Indian J. pure appl. Math., **35**(9) : 1047-1062, September 2004
© Printed in India.

ON RAMANUJAN'S CUBIC CONTINUED FRACTION AND EXPLICIT EVALUATIONS OF THETA-FUNCTIONS

CHANDRASHEKAR ADIGA, TAEKYUN KIM*, M. S. MAHADEVA NAIKA AND H. S. MADHUSUDHAN

Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, *India*
**Institute of Science Education, Kongju National University, Kongju* 314 701, *Republic of Korea*

(*Received* 18 *March* 2003; *after final revision* 5 *August* 2004; *accepted* 26 *August* 2004)

In this paper we give two integral representations for the Ramanujan's cubic continued fraction $V(q)$ and also derive a modular equation relating $V(q)$ and $V(q^3)$. We also establish some modular equations and a transformation formula for Ramanujan's theta-function $\psi(-q)$. As an application of these, we compute several new explicit evaluations of theta-functions and Ramanujan's cubic continued fraction.

Key Words: Cubic continued fraction, integral representation, modular equation, theta-function, transformation formula

1. INTRODUCTION

The celebrated Rogers-Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1} \; \frac{q}{1+} \; \frac{q^2}{1+} \; \frac{q^3}{1+} \; \frac{}{+\ldots}, \qquad |q| < 1. \qquad \ldots (1.1)$$

On page 46 in his 'lost' notebook[19] [p. 46], Ramanujan claims that

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left((-1/5) \int_q^1 \frac{(1-t)^5 (1-t^2)^5 \ldots dt}{(1-t^5)(1-t^{10}) \ldots t} \right) \qquad \ldots (1.2)$$

$$= \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2} \exp\left((-1/5) \int_0^q \frac{(1-t)^5 (1-t^2)^5 \ldots dt}{(1-t^{1/5})(1-t^{2/5})^{10} \ldots t^{4/5}} \right)}, \qquad \ldots (1.3)$$

where $0 < q < 1$. The first equality (1.2) was proved by Andrews[3] and the second equality (1.3) was proved by Son[20]. On page 207 of his 'lost' notebook Ramanujan also recorded six identities involving integrals of theta-functions. All these identities were proved by Son[20] and were generalized by Chandrashekar Adiga, Vasuki and Mahadeva Naika[1]. On page 365 of his 'lost' notebook[19], Ramanujan wrote five modular equations relating $R(q)$ with $R(-q), R(q^2), R(q^3), R(q^4)$ and $R(q^5)$.

Ramanujan eventually found several generalizations and ramifications of (1.1) which are recorded in his 'lost' notebook. These and related works may be found in the papers by Bhargava[8], Bhargava and Adiga[9,10] and Denis[13-15].

On page 366 of his 'lost' notebook[19], Ramanujan investigated the continued fraction

$$V(q) := \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+\dots}, \qquad |q|<1. \qquad \dots (1.4)$$

which is known as Ramanujan's cubic continued fraction. Analogous to those relations for $R(q)$ which are mentioned above Chan[12] established several modular equations relating $V(q)$ with $V(-q)$, $V(q^2)$, $V(q^2)$ and $V(q^3)$. An example of these modular equations is

$$V^3(q) = V(q^3)\,\frac{1 - V(q^3) + V^2(q^3)}{1 + 2\,V(q^3) + 4V^2(q^3)}. \qquad \dots (1.5)$$

In Section 2, we will establish two integral representations for $V(q)$, which are analogous to (1.2) and (1.3). We also give a simple proof of (1.5).

In Ramanujan's theory of theta-functions, the three theta-functions that play central roles are defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \qquad \dots (1.6)$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad \dots (1.7)$$

and

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \qquad \dots (1.8)$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1-aq^n), \quad |q|<1.$$

In Chapter 16 of his second notebook[18], Ramanujan records many transformation formulas for $\varphi(q)$, $\psi(q)$ and $f(-q)$. Four of the most important transformation formulas are given by

$$\sqrt{\alpha}\, \varphi(e^{-\alpha^2}) = \sqrt{\beta}\, \varphi(e^{-\beta^2}),\ \alpha\beta = \pi, \qquad \text{... (1.9)}$$

$$2\sqrt{\alpha}\, \psi(e^{-2\alpha^2}) = \sqrt{\beta}\, e^{\alpha^2/4}\, \varphi(-e^{-\beta^2}),\ \alpha\beta = \pi, \qquad \text{... (1.10)}$$

$$e^{-\alpha/12}\, \sqrt[4]{\alpha}\, f(-e^{-2\alpha}) = e^{-\beta/12}\, \sqrt[4]{\beta}\, f(-e^{-2\beta}),\ \alpha\beta = \pi^2, \qquad \text{... (1.11)}$$

and

$$e^{-\alpha/24}\, \sqrt[4]{\alpha}\, f(e^{-\alpha}) = e^{-\beta/24}\, \sqrt[4]{\beta}\, f(e^{-\beta}),\ \alpha\beta = \pi^2. \qquad \text{... (1.12)}$$

In Section 3, we will prove the transformation formula

$$\sqrt[4]{\alpha}\, e^{-\alpha/8}\, \psi(-e^{-\alpha}) = e^{-\beta/8}\, \sqrt[4]{\beta}\, \psi(-e^{-\beta}). \qquad \text{... (1.13)}$$

On page 204 of his second notebook [18, p. 204], Ramanujan claims that

$$\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\alpha})\right)\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\pi\beta})\right) = \frac{5+\sqrt{5}}{2} \qquad \text{... (1.14)}$$

and

$$\left(\frac{\sqrt{5}-1}{2} - R(-e^{-2\pi\alpha})\right)\left(\frac{\sqrt{5}-1}{2} - R(-e^{-2\pi\beta})\right) = \frac{5-\sqrt{5}}{2}. \qquad \text{... (1.15)}$$

Identities (1.14) and (1.15) were first proved by Watson[21]. Chan[12] has proved several identities for $V(q)$ which are similar to (1.14) and (1.15). In Section 4, we will establish three reciprocity theorems for $V(q)$ which are also similar to (1.14) and (1.15).

Ramanujan has recorded many modular equations in his notebooks[5,6] [pp. 204-237], [pp. 156-160] which are very useful in the computation of class invariants and the values of theta-functions. In the literature not much attention has been given to find the values of $\psi(q)$ and $\varphi(q)$. But Ramanujan recorded several values of $\varphi(q)$ in his notebooks. For example

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)},$$

$$\psi(e^{-\pi}) = 2^{-5/8}\, e^{\pi/8}\, \frac{\pi^{1/4}}{\Gamma(3/4)},$$

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9},$$

and

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5}} - 10.$$

Borwein and Borwein[11] first observed that class invariants could be used to calculate certain values of $\varphi(e^{-n\pi})$. Berndt[6] has verified all values for $\varphi(e^{-n\pi})$ claimed by Ramanujan by combining Ramanujan's class invariants with modular equations.

In Section 5, we derive some modular equations and briefly discuss evaluations of theta-functions $\psi(q)$ and $\varphi(q)$. At the end of this section we compute some interesting new explicit evaluations of $V(q)$. Our work is sequel to the works of Berndt[7] *et al.*, Chan[12], Ramachandra[16], Ramanathan[17] and Adiga, Vasuki and Mahadeva Naika[2]. In this paper we adopt existing methods in the literature and work with $\psi(q)$ instead of $\varphi(q)$ as is done in [12].

2. Integral Representations for $V(q)$ and Modular Equation Relating $V(q)$ with $V(q^3)$

***Theorem* 2.1** — *We have*

$$V(q) = \frac{1}{\sqrt[3]{-1 + 9\exp\left(\int_q^1 \varphi^2(-t)\,\varphi^2(-t^3)\frac{dt}{t}\right)}} \quad \ldots (2.1)$$

$$= \frac{1}{2}\sqrt[3]{1 - \exp\left(-8\int_0^q \psi^2(t)\,\psi^2(t^3)\right)} \quad \ldots (2.2)$$

Proof of (2.1): Let $F(q) := \dfrac{\psi^4(q)}{q\,\psi^4(q^3)}$. Using (1.7) and then taking logarithm on both sides, we find that

$$\log F(q) = 4\sum_{n=1}^{\infty}\left\{\log(1-q^{2n}) - \log(1-q^{2n-1})\right\}$$

$$+\,4\sum_{n=1}^{\infty}\left\{\log(1-q^{3(2n-1)}) - \log(1-q^{6n})\right\} - \log q.$$

Taking derivative on both sides of the above identity, we deduce that

$$\frac{d}{dq}[\log F(q)] = -\frac{1}{q}\left[1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1-q^n} - 12\sum_{n=1}^{\infty}\frac{(-1)^n nq^{3n}}{1-q^{3n}}\right].$$

Using Entry 3 (iv) of Chapter 19 of Ramanujan's notebooks [4, p. 226] we find that

$$\frac{d}{dq}[\log F(q)] = -\frac{\varphi^2(-q)\,\varphi^2(-q^3)}{q}.$$

Integrating the above identity over $[q, 1]$ on both sides, we obtain

$$\log F(q) = \int_q^1 \varphi^2(-t)\,\varphi^2(-t^3)\frac{dt}{t} + \log 9. \qquad \text{... (2.3)}$$

Using Entry 1 (i) of Chapter 20 of Ramanujan's notebooks [4, p. 345] in (2.3) and then exponentiating, we obtain (2.1).

PROOF OF (2.2) : Let $H(q) := \dfrac{\varphi^4(-q)}{\varphi^4(-q^3)}$. Using (1.6) and then taking logarithm on both sides, we find that

$$\log H(q) = 4\sum_{n=1}^{\infty}\left\{\log(1-q^n) - \log(1+q^n)\right\}$$

$$+4\sum_{n=1}^{\infty}\left\{\log(1+q^{3n}) - \log(1-q^{3n})\right\}.$$

Taking derivative on both sides of the above identity, we see that

$$\frac{d}{dq}[\log H(q)] = -\frac{8}{q}\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} - 3\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{6n}}\right].$$

Using Entry 1 *(iii)* of Chapter 19 of Ramanujan's notebooks [4, p. 225], we deduce that

$$\frac{d}{dq}[\log H(q)] = -8\,\psi^2(q)\,\psi^2(q^3).$$

Integrating the above identity over $[0, q]$ on both sides and then exponentiating, we obtain

$$H(q) = \exp\left(-8\int_0^q \psi^2(t)\,\psi^2(t^3)\,dt\right). \qquad \text{... (2.4)}$$

We deduce (2.2) on employing the following identity[4] [p. 345] in (2.4) :

$$1 - 8V^3(q) = \frac{\varphi(-q)}{\varphi^4(-q^3)}. \qquad \text{... (2.5)}$$

Theorem 2.2 — *Let V (q) be defined as in (1.4). Then*

$$V^3(q) = V(q^3)\,\frac{1 - V(q^3) + V^2(q^3)}{1 + 2V(q^3) + 4V^2(q^3)}. \qquad \text{... (2.6)}$$

PROOF : To prove this theorem we require the following identities [4, p. 345, Entry 1 (*i*), (ii)] :

$$1 + \frac{1}{V(q^3)} = \frac{\psi(q)}{q\,\psi(q^9)}, \qquad \text{... (2.7)}$$

$$1 + \frac{1}{V^3(q)} = \frac{\psi^4(q)}{q\,\psi^4(q^3)}, \qquad \text{... (2.8)}$$

and

$$1 - 3q\,\frac{\psi(q^9)}{\psi(q)} = \left[\,1 - 9q\,\frac{\psi^4(q^3)}{\psi^4(q)}\,\right]^{1/3} \qquad \text{... (2.9)}$$

Using (2.7) and (2.8) in (2.9), we obtain

$$1 - \frac{3V(q^3)}{1 + V(q^3)} = \left[\,1 - \frac{9V^3(q)}{1 + V^3(q)}\,\right]^{1/3}. \qquad \text{... (2.10)}$$

Cubing both sides of the above identity (2.10), we deduce (2.6).

3. TRANSFORMATION FORMULA FOR ψ

In the following theorem we prove a transformation formula for ψ akin to Ramanujan's transformation formulas (1.9)-(1.12)

***Theorem* 3.1** — *If* $\alpha\beta = \pi^2$, *then*

$$\sqrt[4]{\alpha}\, e^{-\alpha/8}\, \psi(-e^{-\alpha}) = e^{-\beta/8}\, \sqrt[4]{\beta}\, \psi(-e^{-\beta}). \qquad \text{... (3.1)}$$

PROOF : Interchanging α and β in (1.10), we find that

$$2\sqrt{\beta}\, \psi(e^{-2\beta^2}) = \sqrt{\alpha}\, e^{\beta^2/4}\, \varphi(-e^{-\alpha^2}). \qquad \text{.. (3.2)}$$

Using (1.10) and (3.2), we deduce that

$$\frac{\psi(e^{-2\alpha^2})}{\psi(e^{-2\beta^2})} = \frac{\beta}{\alpha}\,\frac{e^{\alpha^2/4}}{e^{\beta^2/4}}\,\frac{\varphi(-e^{-\beta^2})}{\varphi(-e^{-\alpha^2})},\ \alpha\beta = \pi. \qquad \text{... (3.3)}$$

Replacing α by $\sqrt{\alpha}$ and β by $\sqrt{\beta}$ in (3.3), we find that

$$\frac{\psi(e^{-2\alpha})\,\varphi(-e^{-\alpha})}{\psi(e^{-2\beta})\,\varphi(-e^{-\beta})}=\sqrt{\frac{\beta}{\alpha}}\,\frac{e^{\alpha/4}}{e^{\beta/4}},\ \alpha\beta=\pi^2. \quad \text{... (3.4)}$$

Using Entry 25 (*iv*) of Chapter 16 of Ramanujan's notebooks [4, p. 40] in (3.4), we obtain (3.1).

Remark : The transformation formula (3.1) can also be proved by using the identities (1.11) and (1.12).

4. *Reciprocity theorems for V(q)*

***Theorem* 4.1** — *We have*

(*i*) If $\alpha\beta=1$, then

$$\left[1+\frac{1}{V(-e^{-\pi\alpha})}\right]\left[1+\frac{1}{V(-e^{-\pi\beta})}\right]=3. \quad \text{... (4.1)}$$

(*ii*) If $3\alpha\beta=1$, then

$$\left[1+\frac{1}{V^3(-e^{-\pi\alpha})}\right]\left[1+\frac{1}{V^3(-e^{-\pi\beta})}\right]=9. \quad \text{(4.2)}$$

(*iii*) If $3\alpha\beta=1$, then

$$\left[1+\frac{1}{V^3(e^{-\sqrt{2}\pi\alpha})}\right][1-8V^3(e^{-\sqrt{2}\pi\beta})]=9. \quad \text{... (4.3)}$$

PROOF OF (4.1) : Using (2.7), we find that

$$\left[1+\frac{1}{V(-e^{-\pi\alpha})}\right]\left[1+\frac{1}{V(-e^{-\pi\beta})}\right]=\frac{\psi(-e^{-\pi\alpha/3})\,\psi(e^{-\pi\beta/3})}{e^{(-\pi/3)(\alpha+\beta)}\,\psi(-e^{-3\pi\alpha})\,\psi(-e^{-3\pi\beta})}. \quad \text{... (4.4)}$$

From the transformation formula (3.1), we deduce that

$$\frac{\psi(-e^{-\pi\alpha/3})}{e^{(\pi/8)((\alpha/3)-3\beta)}\,\psi(-e^{-\pi\beta})}=\sqrt[4]{\frac{9\beta}{\alpha}}. \quad \text{... (4.5)}$$

Interchanging α and β in (4.5), we obtain

$$\frac{\psi(-e^{-\pi\beta/3})}{e^{(\pi/8)((\beta/3)-3\alpha)}\,\psi(-e^{-3\pi\alpha})}=\sqrt[4]{\frac{9\alpha}{\beta}}. \quad \text{... (4.6)}$$

Using (4.5) and (4.6) in (4.4), we obtain (4.1).

PROOF OF (4.2) : Using (2.8), we find that

$$\left[1+\frac{1}{V^3(-e^{-\pi\alpha})}\right]\left[1+\frac{1}{V^3(-e^{-\pi\beta})}\right]=\frac{\psi^4(-e^{-\pi\alpha})\,\psi^4(-e^{-\pi\beta})}{e^{-\pi(\alpha+\beta)}\,\psi^4(-e^{-3\pi\alpha})\,\psi^4(-e^{-3\pi\beta})}. \quad \text{... (4.7)}$$

From the transformation formula (3.1), we deduce that

$$\frac{\psi^4(-e^{-\pi\alpha})}{e^{(\pi/2)(\alpha-3\beta)}\,\psi^4(-e^{-3\pi\beta})}=\frac{3\beta}{\alpha}. \quad \text{... (4.8)}$$

Interchanging α and β in (4.8), we obtain

$$\frac{\psi^4(-e^{-\pi\beta})}{e^{(\pi/2)(\beta-3\alpha)}\,\psi^4(-e^{-3\pi\alpha})}=\frac{3\alpha}{\beta}. \quad \text{... (4.9)}$$

Using (4.8) and (4.9) in (4.7), we obtain (4.2).

PROOF OF (4.3) : Using (2.8), we find that

$$\left[1+\frac{1}{V^3(e^{-\sqrt{2}\pi\alpha})}\right]=\frac{\psi^4(e^{-\sqrt{2}\pi\alpha})}{e^{-\sqrt{2}\pi\alpha}\,\psi^4(e^{-3\sqrt{2}\pi\alpha})}. \quad \text{... (4.10)}$$

Using (1.10) in (4.10), we deduce that

$$\left[1+\frac{1}{V^3(e^{-\sqrt{2}\pi\alpha})}\right]=9\,\frac{\varphi^4(-e^{-\sqrt{2}\pi/\alpha})}{\varphi^4(-e^{-\sqrt{2}\pi/3\alpha})}. \quad \text{... (4.11)}$$

Using (2.5) in (4.11), we find that

$$\left[1+\frac{1}{V^3(e^{-\sqrt{2}\pi\alpha})}\right]=\frac{9}{1-8V^3(e^{-\sqrt{2}\pi/3\alpha})}=\frac{9}{1-8V^3(e^{-\sqrt{2}\pi\beta})}.$$

Hence, we complete the proof.

5. MODULAR EQUATIONS AND EVALUATIONS OF THETA-FUNCTION AND $V(q)$

***Theorem* 5.1** — *Let*

$$P=\frac{\psi(-q)}{q^{1/4}\,\psi(-q^3)} \quad \textit{and} \quad Q=\frac{\varphi(q)}{\varphi(q^3)}.$$

Then

$$Q^4+P^4Q^4=9+P^4. \quad \text{... (5.1)}$$

PROOF : Using Entry 1 (*i*) and (*ii*) of Chapter 20 of Ramanujan's notebooks[4] [p. 345] and (2.5), we find that

$$\frac{\psi^4(q)}{q\psi^4(q^3)}+\frac{\varphi^4(-q)}{\varphi^4(-q^3)}=9+\frac{\psi^4(q)\,\varphi^4(-q)}{q\,\psi^4(q^3)\,\varphi^4(-q^3)}.$$

Replacing q by $-q$ in the above identity, we obtain (5.1).

***Theorem* 5.2** — *Let*

$$P=\frac{\psi(-q)}{q^3\,\psi(-q^3)} \text{ and } Q=\frac{\varphi(q)}{\varphi(q^3)}.$$

Then

$$Q^4+P^4Q^4=9+P^4. \qquad \text{... (5.2)}$$

PROOF : Using Entry 1 (*i*) and (*ii*) of Chapter 20 of Ramanujan's notebooks[4] [p. 345], we deduce that

$$\frac{\psi(q^{1/3})}{q^{1/3}\,\psi(q^3)}+\frac{\varphi(-q^{1/3})}{\varphi(-q^3)}=3+\frac{\psi(q^{1/3})\,\varphi(-q^{1/3})}{q^{1/3}\,\psi(q^3)\,\varphi(-q^3)}.$$

Replacing q by $-q^3$ in the above identity, we obtain (5.2).

***Theorem* 5.3** — *Let*

$$P=\frac{\psi(-q)}{q^{1/2}\,\psi(-q^5)} \text{ and } Q=\frac{\varphi(q)}{\varphi(q^5)}.$$

Then

$$Q^2+P^2Q^2=5+P^2. \qquad \text{... (5.3)}$$

PROOF : Changing q to $-q$ in Entry 9 (*iii*) of Chapter 19 of Ramanujan's notebooks [4, p. 258], we find that

$$\frac{\varphi^2(-q)}{\varphi^2(-q)^5}=1-4W, \qquad \text{... (5.4)}$$

where

$$W=\frac{q\,\chi(-q)\,f(q^5)\,f(-q^{20})}{\varphi^2(-q^5)}.$$

Using Entry 9 (*vii*) of Chapter 19 of Ramanujan's notebooks [4, p. 258], Entry 10 (*v*) of Chapter 19 of Ramanujan's notebooks [4, p. 262] can be written as

$$\frac{\psi^2(q)}{q\,\psi^2(q^5)}=1+\frac{\varphi(-q^5)\,f(-q^5)}{q\,\chi(-q)\,\psi^2(q^5)}. \qquad \text{... (5.5)}$$

The above identity can be written as

$$\frac{\psi^2(q)}{q\,\psi^2(q^5)} = 1 + \frac{1}{W}. \qquad \text{... (5.6)}$$

Using (5.4) and (5.6), we find that

$$\frac{\psi^2(q)}{q\,\psi^2(q^5)} + \frac{\varphi^2(-q)}{\varphi^2(-q^5)} = 5 + \frac{\psi^2(q)\,\varphi^2(-q)}{q\,\psi^2(q^5)\,\varphi^2(-q^5)}.$$

Changing q into $-q$ in the above identity, we obtain (5.3).

***Theorem* 5.4** — *We have*

$$(i)\ \frac{\psi(-e^{-\pi/\sqrt{5}})}{e^{-\pi/2\sqrt{5}}\,\psi(-e^{-\sqrt{5}\,\pi})} = 5^{1/4}, \qquad \text{... (5.7)}$$

$$(ii)\ \frac{\psi(-e^{-3\pi/\sqrt{5}})}{e^{-3\pi/2\sqrt{5}}\,\psi(-e^{-3\sqrt{5}\,\pi})} = \frac{5^{1/4}(\sqrt{3}+1)(\sqrt{5}+\sqrt{3})}{2}, \qquad \text{... (5.8)}$$

$$(iii)\ \frac{\psi(-e^{-\pi/3\sqrt{5}})}{e^{-\pi/3\sqrt{5}}\,\psi(-e^{-3\pi/\sqrt{5}})} = \frac{\sqrt{3}(\sqrt{3}-1)(\sqrt{5}-\sqrt{3})}{2}, \qquad \text{... (5.9)}$$

$$(iv)\ \frac{\psi(-e^{-\pi/3\sqrt{3}})}{e^{-\pi/12\sqrt{3}}\,\psi(-e^{-\pi/\sqrt{3}})} = 3^{-1/4}(\sqrt[3]{4}-1), \qquad \text{... (5.10)}$$

$$(v)\ \frac{\psi(-e^{-\pi})}{e^{(-\pi/4)}\,\psi(-e^{-3\pi})} = \sqrt[4]{3\sqrt{3}\,(2+\sqrt{3})}, \qquad \text{... (5.11)}$$

$$(vi)\ \frac{\psi(-e^{-\pi/3\sqrt{5}})}{e^{-\pi/6\sqrt{5}}\,\psi(-e^{-\sqrt{5}\,\pi/3})} = \frac{5^{1/4}(\sqrt{3}-1)(\sqrt{5}-\sqrt{3})}{2}, \qquad \text{... (5.12)}$$

$$(vii)\ \frac{\psi(-e^{-\sqrt{5}\,\pi/3})}{e^{-\sqrt{5}\,\pi/3}\,\psi(-e^{-3\sqrt{5}\,\pi})} = \frac{\sqrt{3}(\sqrt{3}+1)(\sqrt{5}+\sqrt{3})}{2}, \qquad \text{... (5.13)}$$

$$(viii)\ \frac{\psi(-e^{-\pi/\sqrt{3}})}{e^{-\pi/4\sqrt{3}}\,\psi(-e^{-\sqrt{3}\,\pi})} = 3^{1/4}, \qquad \text{... (5.14)}$$

$$(ix)\ \frac{\psi(-e^{-\pi\sqrt{3}})}{e^{-\pi\sqrt{3}/4}\,\psi(-e^{-3\sqrt{3}\,\pi})} = \frac{\sqrt[4]{27}}{\sqrt[3]{4}-1}, \qquad \text{... (5.15)}$$

$$(x)\ \frac{\psi(-e^{-\pi/3})}{e^{-\pi/3}\,\psi(-e^{-3\pi})} = \sqrt{3}, \qquad \text{... (5.16)}$$

$$(xi)\ \frac{\psi(-e^{-\pi/3})}{e^{-\pi/12}\,\psi(-e^{-\pi})} = \sqrt[4]{\sqrt{3}\,(2-\sqrt{3})}, \qquad \text{... (5.17)}$$

$$(xii)\ \frac{\psi(-e^{-\pi/\sqrt{3}})}{e^{-\pi/\sqrt{3}}\ \psi(-e^{-3\sqrt{3}\,\pi})} = \frac{3}{\sqrt[3]{4}-1}, \qquad \ldots (5.18)$$

and

$$(xiii)\ \frac{\psi(-e^{-\pi/3\sqrt{3}})}{e^{-\pi/3\sqrt{3}}\ \psi(-e^{-\sqrt{3}\,\pi})} = \sqrt[3]{4}-1. \qquad \ldots (5.19)$$

Proofs of the identities (5.7)-(5.19) being similar, for brevity we prove only (5.7)-(5.11).

Proof of (5.7) : Putting $\alpha=\frac{\pi}{\sqrt{5}}$ and $\beta=\pi\sqrt{5}$ in the transformation formula (3.1), we obtain (5.7).

Proof of (5.8) : By Entry 66 of Chapter 25 of Ramanujan's notebooks [5, p. 233] with q replaced by $-q$, we have

$$P^3Q^3+5PQ=Q^4-3PQ^3-3P^3Q-P^4 \qquad \ldots (5.20)$$

where

$$P=\frac{\psi(-q)}{q^{1/2}\ \psi(-q^5)} \quad \text{and} \quad Q=\frac{\psi(-q^3)}{q^{3/2}\ \psi(-q^{15})}$$

Using (5.7) in (5.20) with $q=e^{(-\pi/\sqrt{5})}$, we find that

$$Q^4-5^{1/4}(3+\sqrt{5})\,Q^3-5^{3/4}(3+\sqrt{5})\,Q-5=0. \qquad \ldots (5.21)$$

Putting $Q=i5^{1/4}T$ in (5.21), we deduce that

$$T^4+(3+\sqrt{5})\,iT^3-(3+\sqrt{5})\,iT-1=0. \qquad \ldots (5.22)$$

The eq. (5.22) can be written as

$$(T^2-1)\left(T^2+(3+\sqrt{5})\,iT+1\right)=0.$$

Solving the above equation, we deduce that

$$T=\pm 1, \quad T=\frac{-(3+\sqrt{5})\,i\pm i\sqrt{(3+\sqrt{5})+4}}{2}$$

and hence

$$Q=\pm i5^{1/4}, \quad Q=\frac{5^{1/4}\left((3+\sqrt{5})\pm\sqrt{(3+\sqrt{5})^2+4}\right)}{2}.$$

Since $Q>0$, we obtain (5.8).

Proof of (5.9) : Putting $q = e^{(-\pi/\sqrt{5})}$ in (5.20) and using (5.7), we find that

$$P = 5^{1/4}. \qquad \text{... (5.23)}$$

Also, using (3.1), we see that

$$Q = \frac{\sqrt[4]{45}}{C} \qquad \text{... (5.24)}$$

where

$$C = \frac{\psi(-e^{-\pi/3\sqrt{5}})}{e^{-\pi/3\sqrt{5}}\,\psi(-e^{-3\pi/\sqrt{5}})}.$$

Using (5.23) and (5.24), we find that

$$PQ = \frac{\sqrt{15}}{C} \qquad \text{... (5.25)}$$

and

$$\frac{P}{Q} = \frac{C}{\sqrt{3}}. \qquad \text{... (5.26)}$$

Using (5.25) and (5.26) in (5.20), we deduce that

$$\sqrt{5}\left[\frac{\sqrt{3}}{C} + \frac{C}{\sqrt{3}}\right] = \left[\frac{\sqrt{3}}{C} + \frac{C}{\sqrt{3}}\right]\left[\frac{\sqrt{3}}{C} - \frac{C}{\sqrt{3}}\right] - 3\left[\frac{\sqrt{3}}{C} + \frac{C}{\sqrt{3}}\right].$$

Since $\frac{\sqrt{3}}{C} + \frac{C}{\sqrt{3}} \neq 0$, we obtain

$$\frac{\sqrt{3}}{C} - \frac{C}{\sqrt{3}} = \sqrt{5} + 3.$$

Since $C > 0$, solving the above equation we obtain the required result.

Proof of (5.10) : Let

$$P = \frac{\psi(-q)}{q^{1/4}\,\psi(-q^3)} \quad \text{and} \quad Q = \frac{\psi(-q^3)}{q^{3/4}\,\psi(-q^9)}. \qquad \text{... (5.27)}$$

Using (5.27) in Entry 1 (*ii*) of Chapter 20 of Ramanujan's notebooks [4, p. 345], we find that

$$Q^3 - P^3 Q^2 - 3P^2 Q - 3P = 0. \qquad \text{... (5.28)}$$

Putting $q = e^{(-\pi/3\sqrt{5})}$ in (5.27), and using the transformation formula (3.1), we see that

$$Q = 3^{1/4}. \qquad \text{... (5.29)}$$

Using (5.29) in (5.28), we deduce that

$$P^3 + 3^{3/4} P^2 + \sqrt{3}\, P - 3^{1/4} = 0. \qquad \text{... (5.30)}$$

Putting $P = 3^{-1/4} T$ in (5.30), we find that

$$T^3 + 3T^2 + 3T - 3 = 0. \qquad \text{... (5.31)}$$

Putting $x = T + 1$ in (5.31), we obtain

$$x = \sqrt[3]{4}. \qquad \text{... (5.32)}$$

Using (5.32), we obtain the required result.

Proof of (5.11) : Putting $q = e^{(-\pi/3)}$ in (5.28), we find that

$$P = \frac{\psi(-e^{-\pi/3})}{e^{(-\pi/12)}\,\psi(-e^{-\pi})} \qquad \text{... (5.33)}$$

and

$$Q = \frac{\psi(-e^{-\pi})}{e^{(-\pi/4)}\,\psi(-e^{-3\pi})} \qquad \text{... (5.34)}$$

Putting $\alpha = \pi/3$ and $\beta = 3\pi$ in (3.1), we deduce that

$$\frac{\psi(-e^{-\pi/3})}{e^{(-\pi/4)}\,\psi(-e^{-3\pi})} = \sqrt{3}. \qquad \text{... (5.35)}$$

Using (5.35), we find that

$$PQ = \sqrt{3}. \qquad \text{... (5.36)}$$

Using (5.36) in (5.28), we obtain (5.11).

As the pattern of proof of the following theorem is identical with the proof of Theorem 5.4, we skip the proof.

***Theorem* 5.5** — *We have*

$$(i)\ \frac{\varphi(-e^{-\pi/\sqrt{5}})}{\varphi(e^{-\sqrt{5}\,\pi})} = 5^{1/4}$$

$$(ii)\ \frac{\varphi(-e^{-3\pi/\sqrt{5}})}{\varphi(e^{-3\sqrt{5}\,\pi})} = \sqrt{\frac{10 + 5\sqrt{3} + 4\sqrt{5} + 2\sqrt{15}}{8 + 5\sqrt{3} + 4\sqrt{5} + 2\sqrt{15}}}$$

$$(iii)\ \frac{\varphi(-e^{-\pi/3\sqrt{5}})}{\varphi(e^{-3\pi/\sqrt{5}})} = \frac{9 - 3\sqrt{3} + 3\sqrt{5} - \sqrt{15}}{5 - 3\sqrt{3} + 3\sqrt{5} - \sqrt{15}},$$

$$(iv)\ \frac{\varphi(-e^{-\pi/3\sqrt{3}})}{\varphi(e^{\pi/\sqrt{3}})} = \sqrt[4]{\frac{27+(\sqrt[3]{4}-1)^4}{3+(\sqrt[3]{4}-1)^4}},$$

$$(v)\ \frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[3]{6\sqrt{3}-9},$$

$$(vi)\ \frac{\varphi(-e^{-\pi/3\sqrt{5}})}{\varphi(e^{-\sqrt{5}\pi/3})} = \sqrt{\frac{10-5\sqrt{3}+4\sqrt{5}-2\sqrt{15}}{8-5\sqrt{3}+4\sqrt{5}-2\sqrt{15}}},$$

$$(vii)\ \frac{\varphi(e^{-\sqrt{5}\pi/3})}{\varphi(e^{-3/\sqrt{5}\pi})} = \frac{9+3\sqrt{3}+3\sqrt{5}+\sqrt{15}}{5+3\sqrt{3}+3\sqrt{5}+\sqrt{15}},$$

$$(viii)\ \frac{\varphi(e^{-\pi/\sqrt{3}})}{\varphi(e^{-\sqrt{3}\pi})} = 3^{1/4},$$

$$(ix)\ \frac{\varphi(e^{-\pi\sqrt{3}})}{\varphi(e^{-3\sqrt{3}\pi})} = \sqrt[4]{\frac{27+9(\sqrt[3]{4}-1)^4}{27+(\sqrt[3]{4}-1)^4}},$$

$$(x)\ \frac{\varphi(e^{-\pi/3})}{\varphi(e^{-3\pi})} = \sqrt{3},$$

$$(xi)\ \frac{\varphi(e^{-\pi/3})}{\varphi(e^{-\pi})} = \sqrt[4]{3+2\sqrt{3}},$$

$$(xii)\ \frac{\varphi(e^{-\pi/\sqrt{3}})}{\varphi(e^{-3\sqrt{3}\pi})} = \frac{3\sqrt[3]{4}}{2+\sqrt[3]{4}},$$

and

$$(xiii)\ \frac{\varphi(e^{-\pi/3\sqrt{3}})}{\varphi(e^{-\sqrt{3}\pi})} = 4^{-1/3}(2+\sqrt[3]{4}).$$

Using (2.7) and (2.8) in Theorem 5.4, we obtain the following values of the cubic continued fraction of Ramanujan.

***Theorem* 5.6** — *We have*

$$(i)\ V(-e^{-\pi\sqrt{5}}) = \frac{(3-\sqrt{5})(\sqrt{3}-\sqrt{5})}{4},$$

$$(ii)\ V(-e^{-\pi/\sqrt{5}}) = \frac{(\sqrt{3}+\sqrt{5})(\sqrt{5}-\sqrt{3})}{4},$$

$$(iii)\ V(-e^{-\pi/\sqrt{3}}) = \frac{-1}{\sqrt[3]{4}},$$

$$(iv)\ V(-e^{-\pi\sqrt{3}}) = \frac{1-\sqrt[3]{4}}{2+\sqrt[3]{4}},$$

$$(v)\ V(-e^{-\pi/3\sqrt{3}}) = \frac{-1}{\sqrt[3]{3^{-1}(\sqrt[3]{4}-1)^4+1}},$$

$$(vi)\ V(-e^{-\pi}) = \frac{1-\sqrt{3}}{2},$$

and

$$(vii)\ V(-e^{-\pi/3}) = \frac{-1}{\sqrt[3]{2(\sqrt{3}-1)}}.$$

Remark: A different proof of Theorem 5.6 (*i*) can be found in [12] and Theorem 5.6 (*iii*) was first proved by Adiga *et al.*[2]. Other values of V in Theorem 5.6 appear to be new to literature.

Open Problem: Using (2.10) one can express $V(q3)$ in terms of $V(q)$ which gives a triplication formula for $V(q)$. Triplication formula is important in the development of elliptic functions to alternative bases. The only triplication formula known so far is that of the Borweins. Can one develop a cubic theory associated with $V(q)$ similar to that of Borweins?

ACKNOWLEDGEMENT

Authors are thankful to the referee for his useful suggestions and comments. This work was supported by Korea Research Foundation Grant (KRF-2002-050-C00001).

REFERENCES

1. C. Adiga, K. R. Vasuki and M. S. Mahadeva Naika, On some new identities involving integrals of theta-functions, *Advanced Studies in Contemporary Mathematics*, **3**(2) (2001), 1-11.
2. C. Adiga, K. R. Vasuki and M. S. Mahadeva Naika, Some new explicit evaluations of Ramanujans cubic continued fraction, *New Zealand J. Math.*, **31** (2002), 109-14.
3. G. E. Andrews, *Ramanujan's "lost" notebook, III.* The Rogers-Ramanujan continued fraction, *Adv. Math.*, **41** (1981), 186-208.
4. B. C. Berndt, *Ramanujan's notebooks, part III*, Springer-Verlag, New York, 1991.
5. B. C. Berndt, *Ramanujan's notebooks, part IV*, Springer-Verlag, New York, 1994.
6. B. C. Berndt, *Ramanujan's notebooks, Part V*, Springer-Verlag, New York, 1998.
7. B. C. Berndt, H. H. Chan and L. -C. Zhang, *Ramanujan's class invariants and cubic continued-fraction*, *Acta Arith.*, **73** (1995), 67-85.
8. S. Bhargava, *On Rogers-Ramanujan identities*, in Srinivasa Ramanujan (1887-1920), Mac-Millan India Ltd., Madras, 1988, 75-81.
9. S. Bhargava and C. Adiga, On some continued fraction identities of Srinivasa Ramanujan, *Proc. Amer. Math. Soc.*, **92** (1984), 13-18.
10. S. Bhargava and C. Adiga, *Two generalizations of Ramanujan's continued fraction identities*, in Number Theory, K. Alladi, ed., Lecture Notes in Math. No. 1122, Springer-Verlag, Berlin, 1985, 56-62.
11. J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.
12. H. H. Chan, On Ramanujan's cubic continued fraction, *Acta Arith.*, **73** (1995), 343-55.
13. R. Y. Denis, On certain q-series and continued fractions, *Math. Student*, **44** (1983), 70-76.

14. R. Y. Denis, On basic hypergeometric functions and continued fractions. *Math. Student*, **52** (1984), 129-36.
15. R. Y. Denis, On certain summation of q-series and identities of Rogers-Ramanujan type, *J. Math. Phys. Sci.*, **22** (1988), 87-99.
16. K. Ramachandra, Some applications of Kronecker's limit formulas, *Ann. Math.*, **80** (1964), 104-148.
17. K. G. Ramanathan, On Ramanujan's continued fraction, *Acta Arith.*, **43** (1984), 209-226.
18. S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
19. S. Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi, 1988.
20. Seung Hwan Son, Some integrals of theta-functions in Ramanujan's 'lost' notebook, CRM Proceedings and Lecture Notes, **19** (1999), 323-32.
21. G. N. Watson, Theorems stated by Ramanujan (ix): two continued fractions, *J. Lond. Math. Soc.*, **4** (1929), 231-37.