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SPLINE MODULES FROM A DIVIDED DOMAIN TO A SUBDIVIDED DOMAIN*

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Let Δ be a polyhedral complex embedded in $\mathbb{R}^d$ and $S^r(\Delta)$ denote the R-module of all splines on Δ of smoothness degree $r \geq 0$, where $R = \mathbb{R}[x_1, x_2, ..., x_d]$ is the polynomial ring in d variables. If Δ' is a subdivision of Δ of arbitrary small mesh, then we show that any generating set for $S^r(\Delta)$ can be extended to a generating set for $S^r(\Delta')$. This generalizes a recent result of Ruth Haas who proved the case when the subdivision is obtained from latter by subdividing just one d-face.

Key Words : Polyhedral Complex; Subdivision; Affine Form; Spline Module; Gröbner Basis

1. INTRODUCTION

Let Ω be a compact, simple connected region in the Euclidean space $\mathbb{R}^d$ and Δ be a polyhedral decomposition of Ω into properly joined convex polyhedra. We assume that Δ is a pure polyhedral complex, i.e., all maximal faces are d-dimensional (see[1] for details). Let $r \geq 0$ be a fixed positive integer and $S^r(\Delta)$ denote the set of all splines on Δ of smoothness degree r. This means the elements of $S^r(\Delta)$ are real-valued functions f defined on Ω such that $f|\sigma$ is a polynomial in $\mathbb{R}[x_1, x_2, ..., x_d]$ on each convex polyhedron $\sigma \in \Delta$, and the partial derivatives of f of all orders up to r exist and are continuous on Ω. It is well-known (see Billera and Rose[2]) that $S^r(\Delta)$ is a ring with point-wise operations. Since elements of the ring $R = \mathbb{R}[x_1, x_2, ..., x_d]$ themselves are splines of smoothness r on Δ, we find that R is a subring of $S^r(\Delta)$ and hence $S^r(\Delta)$ is a module over R. It is known (see[2]) that this module is finitely generated, torsion free and of finite rank over R. In case it is free, then finding an R-module basis has been a problem of great interest (see Deo[3], Deo and Mazumdar[7]).

Suppose the polyhedral complex Δ' is obtained from Δ by subdividing some of its d-faces into smaller convex polyhedra. Then $S^r(\Delta')$ is also an R-module. The general question is as to

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how these two R-modules, viz., $S^r(\Delta)$ and $S^r(\Delta')$ are related? This question has been studied by Ruth Haas[8], where she has proved an interesting result that a generating set for the R-module $S^r(\Delta)$ can be extended to a generating set for $S^r(\Delta')$ provided Δ' has been obtained from Δ by subdividing exactly one d-face of Δ. The objective of this paper is to generalize the result of Ruth Haas[8] for those subdivisions Δ' of Δ which can be obtained from Δ by subdividing any finite number of d-faces $\sigma_1, \sigma_2, \ldots, \sigma_t$, $t \geq 1$, of Δ, rather than only one d-face. As a result, we find that Δ' can be chosen as a finer subdivision of Δ having its mesh arbitrarily small. This fact could be of use in approximation theory.

Ruth Haas has also described (see[8]) two distinct ways in which a d-face of Δ can be subdivided: The first method is to subdivide a d-face σ of Δ into further d-faces without subdividing any of the boundary $(d-1)$-faces of σ, the second method is to subdivide a d-face σ of Δ into further d-faces in which some of the boundary $(d-1)$-faces of σ have already been subdivided into smaller $(d-1)$-faces. Let us call the first method as type-I subdivision of Δ, and the second method as type-II subdivision of Δ. She has also proved that if the result is true for type-I subdivisions of Δ, then the result is automatically true for the type-II subdivisions of Δ. Hence, to prove the result for arbitrary subdivisions, it is enough to prove it for the type-I subdivisions. We will, therefore, confine our attention only to type-I subdivisions of Δ.

2. PRELIMINARIES

Let Δ be a fixed polyhedral decomposition of a given domain $\Omega \subset \mathbb{R}^d$. We assume that Δ is strongly connected and so are the stars of any of the d-faces of Δ (see Billera[1]). Suppose $\sigma_1, \sigma_2, \ldots, \sigma_t$ are all the d-faces of Δ. If $f: \Omega \to \mathbb{R}$ is a spline on Δ of smoothness order $r \geq 0$, then f is really a t-tuple of polynomials $(f_1, f_2, \ldots, f_t)$ where $f_i = f \mid \sigma_i$ for $i = 1, 2, \ldots, t$. In addition, the smoothness condition on f can be expressed algebraically by using the algebraic criterion of Billera[1], viz., $f \in C^r$ on the union $\sigma_i \cup \sigma_j$ of any two adjacent faces σ_i, σ_j of Δ iff,

$$f_i - f_j = l^{r+1} f_k,$$

for some polynomial $f_k \in R$, where l is the linear form representing the interior $(d-1)$-face $\sigma_i \cap \sigma_j$. Thus a t-tuple of polynomials $f = (f_1, f_2, \ldots, f_t)$ is a spline on Δ of smoothness order r iff for all possible adjacent faces σ_i, σ_j, $1 \leq i, j \leq t$, the above linear relations in f_i and f_j are satisfied for suitable polynomials f_k.

The above condition can be nicely expressed in the form of a matrix as follows: Consider a matrix with n-rows and $(t+n)$-columns, where n is the number of interior $(d-1)$-faces of Δ and t is the number of its d-faces. To make the definition clear, let us take a simple example and consider the Fig. 1. Let l_1, l_2, l_3 be the linear forms representing the three interior edges in Fig. 1.

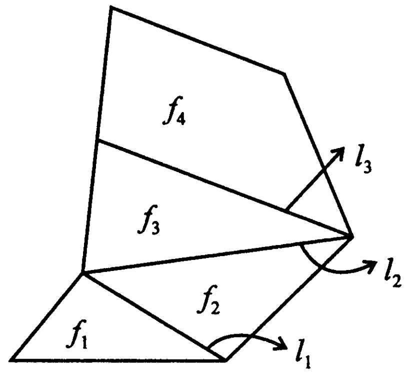

FIG. 1.

Then, in order that the 4-tuple (f_1,f_2,f_3,f_4) of polynomials in two variables represents a spline of smoothness r, all of the following relations must be satisfied:

$$f_1 - f_2 = l_1^{r+1} f_5$$

$$f_2 - f_3 = l_2^{r+1} f_6$$

$$f_3 - f_4 = l_3^{r+1} f_7$$

for some polynomials f_5, f_6 and f_7 in $R = \mathbb{R}[x_1, x_2]$. Now, let us consider the following spline matrix with entries in R,

$$\begin{bmatrix} 1 & -1 & 0 & 0 & l_1^{r+1} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & l_2^{r+1} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & l_3^{r+1} \end{bmatrix}$$

It is clear that (f_1,f_2,f_3,f_4) is a spline on Δ of smoothness order r iff there exists (hence unique) a 7-typle $(f_1,f_2,\dots,f_7)$ of polynomials such that $MX = 0$, where M is the above matrix and $X = [f_1,f_2,\dots,f_7]^T$. It may be mentioned that the spline (f_1,f_2,f_3,f_4) of smoothness order r uniquely determines the other 3-tuple of polynomials satisfying the condition $MX = 0$. This means given M, there is a 1-1 correspondence between the solutions of the matrix equation $MX = 0$ and the spline module $S^r(\Delta)$ of all splines on Δ of smoothness order r. Following Ruth Haas[8], we will use this representation of the spline module and say that the matrix M represents the spline module $S^r(\Delta)$. The entries of the $n \times (t+n)$ matrix M for the general case, when there are n interior

$(d-1)$-faces and t d-faces and Δ is a subdivision of a region in $\mathbb{R}^d$, should now be clear. Note that the numbers n and t are independent of each other.

We also point out that, for obvious reasons, the sum of all the columns of the above spline matrix corresponding to all the d-faces is always a zero column — this fact will be utilized in the proof of our main result.

3. THE MAIN THEOREM

First of all let us explain what do we mean by saying "the polyhedral complex Δ' is obtained from the polyhedral complex Δ by subdividing its d-faces $\sigma_1, \sigma_2, ..., \sigma_t$ by the type-I method". Let us choose a d-face, say σ_1, of Δ and subdivide it into convex polyhedra by the type-I method. Let the resulting complex be denoted by Δ_1. Then we choose a d-face τ of Δ_1. There is no loss in generality in assuming that τ is some d-face of Δ, say $\tau = \sigma_2$, other than σ_1. This is so because if τ were a newly created d-face of Δ_1 not in Δ, then any further subdivision of τ by type-I method could be considered as a part of the subdivision of σ_1 already included while obtaining Δ_1. In other words,

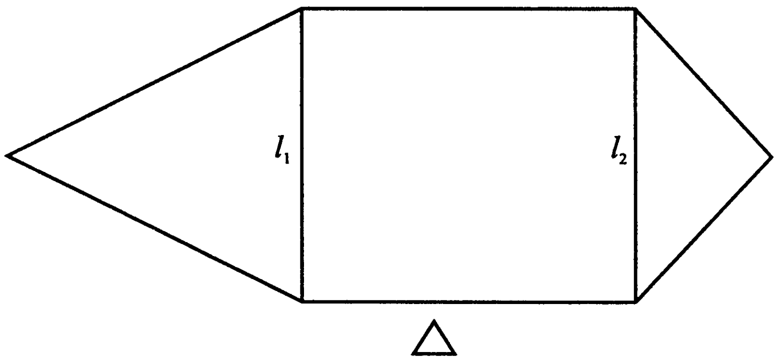

FIG. 2.

we can assume that Δ_1 has been obtained from Δ after subdividing the d-face σ_1 in a maximal manner. Then, we next subdivide σ_2 by the type-I method in a maximal manner. We continue this procedure successively subdividing $\sigma_3, ..., \sigma_{n-1}$ all in a maximal manner to get Δ' in a finite number of steps. This is what we express by saying that Δ' has been obtained from Δ by subdividing its d-faces $\sigma_1, ..., \sigma_t$ by type-I subdivision. Now we have our main result as follows:

***Theorem* 1** — *Let Γ be a finer subdivision of Δ which is obtained from Δ by subdividing any finite number of its d-faces, say $\sigma_1, \sigma_2, ..., \sigma_t$ by type-I subdivision. Let G and H be the matrices for $S^r(\Delta)$ and $S^r(\Gamma)$ respectively. Then there exists a non-singular matrix M such that*

$$HM = \begin{bmatrix} G & 0 \\ 0 & B \end{bmatrix}$$

for some matrix B.

PROOF : We will prove the theorem by the method of induction on the number t of d-faces which have been subdivided. Suppose the original partition Δ has t d-faces and k interior $(d-1)$-faces. Then the matrix G for $S^r(\Delta)$ has k-rows and $(t+k)$-columns. Suppose exactly one d-face of Δ is subdivided in a maximal manner and Δ_1 is the resulting finer subdivision of Δ. Let G_1 be the spline matrix for $S^r(\Delta_1)$. Then we show that there is a non-singular square matrix M_1 such that

$$G_1 M_1 = \begin{bmatrix} G & 0 \\ 0 & B_1 \end{bmatrix}$$

for some matrix B_1.

Let us motivate the proof by a concrete example of a 2-dimensional region in $\mathbb{R}^2$. We consider the partition Δ in Fig. 2. In this Figure, we assume that the two interior edges are embedded on the affine forms l_1 and l_2. A spline of smoothness r on this partition Δ can be represented as an ordered 3-tuple (f_1,f_2,f_3) of polynomials where f_1,f_2,f_3 represent polynomials on the three 2-faces of Δ such that the smoothness conditions are satisfied on the two interior edges. Therefore, by the well known algebraic criterion (see Billera and Rose[2]), we have

$$f_1 - f_2 + l_1^{r+1} f_4 = 0$$

and

$$f_2 - f_3 + l_2^{r+1} f_5 = 0,$$

for some polynomials f_4 and f_5. The spline matrix G for the above system of equations is

$$G = \begin{bmatrix} 1 & -1 & 0 & l_1^{r+1} & 0 \\ 0 & 1 & -1 & 0 & l_2^{r+1} \end{bmatrix}.$$

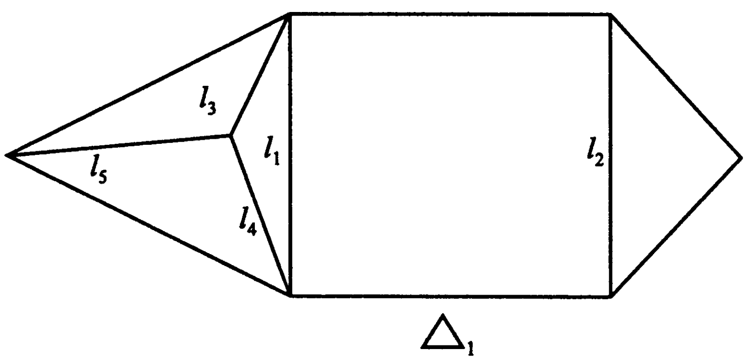

FIG. 3.

Now suppose exactly one d-face (here $d = 2$) of Δ is subdivided in a maximal manner and Δ_1 is the resulting finer subdivision of Δ, as in Fig. 3. Let G_1 be the spline matrix for the system of equations corresponding to the above partition. Here l_i denote the affine forms of interior edges $i = 1, 2, \ldots, 5$.

In general, we obtain matrix G_1 from G as follows: Let the face to be subdivided be represented by the last column and the resulting matrix be $\overline{G}$. Then we drop the last column of $\overline{G}$ and add the new columns corresponding to the new d-faces at the right of the matrix followed by the new columns corresponding to the new $(d-1)$-faces. We then add new rows corresponding to the new $(d-1)$-faces as well and get the matrix G_1.

In our example shown in Fig. 3, Δ_1 has been obtained by subdividing the left 2-face of Δ, which is a triangle. Therefore, we write the first column of the matrix G as the last column and get the new matrix

$$\overline{G} = \begin{bmatrix} -1 & 0 & l_1^{r+1} & 0 & 1 \\ 1 & -1 & 0 & l_2^{r+1} & 0 \end{bmatrix}.$$

To obtain G_1, we drop the last column of the above matrix and add three new columns corresponding to three new 2-faces, followed by three new columns corresponding to three new interior edges. Also, we add three new rows corresponding to three new interior edges. Thus the spline matrix G_1 is as follows:

$$\begin{bmatrix} -1 & 0 & l_1^{r+1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & l_2^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_3^{r+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & l_5^{r+1} \end{bmatrix}$$

When we write the new matrix on the right of $\overline{G}$ replacing its last column, the sum of the columns corresponding to the new d-faces in the new matrix is again zero except for the portion lying in the rows of $\overline{G}$. If we put this sum in place of the replaced column, then note that the old column of G is recovered and everything below that column is zero. Furthermore, if any entry on the right side of $\overline{G}$ on the right of the last column of $\overline{G}$ is still non-zero, we can kill it by adding a column of $\overline{G}$ corresponding to the original d-face of Δ because in every row there is a 1 for -1 and vice-versa. In view of these observations, let us replace the column C_5 by $C_5 + C_6 + C_7$ and then replace C_6 by $C_6 + C_1 + C_2$. Since these are all elementary column operations on the matrix G_1, we find that there exists a non-singular square matrix M'_1, such that

$$G_1 M'_1 = \begin{bmatrix} -1 & 0 & l_1^{r+1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & l_2^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & l_3^{r+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & l_5^{r+1} \end{bmatrix}$$

Now note that the matrix $\overline{G}$ is recovered in the top-left corner of the above matrix, and all other entries below $\overline{G}$ and on the right of $\overline{G}$ are zero. Thus, to get G in the top-left corner in place of $\overline{G}$, we only interchange columns of the above matrix suitably. But this column interchange is equivalent to post-multiplying by a non-singular matrix, say M'_2. Thus, we find that

$$G_1 (M'_1 M'_2) = \begin{bmatrix} 1 & -1 & 0 & l_1^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & l_2^{r+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & l_3^{r+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & l_5^{r+1} \end{bmatrix}.$$

$$= \begin{bmatrix} G & 0 \\ 0 & B_1 \end{bmatrix}$$

where,

$$B_1 = \begin{bmatrix} -1 & 0 & l_3^{r+1} & 0 & 0 \\ 1 & -1 & 0 & l_4^{r+1} & 0 \\ 0 & 1 & 0 & 0 & l_5^{r+1} \end{bmatrix}.$$

Hence by taking $M_1 = M'_1 M'_2$ we find that there exists a non-singular square matrix M_1 such that

$$G_1 M_1 = \begin{bmatrix} G & 0 \\ 0 & B_1 \end{bmatrix}$$

for some matrix B_1. We thus observe that for this example the result is proved for $t = 1$. Since this example is typical of the general case it is now clear that whatever d-face of Δ we choose, we conclude from the above procedure, that the theorem is true for $t = 1$ in the general case also.

Now let $t > 1$, and suppose the result is true for integers less than t. Let $\Delta_t = \Gamma$ be the subdivision of Δ when any t d-faces $\sigma_1, \sigma_2, ..., \sigma_t$ of Δ are subdivided. Let Δ_{t-1} denote the subdivision of Δ when the $(t-1)$ d-faces $\sigma_1, \sigma_2, ..., \sigma_{t-1}$ of Δ have been subdivided. Let G_{t-1}. be the spline matrix for the system of equations corresponding to the partition Δ_{t-1}. Then, by the induction hypothesis, there is a non-singular square matrix M_{t-1} such that

$$G_{t-1}M_{t-1} = \begin{bmatrix} G & 0 \\ 0 & B_{t-1} \end{bmatrix} \qquad \text{... (1)}$$

for some matrix B_{t-1}. We note that Δ_t is obtained from Δ_{t-1} by subdividing exactly one d-face of Δ_{t-1} viz. σ_t. Let G_t be the matrix for the partition Δ_t. Then, by the same argument as in the case $t = 1$, there exists a non-singular square matrix, say M'_t such that

$$G_t M'_t = \begin{bmatrix} G_{t-1} & 0 \\ 0 & B'_t \end{bmatrix} \qquad \text{... (2)}$$

for some matrix B'_t. Clearly the size of the square matrix M'_t is bigger than the size of the square matrix M_{t-1}. Multiply both sides of eq. (2) on the right by the matrix

$$M''_t = \begin{bmatrix} M_{t-1} & 0 \\ 0 & I \end{bmatrix}$$

where I is the $k \times k$ identity matrix and k is the number of columns of B'_t. This gives us that

$$G_t(M'_t M''_t) = \begin{bmatrix} G_{t-1} & 0 \\ 0 & B'_t \end{bmatrix} \begin{bmatrix} M_{t-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} G_{t-1}M_{t-1} & 0 \\ 0 & B'_t \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} G & 0 \\ 0 & B_{t-1} \\ & 0 \end{bmatrix} & 0 \\ & B'_t \end{bmatrix} \text{(from eq. (1))}$$

$$= \begin{bmatrix} G & 0 \\ 0 & B_t \end{bmatrix}$$

where

$$B_t = \begin{bmatrix} B_{t-1} & 0 \\ 0 & B'_t \end{bmatrix}.$$

Thus, by taking $M_t = M'_t M''_t$, we get that there is a non-singular square matrix, viz., M_t such that

$$G_t M_t = \begin{bmatrix} G & 0 \\ 0 & B_t \end{bmatrix}$$

for some matrix B_t This proves the result when the number of d-faces is t, and therefore completes the proof of the theorem. ∎

In view of the above theorem, the following result now follows in a straight-forward manner by using the modified Gröbner Basis Method used by Ruth Haas[8].

Corollary 1 — Let Δ' be a finer subdivision of arbitrarily small mesh of the polyhedral complex Δ obtained by subdividing any finite number of d-faces of Δ. Then a generating set for the spline module $S^r(\Delta)$ can always be extended to a generating set of the spline module $S^r(\Delta')$.

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