*Indian J. pure appl. Math.*, **35**(8) : 969-971, August 2004


# A SIMPLE PROOF OF GENERALIZED ALZER'S INEQUALITY

JEONG SHEOK UME[*], ZEQING LIU[**] AND JOHN N. MCDONALD[***]

[*]*Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea*
*E-mail: jsume@changwon.ac.kr*
[**]*Department of Mathematics, Liaoning Normal University, Dalian, Liaoning*, 116029, *People's Republic of China*
[***]*Department of Mathematics, Arizona State University, Tempe, AZ* 85287-1804, *USA*
*E-mail: mcdonald@math.la.asu.edu*

(*Received* 30 *January* 2002; *after final revision* 16 *December* 2002; *accepted* 22 *April* 2004)

The aim of this paper is to present a more elementary proof of the result of Qi[5] by using differential calculus and mathematical induction.

**Key Words : Inequality; Mathematical Induction**

In[1-7], two inequalities were proved using the mathematical induction and other techniques, which can be expressed as

$$\frac{n}{n+1} \leq \left( \frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r} \leq \frac{\sqrt[n]{n\,!}}{\sqrt[(n+1)]{(n+1)\,!}},$$

where $r > 0$ and $n \in \mathbb{N}$. Recently, Qi[5] proved the generalized Alzer's inequality by using the mathematical induction and Cauchy's mean value theorem.

The aim of this paper is to present more elementary proof of the result of Qi[5] by using differential calculus and mathematical induction.

Throughout this paper we denote by $\mathbb{R}$ the set of all real numbers.

*Lemma* 1 – If $n$ is a positive integer, $k$ is a non-negative integer and $r$ is a positive real number, then

$$n+1 \leq \left(\frac{n+k+2}{n+k+1}\right)^r + n\left(\frac{n+k}{n+k+1}\right)^r \left(\frac{n+k+2}{n+k+1}\right)^r .$$

PROOF : First we shall prove that if $0 < x$ and $0 < a < 1$, then

$$\frac{1}{a} \leq (1+a)^x + \left(\frac{1}{a} - 1\right)(1-a^2)^x. \qquad \text{... (1)}$$

Write (1) in the form

This work was supported by Korea Research Foundation Grant (KRF 2003-015-000039).

$$1 \leq \left(\frac{a}{1+a}\right)(1+a)^{x+1} + \left(\frac{1}{1+a}\right)(1-a^2)^{x+1} \qquad \text{... (2)}$$

Let $f(t) = t^{x+1}$. Then $f$ is a convex function. Thus, (2) follows from

$$1 \leq f(1) = f\left(\frac{a}{1+a}(1+a) + \frac{1}{1+a}(1-a^2)\right)$$

$$\leq \left(\frac{a}{1+a}\right) f(1+a) + \left(\frac{1}{1+a}\right) f(1-a^2).$$

Hence (1) holds.

Next we shall prove that if $0 \leq x \leq \frac{1}{n+1}$ and $0 < r$, then

$$n+1 \leq (1+x)^r + n(1-x^2)^r, \qquad \text{... (3)}$$

where $n$ is a positive integer.

*Case* 1 — $1 \leq r$ : Write (3) in the form

$$1 \leq \frac{1}{n+1}(1+x)^r + \frac{n}{n+1}(1-x^2)^r. \qquad \text{... (4)}$$

Since the function $g(t) = t^r$ is convex, it follows that

$$g\left(1 + \frac{x - nx^2}{n+1}\right) = g\left(\frac{1}{n+1}(1+x) + \frac{n}{n+1}(1-x^2)\right)$$

$$\leq \frac{1}{n+1} g(1+x) + \frac{n}{n+1} g(1-x^2).$$

For $x \in \left[0, \frac{1}{n+1}\right]$ one has $0 < x - nx^2$. Thus,

$$1 = g(1) \leq g\left(1 + \frac{x - nx^2}{n+1}\right)$$

and (4) follows.

*Case* 2 — $0 < r < 1$: Let $h(x) = (1+x)^r + n(1-x^2)^r$. Then $h''(x) < 0$. It follows that the minimum value of $h(x)$ for $x \in \left[0, \frac{1}{n+1}\right]$ is either $h(0) = 1 + n$ or $h\left(\frac{1}{n+1}\right)$. Putting $a = \frac{1}{n+1}$ and $x = r$ in (1), we have $h\left(\frac{1}{n+1}\right) \geq n+1$. Thus (3) follows. Since (3) is equivalent to (4), the result of Lemma 1 follows from Case 1 and Case 2.

***Theorem*** 2 [2] — *Let n and m be natural numbers, k a non-negative integer. Then*

$$\frac{n+k}{n+m+k} < \left( \frac{1}{n} \sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r}$$

*where r is any given positive real number. The lower bounded is best possible.*

PROOF: Let

$$B(k+1, n+k) := \sum_{i=k+1}^{n+k} i^r .$$

Then, the inequality[5] is equivalent to

$$\frac{n(n+k)^r}{(n+m)(n+m+k)^r} < \frac{B(k+1, n+k)}{B(k+1, n+m+k)}. \qquad \text{... (6)}$$

If $m = 1$ in[6], we obtain

$$\frac{n(n+k)^r}{(n+1)(n+k+1)^r} < \frac{B(k+1, n+k)}{B(k+1, n+k+1)}.$$

Using Lemma 1, it is easy to show that the inequality[7] holds by inductive proof for $n$. Suppose inequality[6] holds for $m$. Then, by Lemma 1, it is easy to show that the inequality[6] holds for $m + 1$. Thus the result of Theorem 2 follows.

ACKNOWLEDGEMENT

The authors would like to thank the referee for his helpful suggestions towards the improvement of this paper.

## REFERENCES

1. H. Alzer, *J. Math. Anal. Appl.*, **179** (1993), 396-4-2.
2. N. Elezović and J. Pecarić, *J Math. Anal. Appl.*, **223** (1998), 366-69.
3. J. S. Martins, *Math. Nachr.*, **139** (1988), 281-88.
4. H. Minc and L. Sathre, *Proc. Edinburgh Math. Soc.*, **14** (1964-1965), 41-46.
5. F. Qi, *J. Math. Anal. Appl.*, **240** (1999), 294-97.
6. J. Sandor, *J. Math. Anal. Appl.*, **192** (1995), 1034-35.
7. J. S. Ume, *Math. Japonica*, **44** (1996), 521-22.