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UNIFORM EXTENSION OF A CRACK IN AN ELASTIC WEDGE

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The anti-plane problem of extension of a crack with constant velocity in an elastic wedge has been solved. A crack is assumed to emanate from the corner of the wedge when the edges of the wedge are subjected to anti-planc surface tractions. The problem has been solved using self-similarity technique, which is based on the observation that for externally applied surface traction the particle velocity is dynamically similar. Expressions for shear stress in the plane of the crack, stress intensity factor at the crack tip and stresses at the wedge shaped corners have been derived. Finally, the variations of stress intensity factor with different values of the parameters used in the problem have been presented in the form of graphs.

Key Words: **Self-similar; Chaplygin transformation; Schwartz; Christoffel transformation; Stress intensity factor**

1. INTRODUCTION

Several investigations of elastodynamic influence on the stresses and particle velocity in the region surrounding crack tip, moving in an elastic half-space due to tearing have been carried out in recent years[1-5]. As wedge shaped elastic media are also practical available, it is of interest to investigate the cases of extension of crack in an elastic wedge. For deformation in anti-plane strain the author has recently solved a few problem on extension of crack(s) in a wedge[6-9].

In this paper, the dynamic anti-plane problem of extension of a crack from the corner of an elastic wedge with constant subsonic velocity in an elastic wedge due to sudden application of shear stress in the outer surfaces of the wedge has been considered. The problem is solved by the method of homogeneous solutions which is based on the observation that for externally applied surface traction, the particle velocity is self-similar which allow Chaplygin's transformation to reduce the problem to the solution of Laplace equation in semi-infinite strip containing a slit. The Schwartz-Christoffel transformation is employed to map the semi-infinite strip on a half-space. Expressions for shear stress in the plane of the crack, stress intensity factor at the crack tip and stresses at the corners of the wedge have been derived. Finally, the variations of stress intensity factor with different values of the parameters used in the problem have been presented in the form of graphs.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

We consider the applied surface tractions

$$r > 0,\ \theta = 0,\ \tau_{\theta z} = \sigma_1 H(t)$$

$$r > 0,\ k\pi,\ \theta = \tau_{\theta z} = \sigma_2 H(t) \qquad \dots (1a,b)$$

where k, σ_1, σ_2 are constants and $H(\,)$ is the Heaviside's unit function.

These surface tractions produce plane waves in addition to a cylindrical wave with center at the point of discontinuity of surface traction, which propagate into the medium. It is assumed that due to discontinuity in surface traction a crack appears at the singular point which starts to extend with constant subsonic velocity such that at time $t > 0$, the crack tip is defined by $r = ut, \theta = \alpha\pi$.

The solutions of the plane wave fronts outside the cylindrical region follow easily from (1) as

$$r > 0, 0 < \theta < \pi/2, w = -\frac{c\sigma_1}{\mu}\left[t - \frac{r\sin\theta}{c}\right]H\left[t - \frac{r\sin\theta}{c}\right]$$

$$r > 0, k\pi - \pi/2 < \theta < k\pi, \ w = -\frac{c\,\sigma_2}{\mu}\left[t - \frac{r\sin(k\pi - \theta)}{c}\right]H\left[t - \frac{r\sin(k\pi - \theta)}{c}\right].$$

... (2*a*, *b*)

The boundary conditions in $r - \theta$ plane are

$$\theta = 0, 0 < r < ct, \ \partial\dot{w}/\partial\theta = 0$$

$$\theta = \alpha\pi \pm \varepsilon, 0 < r < ut, \ \partial w/\partial\theta = 0 \rightarrow \partial\dot{w}/\partial\theta = 0$$

$$\theta = k\pi, 0 < r < ct, \ \partial\dot{w}/\partial\theta = 0$$

$$r = ct, \pi/2 < \theta < k\pi, \ \dot{w} = -c\sigma_2/\mu$$

$$r = ct, k\pi - \pi/2 < \theta < \pi/2, \ \dot{w} = -c(\sigma_2 + \sigma_1)/\mu$$

$$r = ct, 0 < \theta < k\pi - \pi/2, \ \dot{w} = -c\sigma_1/\mu.$$

... (3*a-f*)

Absence of any characteristic length and because of applied load, the particle velocity is considered as self-similar. Introduction of $s = r/t$ makes $\partial w/\partial t = \dot{w}$ as a function of s and θ. For $s < c$, Chaplygin's transformation $\beta = \cosh^{-1}(c/s)$ with $c^2 = \mu/\rho$, map the region ABCDEFGA of the physical plane into a semi-infinite strip $[0 \leq \theta \leq k\pi, 0 \leq \beta < \infty]$ containing slit in $\gamma\ \{= \beta + i\,\theta\}$ plane as shown in Fig. 2. The governing equation of motion, expressed in polar coordinates and rewritten in terms of the preceding transformation reveals that $\dot{w}(\beta, \theta)$ satisfies Laplace equation in $\beta - \theta$ plane. The domain in γ-plane can be related to the upper half of the $\zeta = (\zeta + i\,\eta)$ plane by means of Schwartz-Christoffel transformation (Fig. 3). Now, if we take $\partial w/\partial t = \operatorname{Re} F(\zeta)$, where γ is related to ζ by

$$\gamma = \alpha\left[\ln\left\{\sqrt{(1 - \xi_c^2)(1 - \zeta^2)} + \zeta\xi_c + .1\right\} - \ln(\zeta + \xi_c)\right]$$

$$-(k - a)\left[\ln\left\{\sqrt{(1 - \xi_M^2)(1 - \zeta^2)} - \zeta\xi_M + 1\right\} - \ln(\zeta - \xi_M)\right] + ik\pi$$

in which

$$ch^{-1}(c/u) = \alpha\, ch^{-1}(1/\xi_c) + (k-\alpha)\, ch^{-1}(1/\xi_M)$$

$$\alpha \sin^{-1}\left[\frac{1+\xi_c\,\xi_B}{\xi_c+\xi_B}\right] + (k-\alpha)\sin^{-1}\left[\frac{1-\xi_M\,\xi_B}{\xi_B-\xi_M}\right] = (1-k)\,\pi/2$$

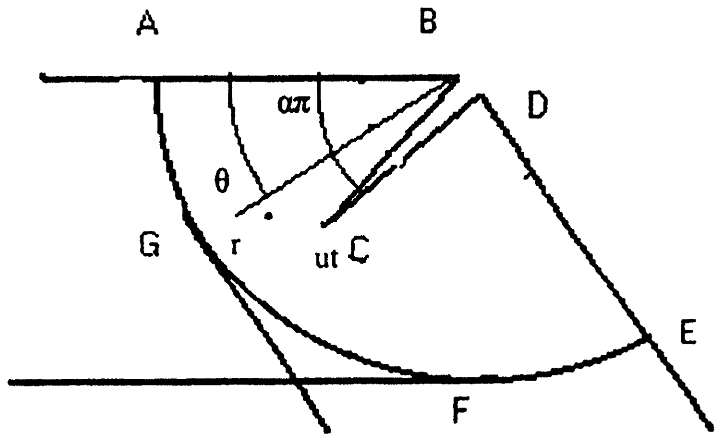

FIG. 1. Geometry and coordinate system

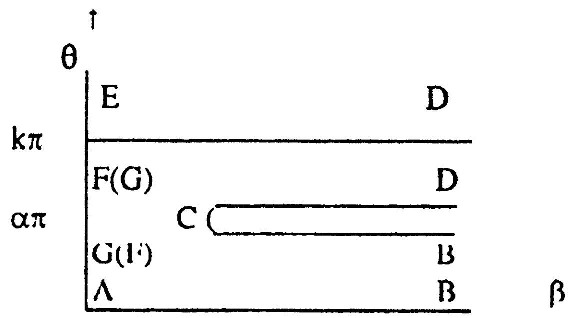

FIG. 2. The $\beta-\theta$ plane

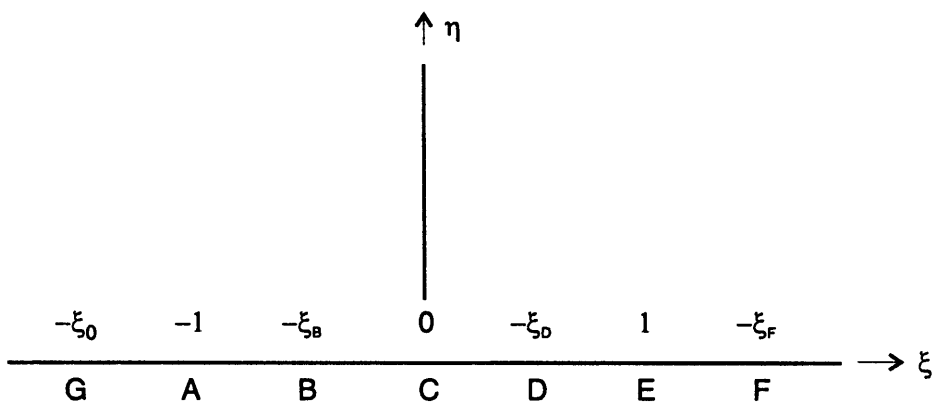

FIG. 3. The $\xi-\eta$ plane

$$\alpha \sin^{-1}\left[\frac{1-\xi_c\,\xi_F}{\xi_F-\xi_c}\right] + (k-\alpha)\sin^{-1}\left[\frac{1+\xi_M\,\xi_F}{\xi_F+\xi_M}\right] = (1-k)\,\pi/2$$

and

$$\alpha\,\xi_M \sqrt{(1-\xi_c^2)} = (k-\alpha)\xi_c \sqrt{(1-\xi_M^2)} \qquad \text{... (4a-e)}$$

For known values of k, α, u/c eq. (4) may be used to determine ξ_M, ξ_c, ξ_B, ξ_F. The boundary conditions given by (3) turn into the following conditions in ζ-plane.

$$\eta = 0, -1 < \xi < 1,\ \partial \dot{w}/\partial \eta = 0$$

$$\eta = 0, 1 < \xi < \xi_B, \dot{w} = -c\sigma_2/\mu \rightarrow \partial \dot{w}/\partial \xi = 0$$

$$\eta = 0, -\infty < \xi < -\xi_F, \xi_B < \xi < \infty,\ \dot{w} = -c\,(\sigma_2+\sigma_1)/\mu \rightarrow \partial \dot{w}/\partial \xi = 0$$

$$\eta = 0, -\xi_F < \xi < 1,\ \dot{w} = -c\,\sigma_1/\mu \rightarrow \partial \dot{w}/\partial \xi = 0. \qquad \text{... (5a-d)}$$

Now, it can be easily shown that [10] for $|\zeta| << 1$, we have the following relations

$$\theta = \alpha\,\pi,\ \gamma - \gamma_D = 0.5\,\omega\,\zeta^2$$

with

$$\omega = \alpha\sqrt{1-\xi_2^2}\,/\xi_c + (k-\alpha)\sqrt{1-\xi_M^2}\,/\xi_M. \qquad \text{... (6)}$$

In view of the conditions given by the eqs. (5) and (6) it is found convenient to work with $F'(\zeta)$. Accordingly we consider $F'(\zeta) = F_1'(\zeta) + F_2'(\zeta)$, where

$$F_1'(\zeta) = \frac{1}{\sqrt{(1-\zeta^2)}}\left[\frac{A}{\zeta-\xi_B} + \frac{B}{\zeta+\xi_F}\right]$$

$$F_2'(\zeta) = \frac{1}{\sqrt{(1-\zeta^2)}}\left[\frac{C}{\zeta} + \frac{D}{\zeta^2}\right]. \qquad \text{... (7a, b)}$$

Integrating (7*a*) and using the conditions (5*b-d*), we find that

$$A = -\frac{c\sigma_1}{\pi\mu}\sqrt{\xi_B^2-1}\,, \qquad B = -\frac{c\sigma_2}{\pi\mu}\sqrt{\xi_F^2-1}. \qquad \text{... (8a,b)}$$

Next, integrating (7*b*) and neglecting the logarithmic singularity at the moving crack tip, we obtain

$$F_2(\zeta) = -D\sqrt{(1-\zeta^2)}\,/\zeta \text{ and } C = 0.$$

The shear stress at $r > ut$, $\theta = \alpha\,\pi$ can be obtained using the relation

$$\frac{\partial w}{\partial \theta} = -\operatorname{Im}\left[F'(\zeta)\frac{d\zeta}{d\gamma} \right]$$

as

$$\tau_{\theta z} = -\frac{\mu}{s}\left[1-\frac{s^2}{c^2}\right] \operatorname{Im} F_2(\zeta) - \mu I + \Sigma$$

where

$$I = I_3 + DI_4$$

$$I_3 = \operatorname{Im}\int_S^c s^{-2}\frac{d\zeta}{d\gamma} F_1'(\zeta)\, ds$$

$$I_4 = \operatorname{Im}\int_S^c \frac{\sqrt{(1-\zeta^2)}}{\zeta}\frac{ds}{s^2\sqrt{(1-s^2/c^2)}} \qquad \ldots (9a\text{-}d)$$

and

$$\Sigma = \sigma_1 \cos\alpha\pi, \qquad 0<\alpha<k-1/2$$

$$= \sigma_1\cos\alpha\pi - \sigma_2\cos(k-\alpha)\pi, \qquad k-1/2<\alpha<1/2$$

$$= -\sigma_2\cos(k-\alpha)/\pi, \qquad 1/2<\alpha<k.$$

The stress intensity factor at the crack tip is obtained as

$$N = \lim_{r\to ut}\sqrt{2\pi(r-ut)}\;\tau_{\theta z}\Big|_u = -\left[\frac{\pi\omega}{uc}\left\{1-\frac{u^2}{c^2}\right\}^{3/2}\right]\mu D\sqrt{ct} \qquad \ldots (10)$$

where the constant D satisfies the equation

$$I_3\Big|_{s=u} + DI_4\Big|_{s=u} = \Sigma/u. \qquad \ldots (11)$$

3. Reflex Angled Wedge

It can be easily shown that the boundary conditions for this problem are

$$\theta=0,\ 0<r<ct,\ \partial\dot{w}/\partial\theta = 0$$

$$\theta=\alpha\pi\pm\varepsilon,\ 0<r<ut,\ \partial\dot{w}/\partial\theta=0$$

$$\theta=k\pi,\ 0<r<ct,\ \partial\dot{w}/\partial\theta=0$$

$$r=ct,\ k\pi-\pi/2<\theta<k\pi,\ \dot{w}=-c\sigma_2/\mu$$

$$r=ct,\ \pi/2<\theta<k\pi-\pi/2,\ \dot{w}=0.$$

$$r=ct,\ 0<\theta<\pi/2,\ \dot{w}=-c\sigma_1/\mu \qquad \ldots (12a\text{-}f)$$

which are transformed in ζ-plane as follows

$$\eta = 0, -1 < \xi < 1, \ \partial \dot{w}/\partial \eta = 0$$

$$\eta = 0, 1 < \xi < \xi_F, \ \dot{w} = -c\sigma_2/\mu \rightarrow \partial \dot{w}/\partial \xi = 0$$

$$\eta = 0, -\infty < \xi < -\xi_B, \xi_F < \xi < \infty, \ \dot{w} = 0 \rightarrow \partial \dot{w}/\partial \xi = 0$$

$$\eta = 0, -\xi_B < \xi < 1, \ \dot{w} = -c\sigma_1/\mu \rightarrow \partial \dot{w}/\partial \xi = 0. \quad \text{... (13}a\text{-}d\text{)}$$

In this case, the equations for determining ξ_M, ξ_c remain the same. But the equations involving ξ_B, ξ_F are changed to

$$\alpha \sin^{-1}\left[\frac{1+\xi_c\,\xi_F}{\xi_c+\xi_F}\right] + (k-\alpha)\sin^{-1}\left[\frac{1-\xi_M\,\xi_F}{\xi_F-\xi_M}\right] = (1-k)\,\pi/2$$

$$\alpha \sin^{-1}\left[\frac{1-\xi_c\,\xi_B}{\xi_B-\xi_c}\right] + (k-\alpha)\sin^{-1}\left[\frac{1+\xi_M\,\xi_B}{\xi_B+\xi_M}\right] = (1-k)\,\pi/2. \quad \text{... (14}a,b\text{)}$$

In the same manner as for problem of obtuse angled wedge, we consider in view of the above conditions $\dot{w} = \operatorname{Re} F(\zeta)$,

where

$$F'(\zeta) = F_3'(\zeta) + F_2'(\zeta)$$

and

$$F_3'(\zeta) = \frac{1}{\sqrt{(1-\zeta^2)}}\left[\frac{A'}{\zeta+\xi_B} + \frac{B'}{\zeta-\xi_F}\right] \quad \text{... (15)}$$

with

$$A' = \frac{c\sigma_1}{\pi\mu}\sqrt{\xi_B^2-1} \quad \text{and} \quad B' = \frac{c\sigma_2}{\pi\mu}\sqrt{\xi_F^2-1}.$$

The shear stress at $r > ut, \theta = \alpha\pi$ is remain same as given by (9a) with the exception that $F_1'(\zeta), \Sigma$ have to be changed by $F_3'(\zeta), \Sigma'$ respectively.

Where

$$\begin{aligned} \Sigma' &= \sigma_1 \cos\alpha\pi, \quad && 0 < \alpha < 1/2 \\ &= 0, && 1/2 < \alpha < k - 1/2 \\ &= -\sigma_2 \cos(k-\alpha)\pi, && k - 1/2 < \alpha < k. \end{aligned} \quad \text{... (16)}$$

4. Singularities at the Wedge Shaped Corners

The nature of the stresses in the wedge-shaped corners at C and M are investigated in this section In the region surrounding $\zeta = -\xi_c$, we assume $\zeta = -\xi_c + \varepsilon$, where $|\varepsilon| << 1$. Then the eq. (4a becomes

$$\gamma = \beta + i\theta = -\alpha \ln \varepsilon + i\alpha\pi. \quad \text{... (17)}$$

Let (R, φ, z) be the cylindrical coordinate system at $\zeta = -\xi_c$ in the ζ-plane. Thus $\varepsilon = Re^{i(\pi - \varphi)}$ which gives $\varphi = \theta/\alpha$ and $R = \exp(-\beta/\alpha)$. So from (9a), we get

$$(\tau_{\theta z})_c = -\frac{\mu}{r\alpha} F'(-\xi_c) \sin(\theta/\alpha) \int_{r/c}^{t} R\, dt + \Sigma.$$

Next using $\beta = \ln(2ct/r)$, for $r << 1$ we get

$$(\tau_{\theta z})_c = -\frac{\mu}{1-\alpha} F'(-\xi_c) \sin(\theta/\alpha) \left[(r/t)^{\alpha^{-1}-1} - c^{\alpha^{1}-1} \right] (2c)^{-\alpha^{-1}} + \Sigma_1$$

where

$$\Sigma_1 = \sigma_1 \cos\theta, \qquad 0 < \theta < \alpha\pi < k\pi - \pi/2$$

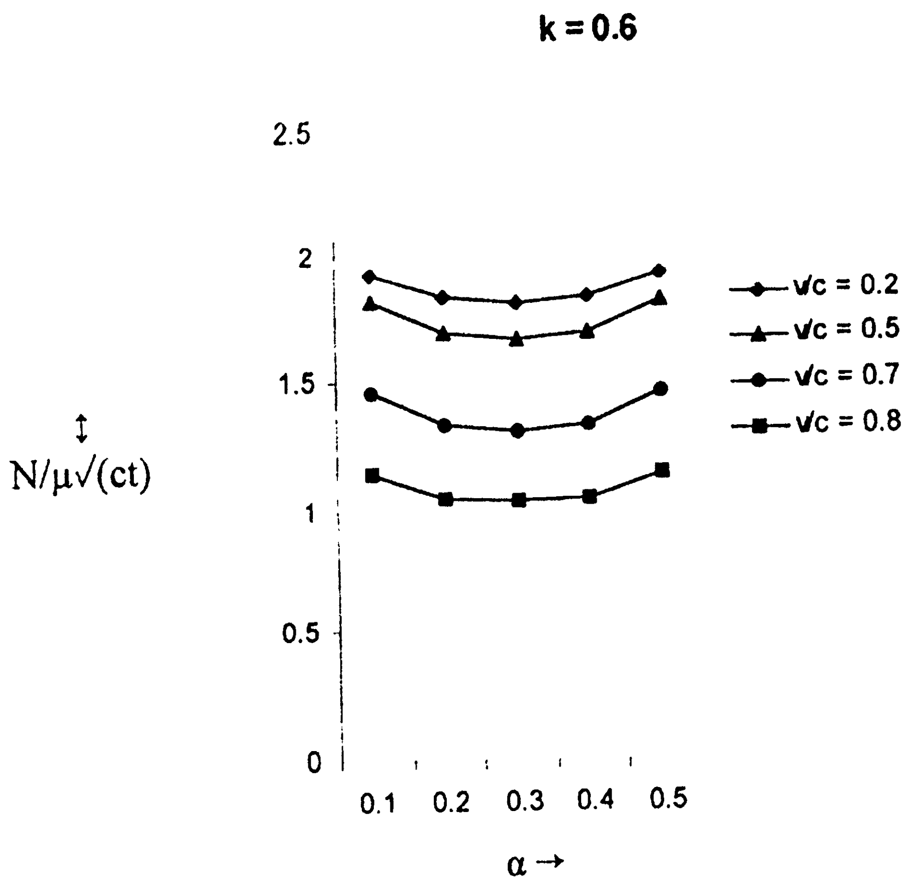

FIG. 4. Variation of $N/\mu\sqrt{(ct)}$ with α for $\sigma_2/\sigma_1 = -1$

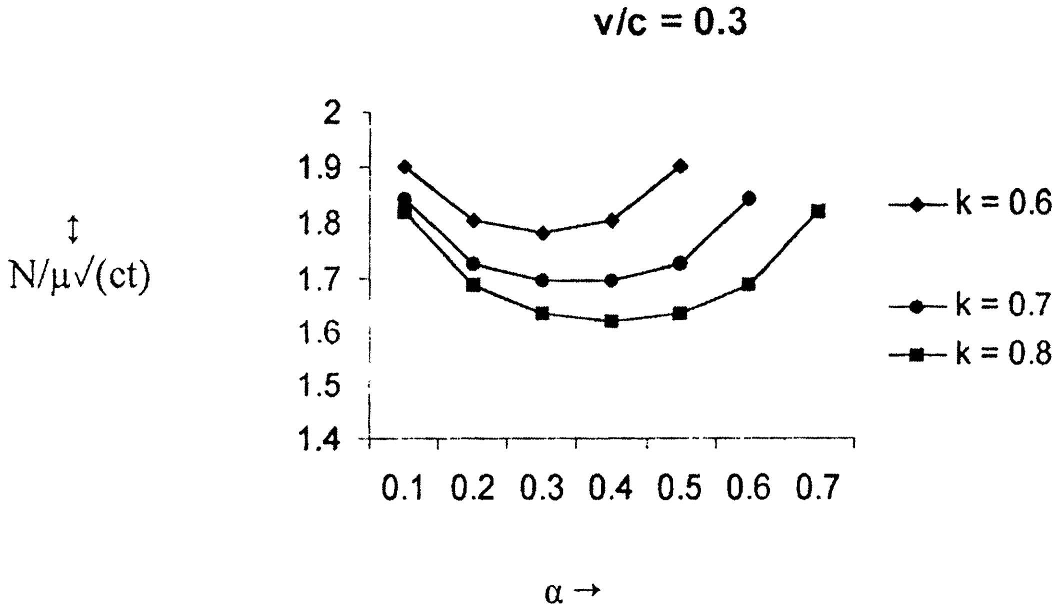

FIG. 5. Variation of $N/\mu\sqrt{(ct)}$ with α for $\sigma_2/\sigma_1 = -1$

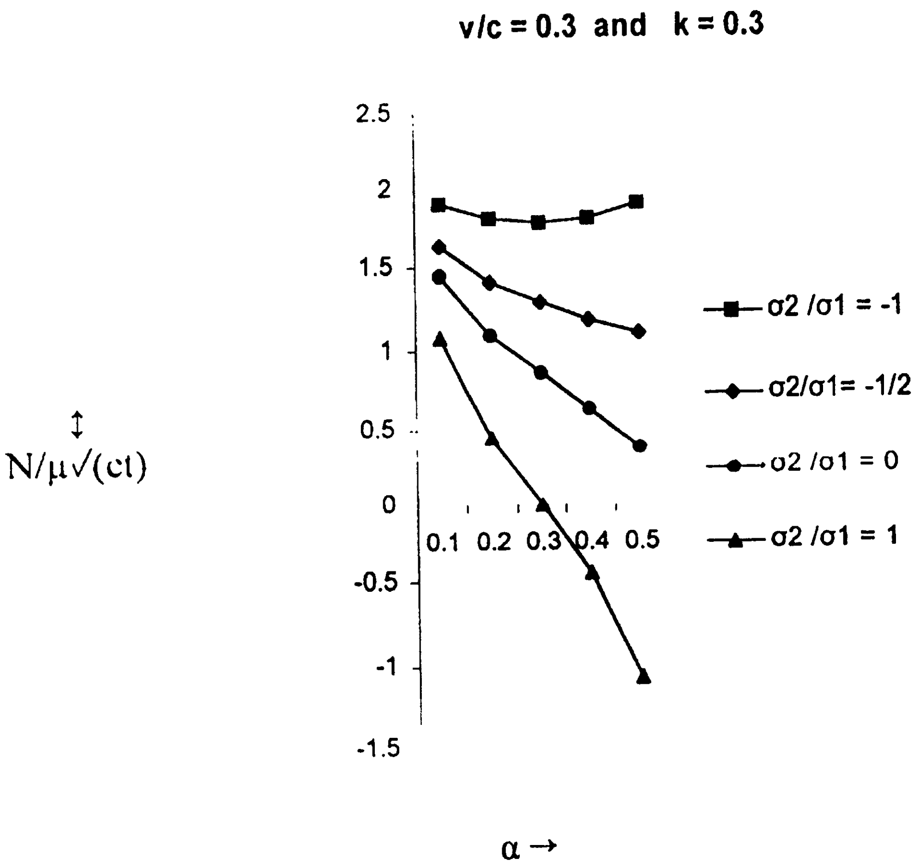

FIG. 6. Variation of $N/\mu\sqrt{(ct)}$ with α

$$= \sigma_1 \cos\theta - \sigma_2 \cos(k\pi - \theta), \qquad k\pi - \pi/2 < \theta < \alpha\pi < \pi/2$$

$$= -\sigma_2 \cos(k\pi - \theta), \qquad \pi/2 < \theta < \alpha\pi < k\pi.$$

Similar techniques are employed to obtain.

$$(\tau_{\theta z})_M = \frac{\mu}{k-\alpha-1} F'(-\xi_M) \sin\left\{\frac{\theta - \alpha\pi}{k\pi - \alpha\pi}\right\}$$

$$\left[(r/t)^{(k-\alpha)^{-1}-1} - c^{(k-\alpha)^{1}-1}\right](2c)^{(k-\alpha)^{-1}} + \Sigma_2$$

where Σ_2 is obtained from Σ_1 just by changing the condition $\theta < \alpha\pi$ by $\alpha\pi < \theta$.

From these results it is clear that the stress remains finite at the corners of an obtuse angled wedge but in the case of reflex angled wedge the stress may have weak singularity at the corners.

5. Numerical Results and Discussion

The numerical calculations of the proposed problem have been presented in this section by means of graphs. The variations of stress intensity factor with the parameters $\alpha, k, v/c$ and σ_2/σ_1 have been shown in Figs. 4-6. In Fig. 4, it is shown that the minimum value of the stress intensity factor at the crack tip occurs when crack plane bisects the wedge angle.

Fig. 4 also depicts that the value of stress intensity factor decreases with the increase in the value of v/c. In Fig. 5, it is shown that as the value of k increases, the value of the stress intensity factor decreases. Finally, the variation of stress intensity factor with σ_2/σ_1 has been presented in Fig. 6.

6. Conclusions

The mathematical model presented in this paper describing the phenomenon of a crack extending from the corner of an elastic wedge whose edges are subjected to anti-plane shear stresses is fully analytical, except at the final stages where the integrals in eq. (11) are to be carried out numerically. To solve the eq. (11), the integrals are evaluated using 16 points Gaussian's quadrature formula. To determine the roots of eq. (4*a*) in complex ζ using Newton's method, different initial approximations are to be given, otherwise the roots may not converge[5,6,9]

As the loading conditions on the boundary of the elastic wedge are not perfectly anti-symmetric, one should expect that due to the discontinuity of surface tractions at the corner of the wedge, a crack may be created at that point and may extend at any arbitrary direction. Other possibilities are the presence of pre-existing flaws or a weak plane in any arbitrary direction.

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