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SOME EVALUATIONS OF RAMANUJAN'S CUBIC CONTINUED FRACTION

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As a sequel to some recent works of Berndt and Baruah and Saikia we evaluate $G(e^{-\Pi\sqrt{n}})$ for certain values of n where $G(q)$ is the cubic continued fraction

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + ..$$

of Ramajunan's. The required modular identities are also obtained.

Key Words : Theta-functions; Modular Equations; Continued Fractions

1. INTRODUCTION

Ramanujan's cubic continued fraction $G(q)$, first introduced by Srinivasa Ramanujan in his second letter to Hardy[15], is defined by

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \ldots, \quad |q| < 1. \qquad \ldots (1.1)$$

Chan[12] and Baruah[4] have proved several elegant theorems for $G(q)$. Berndt, Chan and Zhang[9] have proved some general formulas for $G(e^{-\pi\sqrt{n}})$ and $H(e^{-\pi\sqrt{n}})$ where

$$H(q) := -G(-q), \qquad \ldots (1.2)$$

and n is any positive rational, in terms of Ramanujan-Weber class invariants G_n and g_n:

$$G_n := 2^{-1/4} q^{-1/24} (-q; q^2)_\infty$$

and

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$$g_n := 2^{-1/4} q^{-1/24} (q; q^2)_\infty, \qquad q = e^{-\pi\sqrt{n}}.$$

Here, as usual,

$$(a; q)_\infty := \prod_{m=0}^{\infty} (1 - aq^m), \qquad |q| < 1.$$

Ramanathan[13] has found the value of $G(e^{-\pi\sqrt{10}})$ by using Kronecker's limit formula. Adiga *et al.*[1,2], Vasuki and Shivashankara[18], have obtained evaluation of $G(e^{-\pi\sqrt{n}})$ for several positive rationals n by using modular equations and transformation formulas for the theta-function. Recently, Baruah and Nipen Saikia[5] have found some general theorems for explicit evaluations of $G(e^{-\pi\sqrt{n}})$ by employing modular equations found in[4,6,7,8]

Ramanujan recorded several $P-Q$ eta-function identities in Chapter 25 of his second notebook[14]. Berndt[7], (p. 204-237) and Berndt and Zhang[10] proved these $P-Q$ equations on employing various modular equations belonging to the classical theory. Baruah[3] has given proofs of some of these identities using certain eta-function relations.

Motivated by the above works, we establish in Section 4 of this paper, some general theorems for $G(e^{-\pi\sqrt{n}})$ and thereby evaluate $G(e^{-\pi\sqrt{n}})$ for certain special values of n. The required $P-Q$ modular identities are obtained in Section 3.

2. Some Preliminary Results

Let

$$(a)_0 := 1$$

and

$$(a)_k := a(a+1)(a+2)\ldots(a+k-1)$$

for any positive integer k.

A modular equation of degree n is an equation relating α and β that is induced by

$$n\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(1-\alpha)\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = n\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(1-\beta)\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} x^k, \qquad |x| < 1.$$

Then, we also say that β is of nth degree over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 :={}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ and $z_n :={}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

In the sequel, we need the following modular equation of degree[3] (Berndt[6], p. 234),

$$\left(\frac{1-\beta}{1-\alpha}\right)^{1/2} = \frac{m(m+1)}{3-m}. \qquad \text{... (2.1)}$$

***Theorem* 2.1** — *If* $f(-q) := \prod_{n=1}^{\infty} (1-q^n)$,

$$P := \frac{f^2(-q)}{q^{1/6} f^2(-q^3)} \quad \textit{and} \quad Q := \frac{f^2(-q^2)}{q^{1/3} f^2(-q^6)},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \qquad \text{... (2.2)}$$

For a proof, see[7] (p. 204, entry 51).

***Theorem* 2.2** — *If*

$$X = \frac{f^2(-q^3)}{q^{1/6} f(-q) f(-q^9)} \quad \textit{and} \quad Y = \frac{f^2(q^6)}{q^{1/3} f(-q^2) f(-q^{18})},$$

then

$$(XY)^4 - 3(XY)^2 = X^6 + Y^6. \qquad \text{... (2.3)}$$

For a proof, see[11] (Theorem 3.1 (iv)).

***Theorem* 2.3** — *If*

$$X = \frac{f(-q^3) f(-q^5)}{q^{1/3} f(-q) f(-q^{15})} \quad \textit{and} \quad Y = \frac{f(-q^6) f(-q^{10})}{q^{2/3} f(-q^2) f(-q^{30})},$$

then

$$(XY)^2 - XY = X^3 + Y^3. \qquad \text{... (2.4)}$$

For a proof, see[7] (p. 215, Eq. 59.10).

***Theorem* 2.4** — *If* $\alpha\beta = \pi$, *then*

$$\sqrt{\alpha}\, \varphi(e^{-\alpha^2}) = \sqrt{\beta}\, \varphi(e^{-\beta^2}), \qquad \text{... (2.5)}$$

where

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

For a proof, see[6] (p. 43, Entry 27(i)).

Theorem 2.5 — *If* $\alpha\beta = \pi^2$, *then*

$$e^{-\alpha/8} \sqrt[4]{\alpha}\, \psi(-e^{-\alpha}) = e^{-\beta/8} \sqrt[4]{\beta}\, \psi(-e^{-\beta}), \qquad \text{... (2.6)}$$

where

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

For proof, see[1].

Theorem 2.6 — *We have*

$$G(q) = \frac{1}{2}\left(1 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}\right)^{1/3} \qquad \text{... (2.7)}$$

For a proof, see[6] (p. 347).

Theorem 2.7 — *We have*

$$H(q) = \frac{1}{2}\left(\frac{\varphi^4(q)}{\varphi^4(^3)} - 1\right)^{1/3} \qquad \text{... (2.8)}$$

For a proof of (2.8), change q to $-q$ in (2.7).

Theorem 2.8 — *We have*

$$H(q) = \frac{1}{\left(1 + \dfrac{\varphi^4(-q)}{q\varphi^4(-q^3)}\right)^{1/3}}. \qquad \text{... (2.9)}$$

For a proof, see the second of the set of entries 1(i) of[6] (p. 345).

3. Modular Equations

Theorem 3.1 — *If*

$$P := \frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q := \frac{\varphi(-q)}{\varphi(-q^3)},$$

then

$$\frac{P}{Q} + \frac{Q}{P} = \frac{3}{PQ} - PQ.$$

PROOF : If

$$y = \pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; (1-x)\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}$$

and $q = e^{-y}$, then from Entries 10(i), (ii) of Chapter 17[6] (p. 122), we have

$$\varphi(e^{-y}) = \sqrt{z} \qquad \text{... (3.1)}$$

and

$$\varphi(-e^{-y}) = \sqrt{z}\,(1-x)^{1/4} \qquad \text{... (3.2)}$$

Let β have degree 3 over α, then from (3.1) and (3.2), we deduce

$$m = P^2 \quad \text{and} \quad \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} = \frac{P^2}{Q^2}. \qquad \text{... (3.3)}$$

Substituting (3.3) in (2.1), we obtain

$$Q^2 = \frac{3-P^2}{1+P^2}, \qquad \text{... (3.4)}$$

which is equivalent to the required result.

Theorem 3.2 — *Let*

$$P := \frac{\varphi(-q)}{\varphi(-q^3)} \quad \textit{and} \quad Q := \frac{\varphi(-q^2)}{\varphi(-q^6)}.$$

Then

$$\frac{3}{Q^2} - Q^2 = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2.$$

PROOF : If x, y and z are as in Theorem 3.1, then from Entry 10(iii) of Chapter 17[6] (p. 122), we have

$$\varphi(-e^{-2y}) = \sqrt{z}\,(1-x)^{1/8}. \qquad \text{... (3.5)}$$

With α, β and m as in Theorem 3.1, we have from (3.2) and (3.5)

$$\frac{Q^4}{P^2} = m \quad \text{and} \quad \frac{Q^4}{P^4} = \left(\frac{1-\beta}{1-\alpha}\right)^{1/2}.$$

Substituting the above in (2.1), we have

$$P^2 = \frac{3P^2 - Q^4}{P^2 + Q^4}. \qquad \text{... (3.6)}$$

The identity (3.6) is readily seen to be equivalent to the required result.

Theorem 3.3 — *Let*

$$P := \frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q := \frac{\varphi(q^2)}{\varphi(q^6)}.$$

Then

$$(PQ)^2 + \frac{9}{(PQ)^2} + 2(P^2 - Q^2) = 6\left(\frac{1}{P^2} - \frac{1}{Q^2}\right) - \left(\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right) + 12.$$

PROOF : If $S = \frac{\varphi(-q^2)}{\varphi(-q^6)}$, we have from (3.6)

$$S^4 = \frac{3P^2 - P^4}{P^2 + 1}. \qquad \ldots (3.7)$$

Setting q to q^2 in (3.4), we have

$$S^2 = \frac{3 - Q^2}{1 + Q^2}. \qquad \ldots (3.8)$$

Eliminating S between (3.7) and (3.8), we have the required result.

Theorem 3.4 — *Let*

$$P := \frac{\varphi^2(-q)}{\varphi^2(-q^3)} \quad \textit{and} \quad Q := \frac{\varphi^2(-q^3)}{\varphi^2(-q^9)}.$$

Then

$$PQ + \frac{9}{PQ} = 3 + 6\frac{Q}{P} + \frac{Q^2}{P^2}.$$

PROOF : We have from[6] (p. 39, Entry 24(*iii*)).

$$\varphi(-q) = \frac{f^2(-q)}{f(-q^2)}.$$

Hence, we can rewrite P and Q as

$$P = \frac{f^4(-q) f^2(-q^6)}{f^2(-q^2)(f^4(-q^3)} \quad \text{and} \quad Q = \frac{f^4(-q) f^2(-q^{18})}{f^2(-q^6)(f^4(-q)^9}.$$

Let

$$L_1 := \frac{f^2(-q)}{q^{1/6} f^2(-q^3)}, \quad L_2 := \frac{f^2(-q^3)}{q^{1/2} f^2(-q^9)},$$

$$M_1 := \frac{f^2(-q^2)}{q^{1/3} f^2(-q^6)} \quad \text{and} \quad M_2 := \frac{f^2(-q^6)}{q f(-q^{18})}, \qquad \text{... (3.9)}$$

so that, on employing Theorem 2.1, twice, we have

$$L_i M_i + \frac{9}{L_i M_i} = \left(\frac{M_i}{L_i}\right)^3 + \left(\frac{L_i}{M_i}\right)^3, \quad i = 1, 2. \qquad \text{... (3.10)}$$

Further, we have

$$P = \frac{L_1^2}{M_1} \quad \text{and} \quad Q = \frac{L_2^2}{M_2}.$$

These give

$$L_1^6 = \frac{P^4 (P^2 - 9)}{P^2 - 1} \qquad \text{... (3.11)}$$

and

$$L_2^6 = \frac{Q^2 (Q^2 - 9)}{Q^2 - 1}. \qquad \text{... (3.12)}$$

From (3.11) and (3.12), we have

$$\left(\frac{L_2}{L_1}\right)^6 = \frac{Q^4 (Q^2 - 9)(P^2 - 1)}{P^4 (P^2 - 9)(Q^2 - 1)}. \qquad \text{... (3.13)}$$

It is clear from (3.9) that

$$X = \left(\frac{L_2}{L_1}\right)^{1/2} \quad \text{and} \quad Y = \left(\frac{M_2}{M_1}\right)^{1/2},$$

where X and Y are as in Theorem 2.2. Thus from (2.3), we have

$$\left(\frac{L_2 M_2}{L_1 M_1}\right)^2 - 3\,\frac{L_2 M_2}{L_1 M_1} = \left(\frac{L_2}{L_1}\right)^3 + \left(\frac{M_2}{M_1}\right)^3.$$

Using (3.10) in the above identity, we deduce that

$$\left(\frac{L_2}{L_1}\right)^6 = \frac{Q^4}{P^4}\left(\frac{Q + 3P}{Q - P}\right)^2. \qquad \text{... (3.14)}$$

From (3.13) and (3.14), we obtain

$$\left(\frac{Q + 3P}{Q - P}\right)^2 = \frac{(Q^2 - 9)(P^2 - 1)}{(Q^2 - 1)(P^2 - 9)} \qquad \text{... (3.15)}$$

The identity (3.15) is readily seen to be equivalent to the required result.

We note the following corollary for later use.

Corollary 3.1 — Let

$$P := \frac{\varphi^2(q)}{\varphi^2(q^3)} \quad \text{and} \quad Q := \frac{\varphi^2(q^3)}{\varphi^2(q^9)}.$$

Then

$$PQ + \frac{9}{PQ} = 3 + 6\frac{Q}{P} + \frac{Q^2}{P^2}. \qquad \text{... (3.16)}$$

PROOF : Changing q to $-q$ in Theorem 3.4, we obtain (3.16).

Theorem* 3.5 — *Let

$$P := \frac{\psi(q)}{q^{1/4}\,\psi(q^3)} \quad \text{and} \quad Q := \frac{\psi(q^3)}{q^{3/4}\,\psi(q^9)}.$$

Then

$$(PQ)^2 + \frac{9}{(PQ)^2} = 3 + 6\frac{Q^2}{P^2} + \frac{Q^4}{P^4}.$$

PROOF : Using the result[6] (p. 39, Entry 24(*iii*)) $\psi(q) = \frac{f^2(-q^2)}{f(-q)}$ in the above expressions for P and Q, we have

$$P := \frac{f(-q^3)f^2(-q^2)}{q^{1/4}f(-q)f^2(-q)^6} \quad \text{and} \quad Q := \frac{f(-q^9)f^2(-q^6)}{q^{3/4}f(-q^3)f^2(-q^{18})}.$$

Let

$$L_1 := \frac{f(-q)}{q^{1/12}f(-q^3)}, \quad L_2 := \frac{f(-q^3)}{q^{1/4}f(-q^9)},$$

$$M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)}, \quad \text{and} \quad M_2 := \frac{f(-q^6)}{q^{1/2}f(-q^{18})}.$$

Thus, we have

$$P = \frac{M_1^2}{L_1} \quad \text{and} \quad Q = \frac{M_2^2}{L_2}. \qquad \text{... (3.17)}$$

Employing (2.3) to L_1 and M_1, we have

$$(L_1 M_1)^2 + \frac{9}{(L_1 M_1)^2} = \left(\frac{M_1}{L_1}\right)^6 + \left(\frac{L_1}{M_1}\right)^6.$$

Using (3.17) in the above identity, we deduce that

$$M_1^{12} = \frac{P^8 (P^4 - 9)}{P^4 - 1}, \quad M_2^{12} = \frac{Q^8 (Q^4 - 9)}{Q^4 - 1}. \qquad \text{... (3.18)}$$

The eq. (3.18) and the above identity imply that

$$\left(\frac{M_2}{M_1}\right)^{12} = \frac{Q^8 (Q^4 - 9)(P^4 - 1)}{P^8 (Q^4 - 1)(P^4 - 9)}. \qquad \text{... (3.19)}$$

We have

$$X = \frac{L_2}{L_1} \quad \text{and} \quad Y = \frac{M_2}{M_1},$$

where X and Y are as in (2.3). Thus, we have

$$\left(\frac{L_2 M_2}{L_1 M_1}\right)^4 - 3\left(\frac{L_2 M_2}{L_1 M_1}\right)^2 = \left(\frac{L_2}{L_1}\right)^6 + \left(\frac{M_2}{M_1}\right)^6.$$

Using (3.17) in the above identity, we deduce that

$$\left(\frac{M_2}{M_1}\right)^{12} = \frac{Q^8}{P^8}\left(\frac{Q^2 + 3P^2}{Q^2 - P^2}\right)^2. \qquad \text{... (3.20)}$$

Thus, from (3.19) and (3.20), we obtain

$$\left(\frac{Q^2 + 3P^2}{Q^2 - P^2}\right)^2 = \frac{(Q^4 - 9)(P^4 - 1)}{(Q^4 - 1)(P^4 - 9)}.$$

We deduce the required result from the above identity.

Corollary 3.2 — Let

$$P := \frac{\psi(-q)}{q^{1/4}\,\psi(-q^3)} \quad \text{and} \quad Q := \frac{\psi(-q^3)}{q^{3/4}\,\psi(-q^9)}.$$

Then

$$(PQ)^2 + \frac{9}{(PQ)^2} = 3 - 6\frac{Q^2}{P^2} + \frac{Q^4}{P^4}. \qquad \text{... (3.21)}$$

PROOF : Changing q to $-q$ in Theorem 3.6, we deduce (3.21).

4. EVALUATIONS OF RAMANUJAN'S CUBIC CONTINUED FRACTION

Lemma 4.1 — For $q = e^{-\pi\sqrt{n/3}}$, let

$$J_n := \frac{1}{\sqrt[4]{3}} P(q)$$

and

$$D_n := \frac{1}{\sqrt[4]{3}} Q(q)$$

Then

$$(i)\ J_n J_{1/n} = 1, \qquad \ldots (4.1)$$

$$(ii)\ J_1 = 1, \qquad \ldots (4.2)$$

$$(iii)\ D_n = \sqrt{\frac{\sqrt{3} - J_n^2}{1 + \sqrt{3}\, J_n^2}}, \qquad \ldots (4.3)$$

$$(iv)\ G(q) = \frac{1}{2} \sqrt[3]{1 - 3 D_n^4}, \qquad \ldots (4.4)$$

and

$$(v)\ H(q) = \frac{1}{2} \sqrt[3]{3 J_n^4 - 1}. \qquad \ldots (4.5)$$

Here $P(q)$ and $Q(q)$ are as in Theorem 3.1 and $G(q)$ and $H(q)$ are as in (1.1) and (1.2).

PROOF : Using the defining expressions for J_n and $J_{1/n}$ in the left side of (4.1) and applying (2.5) twice we readily obtain (4.1). Similarly, (4.3) follows from (3.4) and the definitions of J_n and D_n; eqs. (4.4) and (4.5) follow respectively from (2.7) and (2.8). For (4.2) we need only set $n = 1$ in (4.1).

Theorem 4.1 — *If J_n is as defined in Lemma 4.1, then*

$$3J_n^2 J_{4n}^2 + \frac{3}{J_n^2 J_{4n}^2} + 2\sqrt{3}\left(\frac{1}{J_{4n}^2} - \frac{1}{J_n^2}\right) + \left(\frac{J_{4n}^2}{J_n^2} + \frac{J_n^2}{J_{4n}^2}\right) = 2\sqrt{3}\left(J_{4n}^2 - J_n^2\right) + 12. \qquad \ldots (4.6)$$

PROOF : It is enough to realize that (4.6) is the same as the result of Theorem 3.3 with $P(q) = \sqrt[4]{3}\, J_n$, as defined earlier and $Q(q) = P(q^2) = \sqrt[4]{3}\, J_{4n}$.

Theorem 4.2 — *We have*

$$J_2 = \sqrt{\sqrt{6} + \sqrt{3} - \sqrt{2} - 2}$$

and

$$J_{1/2} = \sqrt{\sqrt{6} + 2 - \sqrt{3} - \sqrt{2}}.$$

PROOF : Setting $n = 1/2$ in (4.6) and using (4.1), we obtain

$$J_2^4 + J_2^{-4} - 44\sqrt{3}\, J_2^2 + 4\sqrt{3}\, J_2^{-2} - 6 = 0.$$

Putting $J_2^2 = ix$, we deduce that

$$x^2 + \frac{1}{x^2} + 4\sqrt{3}\, i \left(x + \frac{1}{x} \right) + 6 = 0.$$

Setting $x + x^{-1} = y$ in the above equation, we have

$$y^2 + 4\sqrt{3}\, iy + 4 = 0.$$

Since J_n is positive and decreasing in n, we have $J_2 < J_1 = 1$.

Hence

$$y = i\,(4 - 2\sqrt{3}), \quad x = -\,i\,(\sqrt{3} - 2 + 2\sqrt{2 - \sqrt{3}}\,)$$

and

$$J_2 = \sqrt{\sqrt{6} + \sqrt{3} - \sqrt{2} - 2}.$$

By (4.1), it follows that

$$J_{1/2} = \sqrt{\sqrt{6} + 2 - \sqrt{3} - \sqrt{2}}.$$

Corollary 4.1 — We have

$$(i)\ D_1 = \frac{\sqrt{3} - 1}{\sqrt{2}},$$

$$(ii)\ D_2 = \sqrt{\sqrt{2} - 1},$$

and

$$(iii)\ D_{1/2} = \sqrt{\sqrt{6} + \sqrt{2} - \sqrt{3} - 2}.$$

PROOF : Using (4.2) in (4.3) with $n = 1$, we obtain part (*i*). Similarly, (4.3) and Theorem (4.2) give parts (*ii*) and (*iii*).

Corollary 4.2 — We have

$$(i)\ G\,(e^{-\pi\sqrt{3}}) = 2\,\sqrt[3]{12\sqrt{3} - 20},$$

$$(ii)\ G\,(e^{-\pi\sqrt{2/3}}) = \frac{1}{2}\,\sqrt[3]{6\sqrt{2} - 8},$$

$$(iii)\ G\,(e^{-\pi\sqrt{6}}) = \frac{1}{2}\,\sqrt[3]{30\sqrt{2} - 24\sqrt{3} + 18\sqrt{6} - 44},$$

$$(iv)\ H\,(e^{-\pi\sqrt{2/3}}) = \frac{1}{2}\,\sqrt[3]{44 + 30\sqrt{2} - 24\sqrt{3} + 18\sqrt{6}},$$

and

$$(v)\ H\,(e^{-\pi\sqrt{6}}) = \frac{1}{2}\,\sqrt[3]{44 - 30\sqrt{2} - 24\sqrt{3} + 18\sqrt{6}}.$$

PROOF : For proving part (*i*), we need only apply part (*i*) of Corollary 4.1 to (4.4). Similarly, for obtaining parts (*ii*) and (*iii*), we apply parts (*ii*) and (*iii*) of Corollary 4.1 to eq. (4.4). Finally, application of results of Theorem 4.2 to (4.5) yields parts (*iv*) and (*v*).

***Theorem* 4.3** — *We have*

$$J_4 = \sqrt{3} + 1 - \frac{1}{2}\sqrt{6} - \frac{1}{2}\sqrt{2}$$

and

$$J_{1/4} = \frac{1}{2}\sqrt{6} + \sqrt{3} - \frac{1}{2}\sqrt{2} - 1.$$

PROOF : Setting $n = 1$ in (4.6) and using $J_1 = 1$, we obtain

$$(4 - 2\sqrt{3})\, J_4^2 + (4 + 2\sqrt{3})\, J_4^{-2} - 12 = 0.$$

Putting $J_4^2 = t$, we obtain

$$(4 - 2\sqrt{3})\, t^2 - 12t + 4 + 2\sqrt{3} = 0.$$

Since J_n is positive and decreasing in n, we have $J_4 < J_2 < 1$.

Thus, $t = (3 - 2\sqrt{2})(2 + \sqrt{3})$ and $J_4 = \sqrt{3} + 1 - \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}$, which implies

$$J_{1/4} = \frac{1}{2}\sqrt{6} + \sqrt{3} - \frac{1}{2}\sqrt{2} - 1.$$

Corollary 4.3 — We have

(*i*) $D_4 = (2 - \sqrt{3})^{1/4}$,

and

$$D_{1/4} = \sqrt{6 - 4\sqrt{2} - 3\sqrt{3} + 2\sqrt{6}}.$$

PROOF : By applying the results of Theorem 4.3 to (4.3) we at once have the required results.

Corollary 4.4 — We have

(*i*) $G(e^{-2\pi\sqrt{3}}) = \frac{1}{2}\sqrt[3]{3\sqrt{3} - 5}$,

(*ii*) $G(e^{-\pi\sqrt{12}}) = \frac{1}{2}\sqrt[3]{252\sqrt{2} + 204\sqrt{3} - 144\sqrt{6} - 356}$,

(*iii*) $H(e^{-2\pi\sqrt{3}}) = \frac{1}{2}(356 - 252\sqrt{2} + 204\sqrt{3} - 144\sqrt{6})^{1/3}$,

and

$$(iv)\ H(e^{-\pi\sqrt{12}}) = \frac{1}{2}(356+252\sqrt{2}-204\sqrt{3}-144\sqrt{6})^{1/3}.$$

PROOF : For proving part (*i*), we apply part (*i*) of Corollary 4.3 to eq. (4.4). Similarly, part (*ii*) is obtained from part (*ii*) of Corollary 4.3 and the eq. (4.4). Lastly, parts (*iii*) and (*iv*) follow from the two results of Theorem 4.3 and the eq. (4.5).

Theorem 4.4 — *If* J_n *is as defined in Lemma 4.1, then*

$$3J_n^2 J_{9n}^2 + \frac{3}{J_n^2 J_{9n}^2} = 3 + 6\frac{J_{9n}^2}{J_n^2} + \frac{J_{9n}^4}{J_n^4}. \quad \text{... (4.7)}$$

PROOF : The proof is similar to that of Theorem 4.1. Indeed, (4.7) is the same as the result of Corollary 3.1 with $P(q) = \sqrt{3}\,J_n^2$ and $Q(q) = P(q^3) = \sqrt{3}\,J_{9n}^2$.

Theorem 4.5 — *We have*

$$J_3 = (2\sqrt{3}-3)^{1/4}$$

and

$$J_{1/3} = \frac{(2\sqrt{3}+3)^{1/4}}{3^{1/4}}.$$

PROOF : Setting $n = 1/3$ in (4.7) and using $J_{1/n} = \frac{1}{J_n}$, we have

$$J_3^8 + 6J_3^4 - 3 = 0.$$

Since J_3 is real, we have $J_3 = \sqrt[4]{2\sqrt{3}-3}$.

From (4.1), we have $J_{1/3} = \left(\frac{2\sqrt{3}+3}{3}\right)^{1/4}$

Corollary 4.5 — We have

$$(i)\ D_3 = \sqrt{\frac{\sqrt{3}-a}{1+\sqrt{3}a}},$$

$$(ii)\ D_{1/3} = 3^{-1/4}\sqrt{\frac{\sqrt{3}-b}{1+b}},$$

where $a = \sqrt{2\sqrt{3}-3}$ and $b = \sqrt{2\sqrt{3}+3}$.

PROOF : For proving part (*i*), we need only apply the first part of Theorem 4.5 to eq. (4.3). Part (*ii*) follows similarly.

Corollary 4.6 — We have

$$G(e^{-\pi}) = \frac{1}{2}\sqrt[3]{1-3a_1},$$

$$G(e^{-\pi/3}) = \frac{1}{2}\sqrt[3]{1-3b_1},$$

$$H(e^{-\pi}) = \frac{1}{2}\sqrt[3]{6\sqrt{3}-10},$$

$$H(e^{-\pi/3}) = \frac{1}{2}\sqrt[3]{2\sqrt{3}+2},$$

where $a_1 = \left(\frac{\sqrt{3}-a}{1+\sqrt{3}\,a}\right)^2$ and $b_1 = \frac{1}{3}\left(\frac{3-b}{1+b}\right)^2$, with a and b are as in Corollary 4.5.

Theorem 4.5 — *We have*

$$J_9 = \sqrt{\sqrt[3]{4}-1}$$

and

$$J_{1/9} = \frac{1}{\sqrt{\sqrt[3]{4}-1}}.$$

PROOF : Setting $n = 1$ in (4.7) and using $J_1 = 1$, we have

$$J_9^6 + 3J_9^4 + 3J_9^2 - 3 = 0.$$

Putting $J_9^2 = t$, we have

$$t^3 + 3t^2 + 3t - 3 = 0.$$

Solving the above equations for real root[17], we find that

$$t = 4^{1/3} - 1.$$

This implies $J_9 = \sqrt{\sqrt[3]{4}-1}$ and $J_{1/9} = \frac{1}{\sqrt{\sqrt[3]{4}-1}}$.

Corollary 4.7 — We have

(*i*) $D_9 = \sqrt{\frac{1+\sqrt{3}-\sqrt[3]{4}}{1-\sqrt{3}-\sqrt{3}\,\sqrt[3]{4}}}$,

(*ii*) $D_{1/9} = \sqrt{\frac{1+\sqrt{3}-\sqrt{3}\,\sqrt[3]{4}}{1-\sqrt{3}-\sqrt[3]{4}}}$.

PROOF : By applying the results of Theorem 4.6 to (4.3), we at once have the required results.

Corollary 4.8 — We have

(*i*) $G(e^{-\pi\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3a},$

$$(ii)\ G(e^{-\pi/3\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3b},$$

$$(iii)\ H(e^{-\pi\sqrt{3}}) = \frac{1}{2}\sqrt[3]{3(\sqrt[3]{4}-1)^2-1},$$

and

$$(iv)\ H(e^{-\pi/3\sqrt{3}}) = \frac{1}{2}\sqrt[3]{\frac{3}{(\sqrt[3]{4}-1)^2}-1},$$

where

$$a=\left(\frac{1+\sqrt{3}-\sqrt[3]{4}}{1-\sqrt{3}+\sqrt{3}\ \sqrt[3]{4}}\right)^2 \quad \text{and} \quad b=\left(\frac{1+\sqrt{3}-\sqrt{3}\ \sqrt[3]{4}}{1-\sqrt{3}-\sqrt[3]{4}}\right)^2.$$

PROOF : For proving part (*i*), we need only apply part (*i*) of Corollary 4.7 to (4.4). Similarly, for part (*ii*), we apply part (*ii*) of Corollary 4.7 to eq. (4.4). Finally, application of results of Theorem 4.6 to (4.5) yields parts (*iii*) and (*iv*).

For proving our Theorem 4.5 below, we need the following identity of Baruah[4], which can be proved on the same lines as Theorem 3.3.

If $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ and $Q := \dfrac{\varphi(q^5)}{q^{15}}$, then

$$(PQ)^2+\frac{9}{(PQ)^2}=\left(\frac{Q}{P}\right)^3+5\left(\frac{Q}{P}\right)^2+5\left(\frac{Q}{P}\right)-5\left(\frac{P}{Q}\right)-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3. \qquad \text{... (4.8)}$$

***Theorem* 4.7** — *J_n satisfies the following*

$$3(J_nJ_{25n})^2+\frac{3}{(J_nJ_{25n})^2}$$

$$=\left(\frac{J_{25n}}{J_n}\right)^3+5\left(\frac{J_{25n}}{J_n}\right)^2+5\left(\frac{J_{25n}}{J_n}\right)-5\left(\frac{J_n}{J_{25n}}\right)+5\left(\frac{J_n}{J_{25n}}\right)^2-\left(\frac{J_n}{J_{25n}}\right)^3. \qquad \text{... (4.9)}$$

PROOF : Proof is similar to that of Theorem 4.1. Indeed, (4.9) is the same as (4.8) with $P(q)=\sqrt[4]{3}\,J_n$, as defined earlier, and $Q(q)=P(q^5)=\sqrt[4]{3}\,J_{25n}$.

***Theorem* 4.8** — *We have*

$$(i)\quad J_5 = \sqrt{\frac{\sqrt{5}-1}{2}},$$

and

$$(ii)\quad J_{1/5} = 3^{-1/4}\sqrt{\frac{\sqrt{5}+1}{2}}$$

PROOF : Setting $n = 1/5$ in (4.9) and then using (4.1) with $J_5^2 = it$ we find that

$$t^6 - 5it^5 - 5t^4 - 6it^3 - 5t^2 - 5it + 1 = 0.$$

Setting $t + t^{-1} = x$ in the above, we have

$$x^3 - 5\,ix^2 - 8x + 4i = 0.$$

By the standard method of solving this eq. [17, p. 27] and since J_5 is real and $0 < J_5 < 1$, we have

$$t = i\left(\frac{1-\sqrt{5}}{2}\right)$$

and

$$J_5 = \sqrt{\frac{\sqrt{5}-1}{2}},$$

where we have chosen the root which leads to $0 < J_5 < 1$.

Corollary 4.9 — We have

$$(i)\quad D_5 = \sqrt{\frac{1+2\sqrt{3}-\sqrt{5}}{2-\sqrt{3}+\sqrt{15}}},$$

and

$$(ii)\quad D_{1/5} = \sqrt{\frac{2\sqrt{3}-\sqrt{5}-1}{2+\sqrt{3}+\sqrt{15}}}.$$

PROOF : For proving part (*i*), we apply part (*i*) of Theorem 4.8 to (4.3). Similarly, part (*ii*) is obtained from part (*ii*) of Theorem 4.8 and eq. (4.3).

Corollary 4.10 — We have

$$(i)\ G\left(e^{\pi\sqrt{5/3}}\right) = \frac{1}{2}\sqrt[3]{1-3a},$$

where

$$a = \left(\frac{1+2\sqrt{3}-\sqrt{5}}{2-\sqrt{3}+\sqrt{15}}\right)^2.$$

$$(ii)\ G\left(e^{\pi\sqrt{15}}\right) = \frac{1}{2}\sqrt[3]{1-3b},$$

where

$$b = \left(\frac{2\sqrt{3}-\sqrt{5}-1}{2+\sqrt{3}+\sqrt{15}}\right)^2.$$

(*iii*) $H(e^{\pi\sqrt{5/3}}) = \frac{1}{2}\sqrt[3]{\frac{7-3\sqrt{5}}{2}}$

(*iv*) $H(e^{\pi\sqrt{15}}) = \frac{1}{2}\sqrt[3]{\frac{7+3\sqrt{5}}{2}}$

PROOF : For proving part (*i*), we need only apply part (*i*) of Corollary 4.9 to (4.4). Similarly, for part (*ii*), we apply part (*ii*) of corollary 4.9 to eq. (4.4). Finally, application of results of Theorem 4.7 to (4.5) yields parts (*iii*) and (*iv*).

Theorem* 4.9 — *We have

$$J_{25} = \frac{\sqrt{a^2+4}-a}{2} \quad \text{and} \quad J_{1/25} = \frac{2}{\sqrt{a^2+4}-a}, \text{ where } a = \frac{2+(80)^{1/3}-(100)^{1/3}}{3}.$$

PROOF : Setting $n = 1$ in (4.8) and then using $J_1 = 1$, we find that

$$J_{25}^6 + 2J_{25}^5 + 5J_{25}^4 - 5J_{25}^2 + 2J_{25} - 1 = 0.$$

Setting $J_{25} = it$, we find that

$$t^6 - 2it^5 - 5t^4 - 5t^2 - 2it + 1 = 0.$$

Setting $t + t^{-1} = x$, in the above we have

$$x^3 - 2ix^2 - 8x + 4i = 0.$$

By the standard method of solving this equation[17] (p. 27) and since $0 < J_{25} < 1$, we obtain

$$J_{25} = \frac{\sqrt{a^2+4}-a}{2} \quad \text{where } a = \frac{2+(80)^{1/3}-(100)^{1/3}}{3}.$$

Corollary 4.11 — We have

(*i*) $D_{25} = \sqrt{\frac{\sqrt{3}-b}{1+\sqrt{3}b}}$,

and

(*ii*) $D_{1/25} = \sqrt{\frac{\sqrt{3}b-1}{b+\sqrt{3}}}$,

where

$$b = \frac{2+a^2-a\sqrt{a^2+4}}{2},$$

with

$$a = \frac{2 + (80)^{1/3} - (100)^{1/3}}{3}.$$

PROOF : By applying the results of Theorem 4.9 to (4.3), we have the required results.

Corollary 4.12 — We have

$$(i)\ G(e^{-5\pi\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3c},$$

where

$$c = \left(\frac{2\sqrt{3} - 2 - a^2 + a\sqrt{a^2+4}}{2 + \sqrt{3}\,(2 + a^2 - a\sqrt{a^2+4})} \right)^2.$$

$$(ii)\ G(e^{-\pi/5\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3d},$$

where

$$d = \left(\frac{\sqrt{3}\,(2 + a^2 - a\sqrt{a^2+4}) - 2}{2 + a^2 - a\sqrt{a^2+4} + 2\sqrt{3}} \right)^2.$$

$$(iii)\ H(e^{-5\pi/\sqrt{3}}) = \frac{1}{2}\sqrt[3]{3c_1 - 1},$$

and

$$(iv)\ H(e^{-\pi/5\sqrt{3}}) = \frac{1}{2}\sqrt[3]{\frac{9}{c_1} - 1},$$

where

$$c_1 = \left(\frac{\sqrt{a^2+4} - a}{2} \right)^4,$$

with

$$a = \frac{2 + (80)^{1/3} - (100)^{1/3}}{3}.$$

PROOF : For proving part (*i*), we apply part (*i*) of Corollary 4.11 to (4.4). Similarly, for part (*ii*), we apply part (*ii*) of Corollary 4.11 to eq. (4.4). Finally, application of results of Theorem 4.9 to (4.5) yields parts (*iii*) and (*iv*).

The following lemma is due to Baruah and Saikia[5].

Lemma 4.2. — (Baruah and Saikia[5]). If $q = e^{-\pi\sqrt{n/3}}$ and

$$A_n := \frac{\psi(-q)}{\sqrt[3]{3}\, q^{1/4}\, \psi(-q^3)},$$

then

$$(i)\ A_n A_{1/n} = 1, \qquad \ldots (4.10)$$

$$(ii)\ A_1 = 1, \qquad \ldots (4.11)$$

$$(iii)\ H(q) = \frac{1}{\sqrt[3]{3A_n^4+1}}. \qquad \ldots (4.12)$$

PROOF : Using the defining expressions for A_n and $A_{1/n}$ in the left side of (4.10) and applying (2.6) twice we readily obtain (4.10). Similarly, (4.12) follows from (2.9) and the definition of A_n. For (4.11) we need to set $n = 1$ in (4.10).

While Baruah and Saikia[5] use their above lemma to obtain evaluations of A_5, A_{15}, A_{25} and $A_{1/25}$ and related evaluations, we employ it for evaluations of $A_3, A_{1/3}, A_9, A_{1/9}$ and $H(e^{-\pi\sqrt{3}})$ and $H(e^{-\pi/\sqrt{3}})$.

Theorem 4.10 — *If A_n is as defined in Lemma 4.2, then*

$$3A_n^2 A_{9n}^2 + \frac{2}{A_n^2 A_{9n}^2} = 3 - 6\frac{A_{9n}^2}{A_n^2} + \frac{A_{9n}^4}{A_n^4}. \qquad \ldots (4.13)$$

PROOF : Setting $q = e^{-\pi\sqrt{n/3}}$ in (3.21), we deduce (4.13).

Theorem 4.11 — *We have*

$$A_3 = (3 + 2\sqrt{3})^{1/4}$$

and

$$A_{1/3} = \left(\frac{2\sqrt{3}-3}{3}\right)^{1/4}$$

PROOF : Setting $n = 1/3$ in (4.13) and upon using (4.10), we find that

$$A_3^8 - 6A_3^4 - 3 = 0.$$

Since A_n is real and increasing in n, we have $A_3 > 1$. Hence

$$A_3 = (3 + 2\sqrt{3})^{1/4}.$$

This implies, by (4.10),

$$A_{1/3} = \left(\frac{2\sqrt{3}-3}{3}\right)^{1/4}.$$

Corollary 4.13 — We have

$$H(e^{-\pi}) = \frac{1}{\sqrt[3]{10+6\sqrt{3}}}$$

and

$$H(e^{-\pi/3}) = \frac{1}{\sqrt[3]{\sqrt{2}\sqrt{3}-2}}.$$

PROOF : The results follow from Theorem 4.11 and (4.12).

Theorem 4.12 — *We have*

$$A_9 = \sqrt{3+2\sqrt[3]{2}(\sqrt[3]{2}+1)}$$

and

$$A_{1/9} = \frac{1}{\sqrt{3+2\sqrt[3]{2}(\sqrt[3]{2}+1)}}.$$

PROOF : Setting $n = 1$ in (4.13) and using $A_1 = 1$, we have

$$A_9^6 - 9A_9^4 + 3A_9^2 - 3 = 0.$$

Setting $A_9^2 = t$, we have

$$t^3 - 9t^2 + 3t - 3 = 0.$$

Since $J_9 > 1$ and real, by solving above equation by a standard method[17], (p. 27), we have

$$t = \sqrt{3+2\sqrt[3]{2}(\sqrt[3]{2}+1)}, \qquad A_9 = \sqrt{\sqrt{3+2\sqrt[3]{2}(\sqrt[3]{2}+1)}},$$

and

$$A_{1/9} = 1/\sqrt{\sqrt{3+2\sqrt[3]{2}(\sqrt[3]{2}+1)}}.$$

Corollary 4.14 — We have

$$H(e^{-\pi\sqrt{3}}) = \frac{1}{\sqrt[3]{3a^2+1}},$$

and

$$H(e^{-\pi/3\sqrt{3}}) = \frac{1}{\sqrt[3]{3+a^2}},$$

where

$$a = 3+2\sqrt[3]{2}(\sqrt[3]{2}+1).$$

PROOF : The results follow from Theorem 4.12 and (4.12).

Below, we describe briefly how evaluations of $A_5, A_{1/5}, A_{25}, A_{1/25}$ and related evaluations of Baruah and Saikia[5] can be obtained based on our following eq. (4.14), which is of lower degree than theirs.

Theorem 4.13 — *A_n satisfies the following:*

$$3(A_n A_{25n})^2 + \frac{3}{(A_n A_{25n})^2}$$

$$= \left(\frac{A_{25n}}{A_n}\right)^3 + 5\left(\frac{A_{25n}}{A_n}\right)^2 + 5\left(\frac{A_{25n}}{A_n}\right) - 5\left(\frac{A_n}{A_{25n}}\right) + 5\left(\frac{A_n}{A_{25n}}\right)^2 - \left(\frac{A_n}{A_{25n}}\right)^3. \quad \text{... (4.14)}$$

For a proof of (4.14), we employ the following modular identity of Baruah[4] which can be proved on the same lines as Theorem 3.5.

$$\text{If} \quad P := \frac{\psi(-q)}{q^{1/4}\,\psi(-q^3)} \quad \text{and} \quad Q := \frac{\psi(-q^5)}{q^{5/4}\,\psi(-q^{15})},$$

then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{Q}{P}\right) - 5\left(\frac{P}{Q}\right) + 5\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3.$$

Theorem 4.14 — *We have*

$$A_5 = \sqrt{2+\sqrt{5}}$$

and

$$A_{1/5} = \sqrt{\sqrt{5}-2}.$$

PROOF : Setting $n = 1/5$ in Theorem 4.13, then using (4.10) and putting $A_5^2 = t$, we find that

$$t^6 - 5t^5 + 5t^4 - 6t^3 - 5t^2 - 5t - 1 = 0.$$

Setting $t = ix$, we obtain the following reciprocal equation

$$x^6 + 5ix^5 - 5x^4 - 6ix^3 - 5x^2 + 5ix + 1 = 0.$$

Setting $x + x^{-1} = y$, in the above, we find that

$$y^3 + 5iy^2 - 8y - 16i = 0.$$

Solving this equation by a standard method[17] (p. 27), we deduce the required result.

Corollary 4.15 — We have

$$H(e^{-\pi/\sqrt{5/3}}) = \frac{1}{\sqrt[3]{28+12\sqrt{5}}},$$

and

$$H(e^{-\pi/\sqrt{15}}) = \frac{1}{\sqrt[3]{28-12\sqrt{5}}}.$$

PROOF : The results follow from Theorem 4.14 and (4.12).

Theorem 4.15 — *We have*

$$A_{25} = \frac{a+\sqrt{a^2+9}}{3}$$

and

$$A_{1/25} = \frac{3}{a+\sqrt{a^2+9}},$$

where

$$a = 4 + \sqrt[3]{10} + \sqrt[3]{100}.$$

PROOF : Setting n = 1 in Theorem 4.13 and then using (4.11), we find that

$$A_{25}^6 - 8A_{25}^5 + 5A_{25}^4 - 5A_{25}^2 - 8A_{25} - 1 = 0.$$

Setting $A_{25} = it$, we find that

$$t^6 + 8it^5 - 5t^4 - 5t^2 + 8it + 1 = 0.$$

Setting $t + t^{-1} = y$, in the above, we find that

$$y^3 + 8iy^2 - 8y - 16i = 0.$$

Solving this equation by a standard method[17] (p. 27), we deduce the required result.

Corollary 4.16 — We have

$$H(e^{5\pi/\sqrt{3}}) = \frac{1}{\sqrt[3]{3c+1}},$$

and

$$H(e^{\pi/5\sqrt{3}}) = \frac{1}{\sqrt[3]{(3c)+1}},$$

where

$$c = \left[1 + \frac{2a\,[a+\sqrt{a^2+9}]}{9}\right]^2$$

with

$$a = 4 + \sqrt[3]{10} + \sqrt[3]{100}.$$

PROOF : The results follow from Theorem 4.15 and (4.11).

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