Indian J. pure appl. Math., **35**(8) : 1027-1031, August 2004

ON THE VALUE DISTRIBUTION OF A TRANSCENDENTAL MEROMORPHIC FUNCTION AND ITS DERIVATIVES

CHUNG-CHUN YANG

Department of Mathematics, The Hong Kong University of Science and Technology, Kowloon, Hong Kong, China
e-mail: mayang@ust.hk

(*Received* 16 *February* 2004; *accepted* 15 *July 2004)*

Let f denote a transcendental meromorphic function and $b(z)$ be an arbitrary small function of $f, b(z) \not\equiv 0$. If $N\left(r, \frac{1}{f}\right)$ the counting function of the zeroes of f satisfies $N\left(r, \frac{1}{f}\right) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(1)\, T(r,f)$ as $r \to +\infty$ possibly outside a set of r of finite linear measure and $T(r,f)$ is the Nevanlinna characteristic function of f, then, for any positive integer $n, N\left(r, \frac{1}{f^{(n)} - b}\right) \neq S(r,f)$.

Key Words: Differential Polynomial; Value Distribution Zeros

1. INTRODUCTION AND MAIN RESULT

Let F denote a transcendental meromorphic function. As the studies of the possible Picard exceptional values of $F' F^n\ (n \geq 1)$, series of interesting results have been obtained, see e.g.[1,3]. Very recently Bergweiler and Pang, by noting if $f = \frac{1}{n+1} F^{n+1}, (n \geq 1)$, then $f' = F' F^n$, obtained the following result:

Theorem **A** ([2], p. 286) — *Let f be a transcendental meromorphic function and R be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of f are multiple, except possibly finitely many. Then $f' - R$ has infinitely many zeros.*

They also proved in[2] that f is of finite order, then Theorem A still holds without the assumption that the poles be multiple. In the proof of the theorem, the following result was used.

Theorem **B** [4] — *Let f be a transcendental meromorphic function and p be a polynomial, $p \not\equiv 0$. Then for any integer $k > 0$ and $\varepsilon > 0$,*

$$T(r,f) \leq \left(1 + \frac{1}{k} + \varepsilon\right)\left\{\overline{N}\left(r + \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - p}\right)\right\} + S(r,f).$$

In this note, we shall use an argument which is quite different from that of used in the proofs of the above mentioned results and derive the following general result.

Theorem — *Let F be a transcendental meromorphic function satisfying*

$$N\left(r, \frac{1}{F}\right) = S(r, F) \quad \text{... (1)}$$

where $S(r, F)$ denotes any quantity that satisfies $S(r, F) = o(1)\,T(r, F)$ as $r \to +\infty$, except a set of finite linear length. Then, for any $n \geq 1$ and any small function $b(z)$ ($\not\equiv 0$) of F,

$$N\left(r, \frac{1}{F^{(n)} - b}\right) \neq S(r, F). \quad \text{... (2)}$$

Our proof is almost a straight forward application of the so called Tumura-Clunie theorem below.

Lemma ([5], p. 69) — Suppose that $f(z)$ is meromorphic and not constant (in the plane), that

$$g(z) = f(z)^n + P_{n-1}(f), \quad \text{... (3)}$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n - 1$ in f, and that

$$N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f). \quad \text{... (4)}$$

Then $g(z) = h(z)^n$, $h(z) = f(z) + \frac{1}{n}a(z)$ and $h(z)^{n-1}a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h'(z)$ for $f'(z)$, etc., in the terms of degree $n - 1$ in $P_{n-1}(f)$. Thus $g(z)$ is of the form

$$g(z) = \left(f(z) + \frac{a(z)}{n}\right)^n, \quad \text{... (5)}$$

where $a(z)$ (a small function of f) is determined by the terms of degree $n - 1$ in $P_{n-1}(f)$ and by $g(z)$.

Proof the theorem.

Set

$$F = \frac{1}{f}. \quad \text{... (6)}$$

Then

$$T(r, F) = T(r, f) + O(1), N\left(r, \frac{1}{F}\right) = N(r, f). \quad \text{... (7)}$$

Now $F' = \frac{-f'}{f^2}$, $F'' = \frac{-ff' + 2f'^2}{f^3}$ ... etc. Thus, in general,

$$F^{(n)} = \frac{Q_n(f)}{f^{n+1}}, \quad \text{... (8)}$$

where $Q_n(f)$ denotes a homogeneous differential polynomial in f of degree n. Thus

$$F^{(n)} - b(z) = \frac{Q_n(f) - bf^{n+1}}{f^{n+1}}. \qquad \dots (9)$$

If the assertion of the theorem were false, i.e.

$$N\left(r, \frac{1}{F^{(n)} - b}\right) = S(r,f), \qquad \dots (10)$$

then from (9), we would have

$$f^{n+1} - \frac{1}{b} Q_n(f) = g \qquad \dots (11)$$

and, from (7), that

$$N(r,f) + N\left(r, \frac{1}{g}\right) = S(r,f). \qquad \dots (12)$$

Thus by an application of the lemma, we conclude that

$$g = \left(f + \frac{a(z)}{n+1}\right)^{n+1} \qquad \dots (13)$$

where $a(z)$ is determined by the equation: $h(z) = f(z) + \frac{1}{n+1} a(z)$ and $h(z)^n a(z) = -\frac{1}{b} Q_n(h)$. Hence $a(z) = -\frac{1}{b} \frac{Q_n(h)}{h^n} = S(r,h) = S(r,f)$.

Here we use the fact that h satisfies $N(r,h) = S(r,h)$. Therefore, for any l, $T\left(r, \frac{h^{(l)}}{h}\right) = S(r,f)$. Thus from (11) and (13), we have

$$\left(f + \frac{1}{n+1} a\right)^{n+1} = f^{n+1} + af^n + \sum_{k=2}^{n+1} C_k^{n+k} \left(\frac{a}{n+1}\right)^k f^{n+1-k}$$

$$= f^{n+1} - \frac{1}{b} Q_n(f). \qquad \dots (14)$$

Since $af^n \equiv -\frac{1}{b} Q_n(f)$, it follows that

$$\sum_{k=2}^{n+1} C_k^{n+k} \left(\frac{a}{n+1}\right)^k f^{n+1-k} \equiv 0,$$

which is impossible unless $a \equiv 0$. But then, from (14), $-\frac{1}{b} Q_n(f) \equiv 0$ and we would have $F^{(n)} \equiv 0$ which is absurd. The theorem is thus proved.

It's easily seen from the above proof that we can state the following more general result.

***Theorem* 2** — *Let F denote a transcendental meromorphic function satisfying $N\left(r, \frac{1}{F}\right) = S(r, F)$ and $a_i(z), i = 1, 2, \ldots, n-1$ be an arbitrary set of small functions of f and that not all of them are identically zero. Then for any small function $b(z)$ ($\equiv 0$) of F,*

$$N\left(r, \frac{1}{G-b}\right) \neq S(r, F),$$

where

$$G(z) = \sum_{i=1}^{n-1} a_{in}(z) F^{(i)}.$$

As an application of the above two results and arguments, we have the following:

Corollary 1 — Let F be an entire function satisfying $N\left(r, \frac{1}{F}\right) = S(r, F)$ and $b(z)$ be any small function of $F, b(z) \not\equiv 0$. Set $f(z) \equiv \int_{z_0}^{z} [F(z) - b(z)]\, dz$. Then $N\left(r, \frac{1}{f}\right) \neq S(r, f)$.

Corollary 2 — Let $h_i, i = 1, 2, 3$ be three entire functions with $h_1(z) h_2(z) h_3(z) \not\equiv 0$. Let n be a positive integer, $P_i(z), i = 1, 2, \ldots, n-1$ be a set of polynomials and α be an entire function. Then the following differential equation:

$$f(z) \left\{ f^{(n)}(z) - \sum_{i=1}^{n-1} P_i(z) f^{(i)}(z) - h_1(z) \right\} = \frac{h_2(z)}{h_3(z)} e^{\alpha(z)}$$

has no transcendental meromorphic solution f that satisfies $T(r, h_1) + T(r, h_2) = S(r, f)$.

Corollary 3 — Let a, b and c be three nonzero constants. Let $R(z)$ be a rational function, $R(z) \not\equiv 0$ and $\alpha(z)$ be an entire function. Then the following functional equation:

$$a(f')^2 + bf + c = \int_{z_0}^{z} \mathrm{Re}^{\alpha}\, dz,$$

where z_0 is an arbitrary point in the complex plane C, has no transcendental meromorphic solution f.

Here we would like to pose a conjecture which relates to the value distribution of the anti-derivative of an entire function.

Conjecture — If α is a nonconstant entire function, and z_0 is an arbitrary point in the plane C, then $F(z) = \int_{z_0}^{z} e^{\alpha(z)}\, dz$ must have infinitely many zeros.

Remark : The conjecture is easily shown to be true when α is a polynomial.

REFERENCES

1. W. Bergweiler and A. Eremenko, *Rev. Mat. Iberoamerican*, **II** (1995), 355-73.
2. W. Bergweiler and X. C. Pang, *JMAA*, **278** (2003), 285-92.
3. H. H. Chen and M. L. Fang, *Sci. Chinese Ser.*, **A38** (1995), 789-98.
4. X. H. Hua, *Kodai Math. J.*, **13** (1990), 386-90.
5. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.