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PERIODIC BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS*

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This paper considers the existence of extreme solutions of the periodic boundary value problem for second order functional differential equations by using the method of lower and upper solutions coupled with monotone iterative technique.

Key Words: Impulsive Functional Differential Equation; Monotone Iterative Technique; Lower (Upper) Solution; Periodic Boundary Value Problem

1. INTRODUCTION

The theory of impulsive differential equations has become an important aspect of differential equations[1]. In papers[2-6], periodic boundary value problem for first order (impulsive) functional differential equations has been studied. Guo[7] consider PBVP for second order impulsive integro-differential equations in Banach spaces. In this paper, we consider the PBVP for the second order impulsive functional differential equations,

$$\begin{cases} -y''(t) = f(t, y(t), y(w(t))), & t \neq t_k, t \in J = [0, T], \\ \Delta y|_{t=t_k} = I_k y'(t_k), & \\ \Delta y'|_{t=t_k} = I_k^* y(t_k), & (k = 1, 2, ..., m), \\ y(0) = y(T), y'(0) = y'(T), & \\ y(t) = y(0), & t \in [-r, 0]. \end{cases} \quad \text{... (1)}$$

where $f \in C(J \times R^2, R)$, $w(t) \in C(J, [-r, T])$, $r > 0$, $0 < t_1 < t_2 < \ldots < t_m < T$, I_k and I_k^* are non-negative constants. $\Delta y|_{t=t_k} = y\left(t_k^+\right) - y\left(t_k^-\right)$, $\Delta y'|_{t=t_k} = y'\left(t_k^+\right) - y'\left(t_k^-\right)$, $k = 1, 2, ..., m$.

In this paper, by means of the method of upper and lower solutions and the monotone iterative technique, the existence of extreme solutions of PBVP (1) is obtained. In section 2, we first show an equivalent condition for two differential equations, then establish a comparison principle,

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finally we discuss the existence and uniqueness of the solutions for impulsive functional differential equations. In Section 3, by use the method of upper and lower solutions and the monotone iterative technique we obtain the existence of extreme solutions for PBVP (1).

2. Some Lemmas

Let $J^+ = [-r, T]$, $J^- = J \setminus \{t_1, t_2, \ldots, t_m\}$, $PC(J^+, R) = \{y : J^+ \to R; y(t)$ is continuous everywhere except for some t_k at which $y\left(t_k^+\right)$ and $y\left(t_k^-\right)$ exist and $y\left(t_k^-\right) = y(t_k)\}$; $PC'(J^+, R) = \{y \in PC(J^+, R); y(t)$ is continuous everywhere except for some t_k at which $y'\left(t_k^+\right)$ and $y'\left(t_k^-\right)$ exist and $y'\left(t_k^-\right) = y'(t_k)\}$. Let $E = \{y \in PC(J^+, R) : y(t) = y(0), t \in [-r, 0]\}$ with norm

$$\|y\|_{PC} = \sup_{t \in J^+} \|y(t)\|,$$

then E is a Banach space. Let $E' = \{y \in PC'(J^+, R) : y(t) = y(0), t \in [-r, 0]\}$ with norm

$$\|y\|_{PC'} = \max\left\{ \|y(t)\|_{PC}, \|y'(t)\|_{PC} \right\},$$

then E' is also a Banach space. A function $y \in PC'[J^+, R] \cap C^2[J^-, R]$ is called a solution of PBVP(1) if it satisfies[1].

Consider the PBVP

$$\begin{cases} -y''(t) + My(t) + Ny(w(t)) = \sigma(t), & t \neq t_k, t \in J, \\ \Delta y|_{t=t_k} = I_k y'(t_k), & \\ \Delta y'|_{t=t_k} = I_k^* y(t_k), & (k = 1, 2, \ldots, m), \\ y(0) = y(T), y'(0) = y'(T), & \\ y(t) = y(0), & t \in [-r, 0]. \end{cases} \quad \ldots (2)$$

where $M > 0, N \geq 0$ are constants and $\sigma(t) \in PC(J, R)$.

Lemma 2.1 — $y \in PC'[J^+, R] \cap C^2[J^-, R]$ is a solution of (2) if and only if $y \in PC(J^+, R)$ is a solution of the impulsive integral equation

$$y(t) = \begin{cases} \int_0^T G_1(t, s)[\sigma(s) - Ny(w(s))]\,ds + \sum_{k=1}^m [G_1(t, t_k)(-I_k^* y(t_k)) + G_2(t, t_k) I_k y'(t_k)], & t \in J; \\ y(0), & t \in [-r, 0]. \end{cases} \quad \ldots (3)$$

Here

$$G_1(t,s) = \left[2\sqrt{M}\,(e^{\sqrt{MT}} - 1) \right]^{-1}$$

$$\begin{cases} e^{\sqrt{M}\,(T-t+s)} + e^{\sqrt{M}\,(t-s)}\,; 0 \le s < t \le T, \\ e^{\sqrt{M}\,(T+t-s)} + e^{\sqrt{M}\,(t-s)}\,; 0 \le t < s \le T, \end{cases}$$

$$G_2(t,s) = \left[2\,(e^{\sqrt{MT}} - 1) \right]^{-1}$$

$$\begin{cases} e^{\sqrt{M}\,(T-t+s)} - e^{\sqrt{M}\,(t-s)}\,; 0 \le s < t \le T, \\ -e^{\sqrt{M}\,(T+t-s)} + e^{\sqrt{M}\,(s-t)}\,; 0 \le t < s \le T. \end{cases}$$

PROOF : Suppose that $y(t)$ is a solution of (2), then

$$y''(t) - My(t) = -(\sigma(t) - Ny(w(t))).$$

$$\left[e^{-2\sqrt{Mt}}\,(e^{\sqrt{Mt}}\,y(t))' \right]' = -M\,e^{-\sqrt{Mt}}\,y(t) + e^{-\sqrt{Mt}}\,y''(t)$$

$$= -e^{-\sqrt{Mt}}\,(\sigma(t) - Ny\,(w(t))).$$

Let

$$u(t) = e^{-2\sqrt{Mt}}\,(e^{\sqrt{Mt}}\,(y(t))',$$

then

$$u'(t) = -e^{-\sqrt{Mt}}\,(\sigma(t) - Ny\,(w(t))) \qquad \dots (4)$$

Integrating (4) from 0 to t_1, then

$$u(t_1) - u(0) = -\int_0^{t_1} e^{-\sqrt{Ms}}\,(\sigma(s) - Ny(w(s)))\,ds.$$

Again integrating (4) from t_1 to t, where $t \in (t_1, t_2]$, then

$$u(t) = u\left(t_1^+\right) - \int_{t_1}^{t} e^{-\sqrt{Ms}}\,(\sigma(s) - Ny\,(w(s)))\,ds$$

$$= u(0) - \int_0^{t} e^{-\sqrt{Ms}}\,(\sigma(s) - Ny\,(w(s)))\,ds + e^{-\sqrt{Mt_1}}\,[\sqrt{M}\,I_1\,y'(t_1) + I_1^*\,y(t_1)].$$

Repeating the above procession, we have

$$u(t) = u(0) - \int_0^t e^{-\sqrt{Ms}} (\sigma(s) - Ny(w(s)))\, ds$$

$$+ \sum_{0 < t_k < t} e^{-\sqrt{Mt_k}} \left[\sqrt{M}\, I_k\, y'(t_k) + I_k^* y(t_k) \right], \; t \in J.$$

Hence,

$$(e^{\sqrt{Mt}}\, y(t))' = e^{2\sqrt{Mt}}$$

$$\left\{ u(0) - \int_0^t e^{-\sqrt{Ms}} (\sigma(s) - Ny(w(s)))\, ds + \sum_{0 < t_k < t} e^{-\sqrt{Mt_k}} [\sqrt{M}\, I_k\, y'(t_k) + I_k^* (t_k)] \right\},$$

Denote

$$\upsilon(t) = e^{\sqrt{M}\, t}\, y(t)$$

and

$$m(t) = e^{2\sqrt{Mt}}$$

$$\left\{ u(0) - \int_0^t e^{-\sqrt{Ms}} (\sigma(s) - Ny(w(s)))\, ds + \sum_{0 < t_k < t} e^{-\sqrt{Mt_k}} [\sqrt{M}\, I_k\, y'(t_k) + I_k^* (t_k)] \right\},$$

then

$$\Delta \upsilon \,|_{t = t_k} = e^{\sqrt{M}\, t_k} I_k\, y'(t_k), \qquad \text{and} \qquad \upsilon'(t) = m(t). \qquad \ldots (5)$$

Integrating (5) from 0 to t_1,

$$\upsilon(t_1) = \upsilon(0) + \int_0^{t_1} m(s)\, ds.$$

Again integrating (5) from t_1^+ to t_2,

$$\upsilon(t_2) = \upsilon\left(t_1^+\right) + \int_{t_1}^{t_2} m(s)\, ds$$

$$= \upsilon\left(t_1^+\right) + \int_0^{t_2} m(s)\, ds - \int_0^{t_1} m(s)\, ds$$

$$= v(0) + \int_0^{t_2} m(s)\, ds + e^{\sqrt{M}\, t_1} I_1 y'(t_1).$$

Repeating the procession, we obtain

$$v(t) = v(0) + \int_0^{t} m(s)\, ds + \sum_{0 < t_k < t} e^{\sqrt{M}\, t_k} I_k y'(t_k), \; t \in J.$$

Thus,

$$y(t) = e^{-\sqrt{M}\, t} \left[v(0) + \int_0^{t} m(s)\, ds + \sum_{0 < t_k < t} e^{\sqrt{M}\, t_k} I_k y'(t_k) \right] \quad \dots (6)$$

Since

$$\int_0^{t} m(s)\, ds$$

$$= \int_0^{t} e^{2\sqrt{M} s} \left[u(0) - \int_0^{s} e^{-\sqrt{M}\,\theta} (\sigma(\theta) - Ny(w(\theta)))\, d\theta \right.$$

$$\left. + \sum_{0 < t_k < s} e^{-\sqrt{M}\, t_k} \left[\sqrt{M}\, I_k y'(t_k) + I_k^{*} y(t_k) \right] ds \right.$$

$$= \frac{1}{2\sqrt{M}} \left[u(0) \left(e^{2\sqrt{M}\, t} - 1 \right) + \int_0^{t} e^{\sqrt{M}\, s} (\sigma(s) - Ny(w(s)))\, ds \right.$$

$$- e^{2\sqrt{M}\, t} \int_0^{t} e^{-\sqrt{M}\, s} (\sigma(s) - Ny(w(s)))\, ds$$

$$\left. + \sum_{0 < t_k < t} \left(e^{2\sqrt{M}\, t} - e^{2\sqrt{M}\, t_k} \right) e^{-\sqrt{M}\, t_k} \left(\sqrt{M}\, I_k y'(t_k) + I_k^{*} y(t_k) \right) \right] \quad \dots (7)$$

and note that

$$v(0) = y(0),\; u(0) = \sqrt{M}\, y(0) + y'(0), \quad \dots (8)$$

thus for $t \in J$, we obtain

$$y(t) = \frac{1}{2\sqrt{M}} e^{-\sqrt{M}\, t} \{ 2\sqrt{M}\, y(0) + (\sqrt{M}\, y(0) + y'(0)) (e^{2\sqrt{M}\, t} - 1$$

$$+ e^{-\sqrt{M}\,t}\int_0^t e^{\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds - e^{2\sqrt{M}\,t}\int_0^t e^{-\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds$$

$$+ e^{-\sqrt{M}\,t}\int_0^t e^{\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds + \sum_{0<t_k<t}(e^{2\sqrt{M}\,t} - e^{2\sqrt{M}\,t_k})$$

$$e^{-\sqrt{M}\,(t_k+t)}(\sqrt{M}\,I_k y'(t_k) + I_k^* y(t_k)) + \sum_{0<t_k<t} e^{\sqrt{M}\,(t_k-t)} I_k y'(t_k)\}$$

$$= \frac{1}{2\sqrt{M}}\left\{(\sqrt{M}\,y(0) - y'(0))\,e^{-\sqrt{M}\,t} + (\sqrt{M}\,y(0) + y'(0))\,e^{\sqrt{M}\,t}\right.$$

$$+ e^{-\sqrt{M}\,t}\int_0^t e^{\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds - e^{\sqrt{M}\,t}\int_0^t e^{-\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds$$

$$+ \sum_{0<t_k<t} e^{\sqrt{M}\,(t-t_k)}(\sqrt{M}\,I_k y'(t_k) + I_k^* y(t_k))$$

$$\left.- \sum_{0<t_k<t} e^{\sqrt{M}\,(t_k-t)}(I_k^* y(t_k) - \sqrt{M}\,I_k y'(t_k))\right\} \quad \ldots (9)$$

and

$$y'(t) = \frac{1}{2}\left\{-(\sqrt{M}\,y(0) - y'(0))\,e^{-\sqrt{M}\,t} + (\sqrt{M}\,y(0) + y'(0))\,e^{M\,t}\right.$$

$$- e^{-\sqrt{M}\,t}\int_0^t e^{\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds$$

$$- e^{\sqrt{M}\,t}\int_0^t e^{-\sqrt{M}\,s}(\sigma(s) - Ny(w(s)))\,ds$$

$$+ \sum_{0<t_k<t} e^{\sqrt{M}\,(t-t_k)}(\sqrt{M}\,I_k y'(t_k) + I_0^* y(t_k))$$

$$\left.- \sum_{0<t_k<t} e^{\sqrt{M}\,(t_k-t)}(I_k^* y(t_k) - \sqrt{M}\,I_k y'(t_k))\right\}.$$

In view of that $y(0) = y(T)$, $y'(0) = y'(T)$, we have

$$\sqrt{M}\, y(0) - y'(0) = (e^{\sqrt{M}\,T} - 1)^{-1} \left(\int_0^T e^{\sqrt{M}\,s} (\sigma(s) - Ny(w(s))\, s \right.$$

$$\left. - \sum_{0 < t_k < T} e^{\sqrt{M}\, t_k} (I_k^* y(t_k) - \sqrt{M}\, I_k y'(t_k)) \right), \qquad \ldots (10)$$

$$\sqrt{M}\, y(0) + y'(0) = (e^{\sqrt{M}\,T} - 1)^{-1} (e^{\sqrt{M}\,T} \int_0^T e^{\sqrt{M}\,s} (\sigma(s) - Ny(w(s)))\, ds$$

$$- \sum_{0 < t_k < T} e^{\sqrt{M}\,(T - t_k)} (I_k^* y(t_k) + \sqrt{M}\, I_k y'(t_k))). \qquad \ldots (11)$$

Since

$$\sum_{k=1}^{m} I_k y'(t_k) = \sum_{0 < t_k < T} I_k y'(t_k) = \sum_{t \le t_k < T} I_k y'(t_k) + \sum_{0 < t_k < t} I_k y'(t_k),$$

$$\sum_{k=1}^{m} I_k^* y(t_k) = \sum_{0 < t_k < T} I_k^* y(t_k) = \sum_{t \le t_k < T} I_k^* y(t_k) + \sum_{0 < t_k < t} I_k^* y(t_k),$$

thus

$$- \sum_{0 < t_k < T} e^{\sqrt{M}\, t_k} (I_k^* y(t_k) - \sqrt{M}\, I_k y'(t_k)))$$

$$- \sum_{0 < t_k < T} e^{\sqrt{M}\,(T - t_k)} (I_k^* y(t_k) + \sqrt{M}\, I_k y'(t_k)))$$

$$+ (e^{\sqrt{M}\,T} - 1) \sum_{0 < t_k < t} e^{\sqrt{M}\,(t_k - k)} (\sqrt{M}\, I_k y'(t_k) + I_k^* y(t_k))$$

$$- (e^{\sqrt{M}\,T} - 1) \sum_{0 < t_k < t} e^{\sqrt{M}\,(t_k - t)} (I_k^* y(t_k) - \sqrt{M}\, I_k y'(t_k))\}$$

$$= - \sum_{t \le t_k < T} e^{\sqrt{M}\,(T + t - t_k)} (I_k^* y(t_k) + \sqrt{M}\, I_k y'(t_k)))$$

$$- \sum_{0 < t_k < t} e^{\sqrt{M}\,(t - t_k)} (I_k^* y(t_k) + \sqrt{M}\, I_k y'(t_k)))$$

$$- \sum_{t \le t_k < T} e^{\sqrt{M}(t-t_k)} (\sqrt{M} I_k y'(t_k) + I_k^* y(t_k))$$

$$- \sum_{t < t_k < T} e^{\sqrt{M}(T+t_k-t)} (I_k^* y(t_k) - \sqrt{M} I_k y'(t_k))$$

$$= \sum_{0 < t_k < t} e^{\sqrt{M}(T+t_k-t)} - e^{\sqrt{M}(t-t_k)} (-I_k^* y(t_k)))$$

$$+ \sum_{t \le t_k < T} (e^{\sqrt{M}(T-t_k+t)} - e^{\sqrt{M}(t_k-t)} (-I_k^* y(t_k)))$$

$$+ \sqrt{M} \left\{ \sum_{0 < t_k < t} (e^{\sqrt{M}(T-t+t_k)} - e^{\sqrt{M}(t-t_k)} I_k y'(t_k)) \right.$$

$$\left. + \sum_{t \le t_k < T} (e^{\sqrt{M}(t_k-t)} - e^{\sqrt{M}(T+t-t_k)} I_k y'(t_k)) \right\}$$

$$= \sum_{k=1}^{m} \left[G_1(t, t_k)(-I_k^* y(t_k)) + G_2(t, t_k) I_k y'(t_k) \right]. \qquad \dots (12)$$

Substituting (10)-(12) into (9), for $t \in J$, we obtain

$$y(t) = \int_0^T G_1(t, s) [\sigma(s) - Ny(w(s))] \, ds$$

$$+ \sum_{k=1}^{m} \left[G_1(t, t_k)(-I_k^* y(t_k)) + G_2(t, t_k) I_k y'(t_k) \right].$$

On the other hand, assume $y(t)$ is a solution of (3), then differentiating on (3) for $t \ne t_k$, we have

$$y'(t) = \int_0^T G_{1t}(t, s) [\sigma(s) - Ny(w(s))] \, ds$$

$$+ \sum_{k=1}^{m} \left[G_{1t}(t, t_k)(-I_k^* y(t_k)) + G_{2t}(t, t_k) I_k y'(t_k) \right].$$

$$G_{1t}(t, s) = \left[2\sqrt{M} (e^{\sqrt{M}T} - 1 \right]^{-1}$$

$$\begin{cases} \sqrt{M}\,(-e^{\sqrt{M}\,(T-t+s)}+e^{\sqrt{M}\,(t-s)})\,;\, 0 \le s < t \le T, \\ \sqrt{M}\,\,(e^{\sqrt{M}\,(T+t+s)}-e^{\sqrt{M}\,(s-t)})\,;\, 0 \le t < s \le T, \end{cases}$$

$$= -\,G_2\,(t, s).$$

and

$$G_{2t}\,(t, s) \;=\; -\,M\Big[\,2\,(e^{\sqrt{M}\,T}-1\,\Big]^{-1}$$

$$\begin{cases} e^{\sqrt{M}\,(T-t+s)}+e^{\sqrt{M}\,(t-s)}\,;\, 0 \le s < t \le T, \\ e^{\sqrt{M}\,(T+t+s)}+e^{\sqrt{M}\,(s-t)}\,;\, 0 \le t < s \le T. \end{cases}$$

$$= -\,MG_1\,(t, s).$$

Thus

$$y'\,(t) = \int_0^T G_2\,(t, s)\,[\sigma\,(s) - Ny(w(s))]\,ds$$

$$+ \sum_{k=1}^{m} \Big[\,G_2\,(t, t_k)\;I_k^*\,y(t_k) + MG_1\,(t, t_k)\,I_k\,y'\,(t_k)\Big] \qquad \ldots (13)$$

Differentiating on (13) for $t \ne t_k$, $t \in J$, then

$$y''\,(t) \;=\; Ny(w(t)) - \sigma\,(t) - \int_0^T G_{2t}\,(t, s)\,[\sigma\,(s) - Ny(w(s))]\,ds$$

$$+ \sum_{k=1}^{m} \Big[\,G_{2t}\,(t, t_k)\,(I_k^*\,y(t_k) - MG_{1t}\,(t, t_k)\,I_k\,y'\,(t_k)\Big]$$

$$= \int_0^T MG_1\,(t, s)\,[\sigma\,(s) - Ny(w(s))]\,ds$$

$$+ \sum_{k=1}^{m} \Big[-MG_1\,(t, t_k)\;I_k^*\,y(t_k) + MG_2\,(t, t_k)\,I_k\,y'\,(t_k)\Big] + Ny(w(t)) - \sigma\,(t)$$

$$= My(t) + Ny(w(t)) - \sigma\,(t),$$

i.e.,

$$-\,y''\,(t) + My(t) + Ny(w(t)) = \sigma\,(t).$$

By direct computing, we get

$$\Delta \sum_{k=1}^{m} G_1(t, t_k)(-I_k^* y(t_k)) \mid_{t=t_k} = 0;$$

$$\Delta \sum_{k=1}^{m} G_2(t, t_k) I_k^* y'(t_k) \mid_{t=t_k} = I_k y'(t_k),$$

then

$$\Delta y \mid_{t=t_k} = I_k y'(t_k);$$

$$\Delta y' \mid_{t=t_k} = I_k^* y(t_k).$$

It is easy to see $G_i(0, s) = G_i(T, s)$, for $s \in J$, $i = 1, 2$, then

$$y(0) = y(T), y'(0) = y'(T), y(t) = y(0), \text{ for } t \in [-r, 0].$$

This completes the proof.

In the following, we denote

$$a = \max_k \left\{ t_{k+1} - t_k, k = 0, 1, ..., m \right\}, \text{ where } t_0 = 0, t_{m+1} = T.$$

Lemma 2.2 — Assume that $y \in PC'[J^+, R] \cap C^2[J^-, R]$ satisfy

$$\begin{cases} -y''(t) \le -My(t) - Ny(w(t)), & t \ne t_k, t \in J, \\ \Delta y \mid_{t=t_k} = I_k y'(t_k), & \\ \Delta y' \mid_{t=t_k} \ge I_k^* y(t_k), & (k = 1, 2, ..., m), \\ y(0) = y(T), y'(0) = y'(T), & \\ y(t) = y(0), & t \in [-r, 0]. \end{cases} \quad ... (14)$$

where constants $M > 0, N \ge 0, I_k \ge 0, I_k^* \ge 0$ $(k = 1, 2, ..., m)$, and they satisfy

$$\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N) \right) \left(\sum_{k=1}^{m} I_k + a(m+1) \right) < \frac{1}{2}. \quad ... (15)$$

Then $y(t) \le 0$ for all $t \in J^+$

PROOF : Suppose, to the contrary, that $y(t) > 0$ for some $t \in J^+$. It is enough to consider the following cases,

(*i*) there exists a $\bar{t} \in J$, such that $y(\bar{t}) > 0$ and $y(t) \geq 0$ for all $t \in J$;

(*ii*) there exist $t^*, t_* \in J$, such that $y(t^*) > 0, y(t_*) < 0$.

In case of (*i*), we have

$$-y''(t) \leq -My(t) - Ny(w(t)) \leq 0,$$

which implies $y''(t) \geq 0$, for $t \in J$. And $\Delta y' |_{t=t_k} \geq I_k^* y(t_k) \geq 0$, hence $y'(t)$ is nondecreasing in $t \in J$. So $y'(0) \leq y'(T)$. However, by[14] $y'(0) \geq y'(T)$, hence $y'(0) = y'(T)$, which implies $y'(t)$ = constant, for all $t \in J$. Therefore, $0 = -y''(t) \leq -My(\bar{t}) < 0$, a contradiction.

In case of (*ii*), let $\inf \{y(t) : t \in J\} = -b$, then we can assert that $b > 0$, and there exists $t_* \in (t_j, t_{j+1}]$ such that $y(t_*) = -b$ or $y\left(t_j^+\right) = -b$, $j \in [1, 2, ..., m]$, and

$$-y''(t) \leq -My(t) - Ny(w(t)) \leq (M+N)\, b.$$

We only consider $y(t_*) = -b$, for the case $y\left(t_j^+\right) = -b$, the proof is similar.

If $y'(t) > 0$ for all $t \in J$, then $\Delta y |_{t=t_k} = I_k y'(t_k) \geq 0, k = 1, 2, ..., m$. All these imply $y(t)$ is strictly increasing on J, which contradicts $y(0) = y(T)$. So there exists a $\bar{t} \in J$, such that $y'(\bar{t}) \leq 0$.

Let $\bar{t} \in (t_i, t_{i+1}]$, $i \in [0, m]$, by mean value theorem,

$$y'\left(t_i^-\right) - bI_i^* - y'(\bar{t}) \leq y'\left(t_i^+\right) - y'(\bar{t})$$

$$= -y''(s_i)\left(\bar{t} - t_i^+\right) \leq ab\,(M+N),\ s_i \in (t_i, \bar{t});$$

$$y'\left(t_{i-1}^-\right) - bI_{i-1}^* - y'(\bar{t_i}) \leq y'\left(t_{i-1}^+\right) - y'(t_i)$$

$$= -y''(s_{i-1})\left(t_i - t_{i-1}^+\right) \leq ab\,(M+N),\ s_{i-1} \in (t_{i-1}, t_i);$$

$$y'\left(t_1^-\right) - bI_1^* - y'(t_2) \leq y'\left(t_i^+\right) - y'(t_2)$$

....

$$= -y''(s_1)\left(t_2 - t_1^+\right) \leq ab\,(M+N),\ s_1 \in (t_1, t_2);$$

$$y'(0) - y'(t_1) = -y''(s_0)\, t_1) \leq ab\,(M+N),\ s_0 \in (0, t_1).$$

Summing up the above inequalities, we obtain

$$y'(0) \le y'(\bar{t}) + b\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right)$$

$$\le b\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right)$$

Similarly, we have

$$y'(t) \le y'(T) + b\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right)$$

$$\le y'(0) + b\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right)$$

$$\le 2b\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right)$$

Let $t^* \in (t_j, t_{j+1})$ for some j. First assume $t_* < t^*$, then $i \le j$. By mean value theorem, w have

$$y(t^*) - y(t_j) = y(t^*) - y\left(t_j^+\right) + I_j y'(t_j) = y'(r_j)(t^* - t_j) + I_j y'(t_j)$$

$$\le 2b(I_j + a)\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right), \quad r_j \in (t_j, t^*);$$

$$y(t_j) - y(t_{j-1}) = y(t_j) - y\left(t_{j-1}^+\right) + I_{j-1} y'(t_{j-1})$$

$$= y'(r_{j-1})\left(t_j - t_{j-1}^+\right) + I_{j-1} y'(t_{j-1})$$

$$\le 2b(I_{j-1} + a)\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right), \quad r_{j-1} \in (t_{j-1}, t_j);$$

....

$$y(t_{i+1}) - y(t_*) \le ab(I_j + a)\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N)\right).$$

Summing up, we obtain

$$0 < y(t^*) \leq -b + 2b \left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N) \right) \left(\sum_{k=1}^{m} I_k + a(m+1) \right)$$

Hence

$$\left(\sum_{k=1}^{m} I_k^* + a(m+1)(M+N) \right) \left(\sum_{k=1}^{m} I_k + a(m+1) \right) \geq \frac{1}{2},$$

which contradicts[15]

For the case $t_* > t^*$, the proof is similar and thus we omit it. This completes the proof.

Lemma 2.3 — Let constant $M > 0, N \geq 0, I_k \geq 0, I_k^* \leq 0$. If

$$\frac{1 + e^{\sqrt{M}\,T}}{2\sqrt{M}\,(e^{\sqrt{M}\,T} - 1)} \left(NT + \sum_{k=1}^{m} I_k^* \right) + \frac{1}{2} \sum_{k=1}^{m} I_k < 1; \quad \ldots (16)$$

$$\frac{1}{2} NT + \frac{1}{2} \sum_{k=1}^{m} I_k^* + \frac{\sqrt{M}\,(1 + e^{\sqrt{M}\,T})}{2\,(e^{\sqrt{M}\,T} - 1)} \sum_{k=1}^{m} I_k < 1, \quad \ldots (17)$$

then the eq. (2) has a unique solution y in $PC'([-r, T], R)$.

PROOF : For any $y \in E'$, define an operator F by

$$(Fy)(t) = \begin{cases} \int_0^T G_1(t,s)\,[\sigma(s) - Ny(w(s))]\,ds + \sum_{k=1}^{m} [G_1(t,t_k)(-I_k^*\,y(t_k)) + G_2(t,t_k)\,I_k\,y'(t_k)], & t \in [0, T], \\ (Fy)'(0), & t \in [-r, 0] \end{cases}$$

where G_1, G_2 are given by Lemma 2.1. Then it is obvious that $Fy \in E'$, and

$$(Fy)'(t) = \begin{cases} -\int_0^T G_2(t,s)\,[\sigma(s) - Ny(w(s))]\,ds + \sum_{k=1}^{m} [G_2(t,t_k)\,I_k^*\,y(t_k)) - MG_2(t,t_k)\,I_k\,y'(t_k)], & t \in [0, T], \\ (0), & t \in [-r, 0]. \end{cases}$$

By direct computing, we have

$$\max_{(t,s) \in J^2} \left\{ G_1(t,s) \right\} = \frac{1 + e^{\sqrt{M}\,T}}{2\sqrt{M}\,(e^{\sqrt{M}\,T} - 1)};$$

$$\max_{(t,s) \in J^2} \left\{ G_2(t,s) \right\} = \frac{1}{2}.$$

For any $x, y \in E_0$,

$$\| (Fx)(t) - (Fy)(t) \|_{PC}$$

$$= \Bigg| \int_0^T G_1(t,s)\,[\sigma(s) - Nx(w(s))]\,ds + \sum_{k=1}^m [G_1(t,t_k)(-I_k^* y(t_k)) + G_2(t,t_k) I_k y'(t_k)$$

$$- \int_0^T G_1(t,s)\,[\sigma(s) - Ny(w(s))]\,ds + \sum_{k=1}^m [G_1(t,t_k)(-I_k^* y(t_k)) + G_2(t,t_k) I_k y'(t_k)] \Bigg|$$

$$= \Bigg| \int_0^T G_1(t,s)\,[Nx(w(t)) - Ny(w(s))]\,ds$$

$$+ \sum_{k=1}^m [G_1(t,t_k)\, I_k^* (-x(t_k) + y(t_k)) + G_2(t,t_k) I_k (x'(t_k) - y'(t_k))] \Bigg|$$

$$\le \frac{1 + e^{\sqrt{M}\,T}}{2\sqrt{M}\,(e^{\sqrt{M}\,T} - 1)} \left(NT + \sum_{k=1}^m I_k^* \right) \| x - y \|_{PC} + \frac{1}{2} \sum_{k=1}^m I_k \| x' - y' \|_{PC}$$

$$\le \left(\frac{1 + e^{\sqrt{M}\,T}}{2\sqrt{M}\,(e^{\sqrt{M}\,T} - 1)} \left(NT + \sum_{k=1}^m I_k^* \right) + \frac{1}{2} \sum_{k=1}^m I_k \right) \| x - y \|_{PC} .$$

Similarly,

$$\| (Fx)'(t) - (Fy)'(t) \|_{PC}$$

$$= \Bigg| \int_0^T G_2(t,s)\,[\sigma(s) - Ny(w(s))]\,ds + \sum_{k=1}^m [G_2(t,t_k)\, I_k^* y(t_k) - MG_2(t,t_k) I_k y'(t_k)]$$

$$+ \int_0^T G_2(t,s)\,[\sigma(s) - Ny(w(s))]\,ds - \sum_{k=1}^m [G_2(t,t_k)\, I_k^* y(t_k) - MG_2(t,t_k) I_k y'(t_k)] \Bigg|$$

$$= \Bigg| - \int_0^T G_2(t,s)\,[-Nx(w(s)) + Ny(w(s))]\,ds$$

$$+ \sum_{k=1}^m [G_2(t,t_k)\, I_k^* (-x(t_k) - y(t_k)) - MG_2(t,t_k) I_k (x'(t_k) - y'(t_k))] \Bigg|$$

$$\leq \left(\frac{1}{2} NT + \frac{1}{2} \sum_{k=1}^{m} I_k^* \right) \| x - y \|_{PC} + M \frac{1 + e^{\sqrt{M}\, T}}{2\sqrt{M}\, (e^{\sqrt{M}\, T} - 1)} \sum_{k=1}^{m} I_k \| x' - y' \|_{PC}$$

$$\leq \left(\frac{1}{2} NT + \frac{1}{2} \sum_{k=1}^{m} I_k^* + M \frac{1 + e^{\sqrt{M}\, T}}{2\sqrt{M}\, (e^{\sqrt{M}\, T} - 1)} \sum_{k=1}^{m} I_k \right) \| x - y \|_{PC}$$

Hence

$$\| Fx - Fy \|_{PC'} \leq h \| x - y \|_{PC'},$$

where

$$h = \max \left\{ \frac{1 + e^{\sqrt{M}\, T}}{2\sqrt{M}\, (e^{\sqrt{M}\, T} - 1)} \left(NT + \sum_{k=1}^{m} I_k^* \right) + \frac{1}{2} \sum_{k=1}^{m} I_k; \right.$$

$$\left. \frac{1}{2} \left(NT + \sum_{k=1}^{m} I_k^* \right) + \frac{\sqrt{M}\, (1 + e^{\sqrt{M}\, T})}{2\, (e^{\sqrt{M}\, T} - 1)} \sum_{k=1}^{m} I_k \right\} < 1.$$

By Banach fixed point theorem, F has a unique fixed point $y \in E'$, which is also the unique solution of (2). This completes the proof.

3. Main Result

In this section, we establish the existence theorem of (3) by method of upper and lower solutions coupled with monotone iterative technique.

For $u_0, v_0 \in E' \cap C^2(J^-, R)$, we write $u_0 \leq v_0$ if $u_0(t) \leq v_0(t)$ for all $t \in [-r, T]$. In such a case, we denote

$$[u_0, v_0] = \left\{ y \in E, u_0(t) < y(t) < v_0(t), t \in [-r, T] \right\}.$$

Definition 3.1 — A function $u_0 \in E' \cap C^2(J^-, R)$ is called a lower solution of PBVP (1) if

$$\begin{cases} -u_0''(t) \le f(t, u_0(t), u_0(w(t))), & t \ne t_k, t \in J, \\ \Delta u_0 |_{t=t_k} = I_k u_0'(t_k), & \\ \Delta u_0' |_{t=t_k} \ge I_k^* u_0(t_k), & (k = 1, 2, ..., m), \\ u_0(0) = u_0(T), u_0'(0) \ge u_0'(T), & \\ u_0(t) = u_0(0), & t \in [-r, 0]. \end{cases}$$

Definition 3.2 — A function $v_0 \in E' \cap C^2(J^-, R)$ is called a lower solution of PBVP (1) if

$$\begin{cases} -v_0''(t) = f(t, v_0(t), v_0(w(t))), & t \ne t_k, t \in J, \\ \Delta v_0 |_{t=t_k} = I_k v_0'(t_k), & \\ \Delta v_0' |_{t=t_k} \le I_k^* v_0(t_k), & (k = 1, 2, ..., m), \\ v_0(0) = v_0(T), v_0'(0) \le v_0'(T), & \\ v_0(t) = v_0(0), & t \in [-r, 0]. \end{cases}$$

Now we are in the position to establish the main result.

Theorem 3.3 — *Let the following conditions hold,*

(H_1) The functions u_0, v_0 are lower and upper solutions of PBVP (1) respectively, such that

$$u_0(t) \le v_0(t);$$

(H_2) The function f satisfies

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \ge -M(x, \bar{x}) - N(y - \bar{y}),$$

$$\text{for } t \in [-r, T], u_0(t) \le \bar{x} \le x \le v_0(t), u_0(w(t)) \le \bar{y} \le y \le v_0(w(t));$$

(H_3) Constant $M > 0$, $N \ge 0, I_k \ge 0, I_k^* \le 0$ and satisfy[15-17]

Then there exist monotone sequences $\{u_n(t)\}$, $\{v_n(t)\} \subset E' \cap C^2(J^-, R)$ which converge in PC' to the extreme solutions of PBVP (1) in $[u_0, v_0]$, respectively.

PROOF : For any $\eta \in [u_0, v_0]$, consider linear PBVP (2) with

$$\sigma(t) = f(t, \eta(t), \eta(w(t))) + M\eta(t) + N\eta(w(t)).$$

By Lemma 2.3 eq. (2), has exactly one solution $y \in E' \cap C^2(J^-, R)$. Denote $y(t) = A\eta(t)$, then A is an operator from $[u_0, v_0]$ to $PC' \cap C^2$

We complete the proof in four steps.

Step 1 : We claim that $u_0 \le Au_0$ and $Av_0 \le v_0$.

Let $u_1 = Au_0$ and $p = u_0 - u_1$. Then u_1 satisfies

$$\begin{cases} -u_1''(t) = Mu_1(t) - Nu_1(w(t)) + f(t, u_0(t), u_0(w(t))) + Mu_0(t) + Nu_0(w(t)) \\ \qquad\qquad\qquad\qquad t \ne t_k, t \in J, \\ \Delta u_1 \big|_{t=t_k} = I_k u_1'(t_k), \\ \Delta u_1' \big|_{t=t_k} = I_k^* u_1(t_k), \qquad (k = 1, 2, \ldots, m), \\ u_1(0) = u_1(T), u_1'(0) \ge u_1'(T), \\ u_1(t) = u_1(0), \qquad t \in [-r, 0]. \end{cases}$$

As u_0 is a lower solution of (1), so for $t \ne t_k, t \in J$,

$$\begin{aligned} -p''(t) &\le f(t, u_0(t), u_0(w(t))) + Mu_1(t) + Nu_1(w(t)) \\ &- f(t, u_0(t), u_0(w(t))) - Mu_0(t) - Nu_0(w(t)) \\ &= -Mp(t) - Np(w(t)). \end{aligned}$$

It is easy to verify

$$\Delta p \big|_{t=t_k} = I_k p'(t_k),$$

$$\Delta p' \big|_{t=t_k} \ge I_k^* p(t_k), \quad (k = 1, 2, \ldots, m),$$

$$p(0) = p(T), p'(0) \ge p'(T),$$

$$p(t) = p(0), t \in [-r, 0].$$

Then by Lemma 2.2, $p(t) \le 0$, which implies $u_0(t) \le Au_0(t)$, i.e. $u_0 \le Au_0$.

Similarly, we can prove $Av_0 \le v_0$.

Step 2 : We show that $A\eta_1 \le A\eta_2$, if $u_0 \le \eta_1 \le \eta_2 \le v_0$.

Let

$$\eta_1^* = A\eta_1, \eta_2^* = A\eta_2, p = \eta_1^* - \eta_2^*,$$

then for $t \ne t_k$, $t \in J$, and by (H_2), we obtain

$$-p''(t) = -Mp(t) - Np(w(t)) - [f(t, \eta_1(t), \eta_1(w(t))) - M\eta_1(t)$$

$$- N\eta_1(w(t)) - f(t, \eta_2(t), \eta_2(w(t))) - M\eta_2(t) - N\eta_2(w(t))]$$

$$\ge -Mp(t) - Np(w(t)).$$

It is easy to verify

$$\Delta p \,|_{t=t_k} = I_k p_0'(t_k),$$

$$\Delta p' \,|_{t=t_k} \ge I_k^* p(t_k), \quad (k = 1, 2, ..., m),$$

$$p(0) = p(T), p'(0) = p'(T),$$

$$p(t) = p(0), t \in [-r, 0].$$

Still by Lemma 2.2 $p(t) \le 0$, which implies $A\eta_1 \le A\eta_2$.

Step 3 : In this part, we show that PBVP (1) has solutions.

Let $u_n = Au_{n-1}$, $v_u = Av_{n-1}$, $n = 1, 2, ...$. Following the first two steps, we have

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le v_n \le \ldots \le v_1 \le v_0.$$

Obviously, each u_i, v_i $(i = 1, 2, ...)$ satisfies

$$\begin{cases} -u_i''(t) = -Mu_i(t) - Nu_i(w(t)) + f(t, u_{i-1}(t), u_{i-1}(w(t))) + Mu_{i-1}(t) + Nu_{i-1}(w(t)) \\ \qquad\qquad t \neq t_k, t \in J, \\ \Delta u_i \big|_{t=t_k} = I_k u_i'(t_k), \\ \Delta u_i' \big|_{t=t_k} = I_k^* u_i(t_k), \qquad (k = 1, 2, ..., m), \\ u_i(0) = u_i(T), u_i'(0) \geq u_i'(T), \\ u_i(t) = u_i(0), \qquad t \in [-r, 0]. \end{cases}$$

$$\begin{cases} -v_i''(t) = -Mv_i(t) - Nv_1(w(t)) + f(t, v_{i-1}(t), v_{i-1}(w(t))) + Mv_{i-1}(t) + Nv_{i-1}(w(t)) \\ \qquad\qquad t \neq t_k, t \in J, \\ \Delta v_i \big|_{t=t_k} = I_k v_i'(t_k), \\ \Delta v_i' \big|_{t=t_k} = I_k^* v_i(t_k), \qquad (k = 1, 2, ..., m), \\ v_i(0) = v_i(T), v_1(0) \geq v_i'(T), \\ v_i(t) = v_i(0), \qquad t \in [-r, 0]. \end{cases}$$

Therefore there exist y_* and y^* such that

$$\lim_{i \to +\infty} u_i(t) = y_*(t), \quad \lim_{i \to +\infty} v_i(t) = y^*(t) \text{ uniformly on } t \in J^+.$$

Clearly, y_*, y^* satisfy PBVP (1).

Step 4 : We prove y_*, y^* are extreme solutions of PBVP (1).

Let $y(t)$ be any solution of PBVP (1), which satisfies $u_0(t) \leq y(t) \leq v_0(t)$, for all $t \in [-r, T]$. Also suppose there exists a positive integer n such that for $t \in [-r, T]$, $u_n(t) \leq y(t) \leq v_n(t)$.

Setting $p = u_{n+1} - y$, then for $t \in J$,

$$\begin{aligned} -p''(t) &= -u_{n+1}'' + y''(t) \\ &= -Mu_{n+1}(t) - Nu_{n+1}(w(t)) + f(t, u_n(t), u_n(w(t))) \end{aligned}$$

$$+ Mu_n(t) + Nu_n(w(t)) - f(t, y(t), y(w(t)))$$

$$= -Mu_{n+1}(t) - Nu_{n+1}(w(t)) + My(t) + Ny(w(t)) + f(t, u_n(t), u_n(w(t)))$$

$$- f(t, y(t), y(w(t))) + Mu_n(t) + Nu_n(w(t)) - My(t) - Ny(w(t))$$

$$\leq - Mp(t) - NP(w(t));$$

and

$$\Delta p \mid_{t=t_k} = I_k u'_{n+1}(t_k) - I_k y'(t_k) = I_k p'(t_k),$$

$$\Delta p' \mid_{t=t_k} = I_k^* u_{n+1}(t_k) - I_k^* y(t_k) = I_k^* p(t_k), \quad (k = , 2, \ldots, m),$$

$$p(0) = p(T), p'(0) = p'(T),$$

$$p(t) = p(0), t \in [-r, 0].$$

By Lemma 2.2, we have for all $t \in J^+, p(t) \leq 0$, i.e. $u_{n+1} \leq y(t)$. Similarly, we can prove $y(t) \leq v_{n+1}$, $t \in [-r, T]$. Thus $u_{n+1} \leq y(t) \leq v_{n+1}$, for all $t \in [-r, T]$, which implies $y_*(t) \leq y(t) \leq y^*(t)$. We complete the proof.

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