \leq V(T^*). \qquad \text{... (3.23)}$$

It follows from (3.22) and (3.23) that

$$V(t) + \sum_{i=1}^{2} \alpha_i \int_{T^*}^{t} \left| \overset{*}{x}_i(s) - y_i(s) \right| ds \leq V(T^*) \text{ for } t \geq T^*. \qquad \text{... (3.24)}$$

Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} \left| \overset{*}{x}_i(s) - y_i(s) \right| ds < \infty, \qquad i = 1, 2. \qquad \text{... (3.25)}$$

By Theorem 3.1, $\left| \overset{*}{x}_i(t) - y_i(t) \right|$ $(i = 1, 2)$ are bounded on $[T^*, \infty)$.

On the other hand, it is easy to see that $\dot{\overset{*}{x}}_i(t)$ and $\dot{y}_i(t)$ $(i = 1, 2)$ are bounded for $t \geq T^*$. Therefore, $\left| \overset{*}{x}_i(t) - y_i(t) \right|$ $(i = 1, 2)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's Lemma (Lemma 1.2.2 and 1.2.3[7]), we can conclude that

$$\lim_{t \to \infty} \left| \overset{*}{x}_i(t) - y_i(t) \right| = 0. \qquad \text{... (3.26)}$$

The proof of the Theorem is complete.

Remark : If $\tau_{ij}(t) \equiv \tau_{ij}$, where τ_{ij} are non-negative constants, $i, j = 1, 2$, then assumption (H3) can be simplified. In this case, $\tau = \max\{\tau_{ij}, i, j = 1, 2\}$. We therefore have the following result.

Corollary 3.1 — Let $\tau_{ij}(t) \equiv \tau_{ij}$, where $\tau_{ij}(i, j = 1, 2)$ are non-negative constants. In addition to (H1), assume further that there exist $k_1 > 0, k_2 > 0$, such that the following hold:

$$\text{(H4)} \liminf_{t \to \infty} \left\{ k_i c_i(t)\, r_i(t) - \frac{k_i r_i(t+\tau_{ii})}{a_i^2(t+\tau_{ii})} [a_i(t+\tau_{ii}) + b_i(t+\tau_{ii}) M_j] \right.$$

$$- \frac{k_j b_j(t+\tau_{ji})\, r_j(t+\tau_{jj}) M_j}{a_j^2(t+\tau_{ji})}$$

$$- k_i r_i(t) \int_t^{t+\tau_{ii}} c_i(s)\, r_i(s)\, ds \left\{ 1 + \frac{M_i}{a_i^2(t)} [a_i(t) + b_i(t) M_j] + c_i(t) M_i \right\}$$

$$- k_i M_i r_i(t+\tau_{ii}) \int_{t+\tau_{ii}}^{t+2\tau_{ii}} c_i(s)\, r_i(s)\, ds$$

$$\times \left\{ \frac{1}{a_i^2(t+\tau_{ii})} [a_i(t+\tau_{ii}) + b_i(t+\tau_{ii}) M_j] + c_i(t+\tau_{ii}) \right\}$$

$$\left. - \frac{k_j M_j^2 r_j(t+\tau_{ji})\, b_j(t+\tau_{ji})}{a_j^2(t+\tau_{ji})} \int_{t+\tau_{ji}}^{t+\tau_{ji}+\tau_{jj}} c_j(s)\, r_j(s)\, ds \right\} > 0,$$

$i \neq j, i, j = 1, 2.$

Then (1.3)-(1.4) has a unique positive ω-periodic solution which is globally asymptotically stable.

Remark 2 : If $\tau_{ij}(t) \equiv 0$, $i, j = 1, 2$, then system (1.3) reduces to an instantaneous system i.e. one without delay

$$\dot{x}_1(t) = r_1(t)\, x_1(t) \left[1 - \frac{x_1(t)}{a_1(t) + b_1(t)\, x_2(t)} - c_1(t)\, x_1(t) \right],$$

$$\dot{x}_2(t) = r_2(t)\, x_2(t) \left[1 - \frac{x_2(t)}{a_2(t) + b_2(t)\, x_1(t)} - c_2(t)\, x_2(t) \right], \qquad \dots (3.27)$$

where $r_i(t), a_i(t), b_i(t)$ and $c_i(t)$ are continuously positive periodic functions with period ω.

On substituting $\tau_{ij}(t) \equiv 0$, $(i, j = 1, 2)$ into (H3), bv Theorem 3.2, we have the following result.

Corollary 3.2 — If there exist $k_1 > 0, k_2 > 0$, such that the following hold:

$$\text{(H5)} \liminf_{t\to\infty} \left\{ k_i c_i(t)\, r_i(t) - \frac{k_i r_i(t)\,[a_i(t)\, c_j^L + b_i(t)]}{a_i^2\,(t)\, c_j^L} - \frac{k_j\, r_j\,(t)\, b_j\,(t)}{a_j^2\,(t)\, c_j^L} \right\} > 0,$$

$$i \neq j,\, i, j = 1,\ 2.$$

Then system (3.27) has a unique positive ω-periodic solution which is globally asymptotically stable.

Corollary 3.3 — System (1.3) with initial conditions (1.4) has a unique periodic solution which is globally asymptotically stable if (H5) holds and delays $\tau_{ij}(t)\,(i, j = 1, 2)$ are sufficiently small.

In this paper, it has been shown that small delays are negligible for the global asymptotic stability of the positive periodic solution of the delayed periodic cooperative systems (1.3) provided that the delayed negative feedbacks dominate other interspecific interaction effects with delays.

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