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FURTHER RESULTS ON SRIVASTAVA'S TRIPLE HYPERGEOMETRIC SERIES H_A AND H_C

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The aim of this research is to provide a large number of interesting reducible cases of triple hypergeometric series H_A and H_C introduced by Srivastava who actually noticed the existence of three additional complete triple hypergeometric functions H_A, H_B and H_C of the second order in the course of an extensive investigation of Lauricella's fourteen hypergeometric functions of three variables. The results are derived with the help of generalized Kummer's summation theorem, generalized Gauss's second summation theorem, generalized Bailey's summation formula and generalized Watson's theorem obtained in earlier works by Lavoie, Grodin and Rathie. A few known results obtained recently by Kim, Rathie and Choi follow special cases of our main findings.

Key Words: Triple Hypergeometric Series H_A and H_C, Apell's Functions; Generalized Kummer's Summation Theorem; Generalized Gauss's Theorem; Generalized Watson's Theorem

1. INTRODUCTION

The triple hypergeometric series H_A and H_C introduced by Srivastava given in Srivastava and Manocha[6], [p. 68-69, eqs. (36)-(38)] are defined as follows:

$$H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p}}{(\gamma)_m (\gamma')_{n+p}\, m!\, n!\, p!} x^m y^n z^p, \quad \ldots (1.1)$$

whose region of convergence is $|x| < r, |y| < s, |z| < t$, and $r + s + t = 1 + st$;

$$H_C(\alpha, \beta, \beta'; \gamma; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p} (\beta)_{m+n} (\beta')_{n+p}}{(\gamma)_{m+n+p}\, m!\, n!\, p!} x^m y^n z^p, \quad \ldots (1.2)$$

for $|x| < 1, |y| < 1$ and $|z| < 1$. Here $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, \ldots\} \end{cases}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where Γ is the well-known Gamma function.

Very recently Kim, Rathie and Choi[2] obtained reducible cases for the series H_A and H_C by employing Kummer's theorem, Gauss's second theorem, Bailey's formula and Watson's theorem, respectively. Some of them are recorded here

$$H_A\left(\alpha,\beta,\beta';1+\beta-\alpha+\frac{1}{2}\beta',1+\beta+\beta'-\alpha;-1,1,-1\right)$$

$$=\frac{\Gamma(1-\alpha)\,\Gamma(1+\frac{1}{2}\beta')\,\Gamma(1+\beta+\beta'-\alpha)\,\Gamma(1+\frac{1}{2}\beta)\,\Gamma(1-\alpha+\beta+\frac{1}{2}\beta')}{\Gamma(1+\beta')\,\Gamma(1+\beta-\alpha)\,\Gamma(1+\frac{1}{2}\beta'-\alpha)\,\Gamma(1+\beta)\,\Gamma(1+\frac{1}{2}\beta-\alpha+\frac{1}{2}\beta')}; \quad \dots (1.3)$$

$$H_A\left(\alpha,\beta,\beta';\frac{1}{2}(1+\alpha+\beta-\frac{1}{2}\beta'),1+\beta+\beta'-\alpha;\frac{1}{2},1,-1\right)$$

$$=\frac{\Gamma(1-\alpha)\,\Gamma(1+\frac{1}{2}\beta')\,\Gamma(1+\beta+\beta'-\alpha)\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{4}\beta'\right)}{\Gamma(1+\beta')\,\Gamma(1+\beta-\alpha)\,\Gamma(1+\frac{1}{2}\beta'-\alpha)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{4}\beta'\right)}; \quad \dots (1.4)$$

$$H_A\left(\alpha,\beta,2\alpha+2\beta-2;\gamma,3\beta+\alpha-1;\frac{1}{2},1,-1\right)$$

$$=\frac{\Gamma(1-\alpha)\,\Gamma(\alpha+\beta)\,\Gamma(3\beta+\alpha-1)\,\Gamma\left(\frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\right)}{\Gamma(\beta)\,\Gamma(1+\beta-\alpha)\,\Gamma(2\alpha+2\beta-1)\,\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)}; \quad \dots (1.5)$$

$$H_C(\alpha,\beta,\beta;3\beta;1,1,1)$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\beta\right)\Gamma(3\beta)\,\Gamma(\beta-\alpha)\,\Gamma(1+\alpha-\beta)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma(2\beta)\,\Gamma\left(2\beta-\frac{\alpha}{2}\right)\Gamma\left(1+\frac{1}{2}\alpha-\beta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\beta\right)}; \quad \dots (1.6)$$

The aim of this paper is to derive a large number of results closely related to (2.3) and (2.5). The results are derived with the help of generalized Kummer's theorem, generalized Gauss's second summation theorem, generalized Bailey's summation formula and generalized Watson's theorem obtained earlier by Lavoie, Grodin and Rathie[4,5]. The results derived in this paper are simple, interesting, easily established and may be useful.

2. Results Required

The following results will be required in our present investigations.

Appell's function F_1[6] [p. 55, eq. (15)].

$$F_1\,[a,b,b';c;1,1] = \frac{\Gamma(c)\,\Gamma(c-a-b-b')}{\Gamma(c-a)\,\Gamma(c-b-b')}$$

$$(\Re(c-a-b-b')>0;\ c\neq 0,-1,-2,\ldots), \qquad \ldots (2.1)$$

$$F_1\,(a,b,b';1+a+b-b';1,-1) = \frac{\Gamma(1-b')\,\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma(1+a+b-b')}{\Gamma(1+a)\,\Gamma(1+b-b')\,\Gamma\left(1+\frac{1}{2}a-b'\right)}, \qquad \ldots (2.2)$$

where F_1 is defined as in[6] [p. 53, eq. (4)].

Generalized Kummer's summation formula[5]:

$${}_2F_1\left[\begin{matrix} a, b; \\ & -1 \\ 1+a-b+i; \end{matrix}\right]$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\,\Gamma(1+a-b+i))}{2^a\,\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$

$$\times\left\{\frac{A_i}{\Gamma\left(\frac{a}{2}-b+\frac{1}{2}i+1\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)}\right\}$$

$$+\left\{\frac{B_i}{\Gamma\left(\frac{a}{2}-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \qquad \ldots (2.3)$$

where $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, $[x]$ denotes the greatest integer less than or equal to x and its absolute value is $|x|$. The coefficients A_i and B_i are given in Table I.

For $i = 0$, we get the following Kummer's theorem[1] :

$${}_2F_1\left[\begin{matrix} a, b; \\ & -1 \\ 1+a-b; \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\,\Gamma(1+a-b+i))}{2^a\,\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)} \qquad \ldots (2.4)$$

Generalized Gauss's second summation formula[5].

$$
{}_2F_1\left[\begin{matrix} a, b; \\ \\ \frac{1}{2}(a+b+i+1); \end{matrix} \;\frac{1}{2}\right]
$$

$$
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+i+1)\right)\Gamma\left(\frac{1}{2}(a-b-i+1)\right)}{\Gamma\left(\frac{1}{2}(a-b+|i|+1)\right)} \qquad \ldots (2.5)
$$

$$
\times \left\{\frac{C_i}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)}\right\}
$$

$$
+ \left\{\frac{D_i}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\},
$$

where $i = 0, \pm 1, \pm 2, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients C_i and D_i are given in Table I.

TABLE I

Coefficients $A_i, B_i, C_i, D_i, E_i, F_i$

i	A_i	B_i	C_i	D_i	E_i	F_i
5	$-4(6+a-b)^2$ $+2b(6+a-b)$ $+ b^2$ $+22(6+a-b)$ $-13b-20$	$4(6+a-b)^2$ $+2b(6+a-b)$ $-b^2$ $-34(6+a-b)$ $-b+62$	$-[(a+b+6)^2$ $-\frac{1}{2}(b-a+6)$ $(b+a+6)$ $-\frac{1}{4}(b-a+6)^2$ $-11(b+a+6)$ $+\frac{13}{2}(b-a+6)$ $+20]$	$(b+a+6)^2$ $+\frac{1}{2}(b-a+6)$ $(b+a+6)$ $-\frac{1}{4}(b-a+6)^2$ $-17(b+a+6)$ $-\frac{1}{2}(b-a+6)$ $+62$	$-(4b^2-2ab-a^2$ $-22b+13a+20)$	$4b^2+2ab-a^2$ $-34b-a+62$
4	$2(a-b+3)$ $(1+a-b)$ $-(b-1)(b-4)$	$-4(a-b+2)$	$\frac{1}{2}(a+b+1)$ $(a+b-3)$ $-\frac{1}{4}(b-a+3)$ $(b-a-3)$	$-2(b+a-1)$	$2(b-2)(b-4)$ $-(a-1)(a-4)$	$12-4b$
3	$3b-2a-5$	$2a-b+1$	$-\frac{1}{2}(b+3a-2)$	$\frac{1}{2}(3b+a-2)$	$a-2b+3$	$a+2b-7$

2	$1+a-b$	-2	$\frac{1}{2}(b+a-1)$	-2	$b-2$	-2
1	-1	1	-1	1	-1	1
0	1	0	1	0	1	0
-1	1	1	1	1	1	1
-2	$a-b-1$	2	$\frac{1}{2}(b+a-1)$	2	b	2
-3	$2a-3b-4$	$2a-b-2$	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$	$2b-a$	$a+2b+2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$	$\frac{1}{2}(b+a-3)(b+a+1)-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$	$2b(b+2)-a(a+3)$	$4(b+1)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2+16(a-b-4)-b+12$	$(b+a-4)^2-\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+4(b+a-4)-\frac{7}{2}(b-a-4)$	$(b+a-4)^2+\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+8(b+a-4)-\frac{1}{2}(b-a-4)+12$	$4b^2-2ab-a^2+8b-7a$	$4b^2+2ab-a^2+16b-a+12$

For $i = 0$, we get the following Gauss's second summation theorem[1].

$$
{}_2F_1\left[\begin{matrix} a, b; \\ \frac{1}{2}(a+b+1); \end{matrix} \; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}. \qquad \text{... (2.6)}
$$

Generalized Bailey's formula[5].

$$ {}_2F_1\left[\begin{matrix} a,\ 1+i-a; & \\ & \tfrac{1}{2} \\ c; & \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\,\Gamma(1-a)}{2^{c-i-1}\,\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)} \qquad \text{... (2.7)} $$

$$ \times\left\{\frac{E_i}{\Gamma\left(\frac{c}{2}-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{c}{2}+\frac{a}{2}-\left[\frac{1+i}{2}\right]\right)}\right. $$

$$ \left.+\frac{F_i}{\Gamma\left(\frac{c}{2}-\frac{a}{2}\right)\Gamma\left(\frac{c}{2}+\frac{a}{2}-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, $$

where $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients E_i and F_i are given in Table I.

For $i = 0$, we get the following Bailey's formula[1] :

$$ {}_2F_1\left[\begin{matrix} a,\ 1-a; & \tfrac{1}{2} \\ & \\ c; & \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}. \qquad \text{... (2.8)} $$

Generalized Watson's summation theorem[4]:

$$ {}_3F_2\left(a, b, c; \frac{1}{2}(a+b+i+1), 2c+j; 1\right) $$

$$ = G_{i,j}\,\frac{2^{a+b+i-2}\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)} $$

$$ \times\frac{\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{|i+j|}{2}+\frac{j}{2}+\frac{1}{2}\right)}{\Gamma(b)} \qquad \text{... (2.9)} $$

$$ \times\left\{H_{i,j}\,\frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]-(-1)^j\,\frac{1-(-1)^i}{4}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. $$

$$+\left\{ I_{i,j} \frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+(-1)^j\frac{1-(-1)^i}{4}\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\}$$

where $i, j = 0, \pm 1, \pm 2$. The coefficients $G_{i,j}, H_{i,j}$ and $I_{i,j}$ are given respectively in Tables II-IV.

TABLE II

Coefficient $G_{i,j}$

i \ j	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-(6c^2-2c(a+b-1)$ $-(a-b)^2+1)$	$-4(2c+a-b+1)$ $(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-\{2c(c+1)$ $+(a-b)(c-a+1)\}$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)$ $+(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)$ $(2c+a-b-3)$	$8c^2-2(c-1)(a+b+7)$ $-(a-b)^2-7$	8	$4c-a-b+1$	4

TABLE III

Coefficient $H_{i,j}$

i \ j	-2	-1	0	1	2
2	$c(a+b-1)$ $-(a+1)(b+1)+2$	$a+b-1$ $+b(2c-b)-2c+1$	$a(2c-a)$ $-(a-b)^2+1$	$2c(a+b-1)$	$A_{2,2}$
1	$c-a-1$	1	1	$2c-a+b$ $-(a-b)(c-b+1)$	$2c(c+1)$
0	$(c-a-1)(c-b-1)$ $+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)$ $+c(c+1)$
-1	$2c(-1)(c-2)$ $-(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$A_{-2,-2}$	$2(c-1)(a+b-1)$ $-(a-b)^2+1$	$a(2c-a)$ $+b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)$ $-(a-1)(b-1)$

$A_{2,2} = 2c(c+1)\ \{(2c+1)(a+b-1)-a(a-1)-B(b-1)\}\ -(a-b-1)(a-b+1)\ \{(c+1)(2c-a-b+1)+ab\}$

$A_{-2,-2} = 2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\}-(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)\ +\ ab.$

TABLE IV

Coefficient $I_{i,j}$

i \ j	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{a-b}$	$\frac{1}{a-b}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{c+1}$

For $i = j = 0$, we get the following Watson's theorem[1]:

$$
{}_3F_2\left[\begin{matrix} a, b\ c; \\ \frac{1}{2}(a+b+1), 2c \end{matrix} ;1\right] \qquad \text{... (2.10)}
$$

$$
= \frac{2^{a+b+-2}\,\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}
$$

provided $\Re e\,(2c-a-b) > -1$.

3. MAIN RESULTS

The following results are derived

$$
H_A\left(\alpha, \beta, \beta'; 1+\beta-\alpha+\frac{1}{2}\beta'+i, 1+\beta+\beta'-\alpha; -1, 1, -1\right)
$$

$$
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-\alpha)\,\Gamma\left(1+\frac{1}{2}\beta'\right)\Gamma(1+\beta+\beta'-\alpha)}{2^{\beta}\,\Gamma(1+\beta')\,\Gamma(1+\beta-\alpha)\,\Gamma\left(1+\frac{1}{2}\beta'-\alpha\right)\Gamma\left(1-\alpha+\frac{1}{2}\beta'+\frac{1}{2}i+\frac{1}{2}|i|\right)}
$$

$$
\times \frac{\Gamma\left(1-\alpha+\frac{1}{2}\beta'\right)\Gamma\left(1-\alpha+\beta+\frac{1}{2}\beta'+i\right)}{\Gamma\left(1+\frac{1}{2}\beta'-\alpha\right)\Gamma\left(1-\alpha+\frac{1}{2}\beta'+\frac{1}{2}i+\frac{1}{2}|i|\right)} \qquad \text{... (3.1)}
$$

$$\times\left\{\frac{A_i}{\Gamma\left(\frac{1}{2}\beta-\alpha+\frac{1}{2}\beta'+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)}\right.$$

$$\left.+\frac{B_i}{\Gamma\left(\frac{1}{2}\beta-\alpha+\frac{1}{2}\beta'+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients A_i and B_i can be obtained from the Table I by replacing a by β and b by $\alpha-\frac{1}{2}\beta'$, respectively.

$$H_A\left(\alpha, \beta, \beta'; \frac{1}{2}\left(1+\alpha+\beta-\frac{1}{2}\beta'+i\right), 1+\beta+\beta'-\alpha; \frac{1}{2}, 1, -1\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-\alpha)\,\Gamma\left(1+\frac{1}{2}\beta'\right)\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\,\Gamma(1+\beta-\alpha)} \qquad \ldots (3.2)$$

$$\times\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}\beta'+\frac{1}{2}i\right)\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\alpha+\frac{1}{4}\beta'-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}\beta'-\alpha\right)\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\alpha+\frac{1}{4}\beta'+\frac{1}{2}+\frac{1}{2}|i|\right)}$$

$$\times\left\{\frac{C_i}{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{4}\beta'+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)}\right.$$

$$\left.+\frac{D_i}{\Gamma\left(\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{4}\beta'+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients C_i and D_i can be obtained from the Table I by simply replacing a by β and b by $\alpha-\frac{1}{2}\beta'$, respectively.

$$H_A\left(\alpha, \beta, 2\alpha+2\beta-2-2i; \gamma, 3\beta+\alpha-2i-1; \frac{1}{2}, 1, -1\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\Gamma\right)(\gamma)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}{2^{\gamma-i-1}\,\Gamma(\beta-i)\,\Gamma(1+\beta-\alpha)}$$

$$\times\frac{\Gamma(\alpha+\beta-i)\,\Gamma(3\beta+\alpha-2i-1)}{\Gamma(2\alpha+2\beta-2i-1)\,\Gamma\left(1-\beta+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$

$$\times\left\{\frac{E_i}{\Gamma\left(\frac{1}{2}\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\beta-\left[\frac{1+i}{2}\right]\right)}\right.$$

$$\left.+\frac{F_i}{\Gamma\left(\frac{1}{2}\gamma-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\beta-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\} \qquad \ldots (3.3)$$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients E_i and F_i can be obtained from the Table I by simply replacing a by β and c by γ, respectively.

$$H_C(\alpha, \beta, \beta+i+j; 3\beta+2j+i; 1, 1, 1)$$

$$= G_{i,j}\frac{2^{2\alpha-2\beta-2j-1}\,\Gamma(\beta-\alpha+j)\,\Gamma(1+\alpha-\beta-j)}{\Gamma(1/2)\,\Gamma(\alpha)\,\Gamma(2\beta+j)}$$

$$\times\frac{\Gamma(3\beta+2j+i)\,\Gamma\left(\beta+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(2\beta-\alpha+\frac{3}{2}j+\frac{1}{2}i-\frac{|i+j|}{2}\right)}{\Gamma(2\beta-\alpha+2j+i)\,\Gamma(1+\alpha-2\beta-2j-i)} \qquad \ldots (3.4)$$

$$\times\left\{H_{i,j}\frac{\Gamma\left(\frac{1}{2}\alpha+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\beta-j-\frac{1}{2}i\right)}{\Gamma\left(2\beta-\frac{1}{2}\alpha+j+\frac{1}{2}i+\left[\frac{j}{2}\right]\right)\Gamma\left(\beta-\frac{1}{2}\alpha+j+\frac{1}{2}+\left[\frac{j}{2}\right]-(-1)^j\frac{1-(-1)^i}{4}\right)}\right.$$

$$\left.+\ I_{i,j}\frac{\Gamma\left(\frac{1}{2}\alpha+\frac{1\,(-1)^i}{4}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\beta-j-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(2\beta-\frac{1}{2}\alpha+j+\frac{i}{2}+\left[\frac{j+1}{2}\right]+\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\alpha+\left[\frac{j+1}{2}\right]+(-1)^j\frac{1-(-1)^i}{4}\right)}\right\}$$

for $i, j = 0, \pm 1, \pm 2$. The coefficients $G_{i,j}$, $H_{i,j}$ and $I_{i,j}$ can be obtained from Tables II-IV by simply replacing a by α, b by $1+\alpha-2\beta-2j-i$ and c by β, respectively.

4. DERIVATIONS

In order to prove (3.1), we start with the left-hand-side of (3.1) and use the definition of H_A given in (2.1). The left hand side is

$$H_A\left(\alpha, \beta, \beta'; 1+\beta-\alpha+\frac{1}{2}\beta'+i, 1+\beta+\beta'-\alpha; -1, 1, -1\right)$$

$$= \sum_{m,n,p=0}^{\infty}\frac{(\alpha)_m(\alpha+m)_p(\beta)_m(\beta+m)_n(\beta')_{n+p}}{\left(1+\beta-\alpha+\frac{1}{2}\beta'+i\right)_m(1+\beta+\beta'-\alpha)_{n+p}\,m!\,n!\,p!}(-1)^m(-1)^p$$

$$= \sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m(-1)^m}{\left(1+\beta-\alpha+\frac{1}{2}\beta'+i\right)_m m!}\left(\sum_{n,p=0}^{\infty}\frac{(\beta')_{n+p}(\alpha+m)_p(\beta+m)_n}{(1+\beta+\beta'-\alpha)_{n+p}\,n!\,p!}(-1)^p\right)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m (-1)^m}{\left(1+\beta-\alpha+\frac{1}{2}\beta'+i\right)_m m!} F_1(\beta', \beta+m, \alpha+m; 1+\beta+\beta'-\alpha; 1, -1).$$

If we use the result (2.2), we have

$$H_A\left(\alpha, \beta, \beta'; 1+\beta-\alpha+\frac{1}{2}\beta'+i, 1+\beta+\beta'-\alpha; -1, 1, -1\right)$$

$$= \frac{\Gamma\left(1+\frac{1}{2}\beta'\right)\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)} \times$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (\alpha)_m (\beta)_m}{\Gamma\left(1+\beta-\alpha+\frac{1}{2}\beta'+i\right)_m m!} \frac{\Gamma(1-\alpha-m)}{\Gamma\left(1+\frac{1}{2}\beta'-\alpha-m\right)}$$

which, upon using the following identity:

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \neq 0, \pm 1, \pm 2, \dots),$$

immediately yields

$$= \frac{\Gamma\left(1+\frac{1}{2}\beta'\right)\Gamma(1+\beta+\beta'-\alpha)\Gamma(1-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma\left(1+\frac{1}{2}\beta'-\alpha\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\alpha-\frac{1}{2}\beta'\right)_m (\beta)_m}{\left(1+\beta-\alpha+\frac{1}{2}\beta'+i\right)_m m!}.$$

$$= \frac{\Gamma(1-\alpha)\Gamma\left(1+\frac{1}{2}\beta'\right)\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma\left(1+\frac{1}{2}\beta'-\alpha\right)} \times$$

$${}_2F_1\left[\begin{matrix} \beta, \alpha-\frac{1}{2}\beta'; \\ \\ 1+\beta-\alpha+\frac{1}{2}\beta'+i \end{matrix} ; -1\right]$$

Now, if we use the result (2.3), then after some simplification, we arrive at the right hand side of (3.1).

In the corresponding method, the results from (3.2) to (3.4) can also be derived with the help of the results (2.5), (2.7) and (2.9), respectively.

5. Special Cases

The formulas (1.3), (1.4), (1.5) and (1.6) recently proved by Kim, Rathie and Choi[2] are seen to be obtained by setting $i = 0$ in (3.1), (3.2), (3.3) and $i = j = 0$ in (3.4), respectively.

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