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SUBGRAPHS OF MAXIMUM MATCHING GRAPHS*

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The maximum matching graph of a graph has a vertex for each maximum matching and an edge for each pair of maximum matchings which differ by exactly one edge. In this paper, we study the common neighbour subgraph (i.e., the subgraph induced by common neighbour vertices of two vertices with distance two) and the neighbourhood subgraph (i.e., the subgraph induced by neighbour vertices of a vertex) of maximum matching graphs.

Key Words: Maximum Matching Graph; Common Neighbour Subgraph; Neighbourhood Subgraph

1. INTRODUCTION

The reader is referred to 1 and 2 for undefined terms and concepts. We shall consider finite, undirected and simple graphs only. Let G be a graph, M_1 and M_2 two maximum matchings of G. Then the symmetric difference of M_1 and M_2, denoted by $M_1 \oplus M_2$, consists of mutually disjoint cycles and paths. The number of cycles in $M_1 \oplus M_2$ is denoted by $r(M_1 \oplus M_2)$. The maximum matching graph $\mathcal{M}(G)$ of G is a graph whose vertices are the maximum matchings of G and M_1 is adjacent to M_2 in $\mathcal{M}(G)$ if and only if $|M_1 - M_2| = 1$; that is $M_1 \oplus M_2$ induces a path of length 2.

The distance between two vertices υ_1 and υ_2 of G is the length of the shortest path joining υ_1 and υ_2, denoted by $d_G(\upsilon_1, \upsilon_2)$. For $X \subseteq V(G)$ and $Z \subseteq E(G)$, the subgraph of G induced by X is denoted by $\langle X \rangle$, the subgraph of G induced by Z is denoted by $G[Z]$. The union of two graphs G_1 and G_2 is a graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The union of m copies of G is denoted as mG.

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Studies on maximum matching graph were initiated by the present authors[6] and Eroh[7] independently.

In combinatorial computation and optimization, we often meet a problem of how to generate all forms of a structure or design, or to find a best one if these forms are weighted. Hence, many transformation graphs are introduced[8]. As a transformation graph, the motivation for working on the maximum matching graph is to know the relation between properties of all maximum matchings of a graph G. On the other hand, it is well known that all matchings of G form an independence system (not necessarily a matroid). Similar to the basis graph of a matroid (see [4], [5]), it would be significant to investigate the adjacency relation among all maximum independent sets for this special independence system. In this paper, we will present some results about the subgraphs of maximum matching graphs.

2. MAIN RESULTS

In this section, we study the common neighbour subgraph and the neighbourhood subgraph of maximum matching graphs.

Definition 1[5] — In a given graph G, suppose that $d_G(v, v') = 2$ and V_1 consists of all vertices adjacent to both v and v'. Then V_1 is called the neighbour intersection, denoted by $CN(v, v')$ and $\langle V_1 \rangle$ is called the common neighbour subgraph about v and v'.

Definition 2[5] — In a given graph G, v is a vertex of G. Suppose that V_2 consists of all vertices adjacent to v. Then $\langle V_2 \rangle$ is called the neighbourhood subgraph about v.

The following lemmas are useful.

Lemma 2.1[3] — Let G be a graph, M_1 and M_2 two maximum matchings of G. Then M_1 and M_2 are adjacent in $\mathcal{M}(G)$ if and only if M_2 is obtained from M_1 by the following pivot:

$$M_2 = M_1 - uv_1 + uv_2,$$

where $uv_1 \in M_1, uv_2 \in M_2$ and v_1 and v_2 are missed by M_2 and M_1, respectively.

Lemma 2.2 — Let G be a graph, M_1 and M_2 two maximum matchings of G. If $d_{\mathcal{M}(G)}(M_1, M_2) = 2$, then $G[M_1 \oplus M_2]$ has no cycles, and either $G[M_1 \oplus M_2]$ is a path P with length 4 or $G[M_1 \oplus M_2]$ consists of two paths with length 2.

PROOF : Suppose, to the contrary, that $G[M_1 \oplus M_2]$ has a cycle C. Since C is even, the length of C is at least 4. Hence

$$\left| M_1 - M_2 \right| \geq \frac{|E(C)|}{2} \geq 2.$$

By the definition of maximum matching graph, clearly $|M_1 - M_2| \le d_{\mathcal{M}(G)}(M_1, M_2) = 2$.

It follows that $|M_1 - M_2| = 2$ and the length of C is 4. Let $C = v_1 v_2 v_3 v_4 v_5$, where $v_1 v_2, v_3 v_4 \in M_1$ and $v_2 v_3, v_1 v_4 \in M_2$. Suppose that $M_1 M M_2$ is a shortest path joining M_1 and M_2 in $\mathcal{M}(G)$. Then $|M_1 - M| = 1$ and $|M - M_2| = 1$. By Lemma 2.1, we can assume that $M = M_1 - uv + uw$, where v and w are missed by M and M_1 respectively. Then $uw \ne v_2 v_3, v_1 v_4$ since every v_i is covered by M_1. Hence $v_2 v_3, v_1 v_4 \notin M$. It follows that $|M_1 - M_2| \ge 2$, which contradicts with $|M - M_2| = 1$. Then $G[M_1 \oplus M_2]$ has no cycles. Hence every component of $G[M_1 \oplus M_2]$ is a path with even length. Since $|M_1 - M_2| = 2$, $G[M_1 \oplus M_2]$ either is a path with length 4 or consists of two paths with length 2. ■

Theorem 2.3 — *Let $\mathcal{M}(G)$ be the maximum matching graph of a graph G. Then each common neighbour subgraph of $\mathcal{M}(G)$ is K_1 or $2K_1$.*

PROOF : Let M_1 and M_2 be two maximum matchings of G such that $d_{\mathcal{M}(G)}(M_1, M_2) = 2$. By Lemma 2.2, $G[M_1 \oplus M_2]$ either is a path P with length 4 or $G[M_1 \oplus M_2]$ consists of two paths with length 2. Let $CN(M_1, M_2)$ be the neighbour intersection of M_1 and M_2.

Case 1 : $G[M_1 \oplus M_2]$ is a path P with length 4. Let $P = v_1 v_2, v_3 v_4, v_5$, where $v_1 v_2, v_3 v_4 \in M_1$ and $v_2 v_3, v_4 v_5 \in M_2$. Then v_1 is missed by M_2 and v_5 is missed by M_1. Let

$$M = M_1 - v_4 v_3 + v_4 v_5.$$

Then

$$M = M_2 - v_2 v_3 + v_2 v_1.$$

Hence $M \in CN(M_1, M_2)$. If M' is another vertex in $CN(M_1, M_2)$, $|M' - M_1| = 1$ and $|M' - M_2| = 1$.

Let

$$M' = M_1 - uv + uw,$$

where w is missed by M_1.

Since $M' \neq M$, $uw \neq v_4v_5$. Since v_2 and v_3 are covered by M_1, $uw \neq v_2v_3$. So $v_4v_5, v_2v_3 \in M_2 \setminus M'$, which contradicts with $|M' - M_2| = 1$. Therefore $|CN(M_1 - M_2)| = 1$. It follows that the common neighbour subgraph of $\mathcal{M}(G)$ is K_1.

Case 2 : $G[M_1 \oplus M_2]$ is disconnected.

In this case, $G[M_1 \oplus M_2]$ has two components which are paths with length 2, say P_1 and P_2.

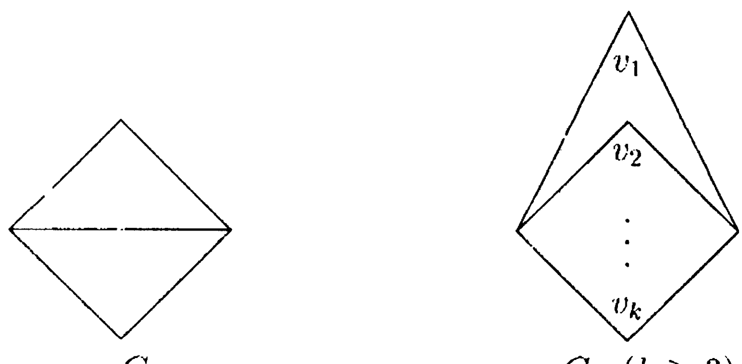
Let

$$M = M_1 \oplus P_1 \text{ and } M' = M_1 \oplus P_2.$$

Then $M, M' \in CN(M_1, M_2)$ and $|M' - M_1| = 2$. Hence M and M' are not adjacent in $\mathcal{M}(G)$. Similar to the proof of Case 1, $CN(M_1 - M_2)$ has no other vertices except M, M'. So the common neighbour subgraph is $2K_1$. The proof is completed. ■

***Theorem* 2.4** — *Let $\mathcal{M}(G)$ be the maximum matching graph of a graph G. Then each component of any neighbourhood subgraph of $\mathcal{M}(G)$ is a complete graph K_m with $m \geq 1$.*

PROOF : Let M be a maximum matching of G and $\mathcal{H}$ a component of the neighbourhood subgraph about M of $\mathcal{M}(G)$. Suppose that $\mathcal{H}$ has at least three vertices without loss of generality. Let $M_1, M_2, M_3 \in V(\mathcal{H})$ and $M_1M_2, M_2M_3 \in E(\mathcal{H})$. Then we will show that $M_1M_3 \in E(\mathcal{H})$. Let $M_i = M - e_i + e_i'$ for $1 \leq i \leq 3$. Clearly $e_i' \neq e_j'$ for $1 \leq i < j \leq 3$. (Otherwise, $M_i = M_j$). This will imply that $e_1 = e_2 = e_3$. Suppose that $e_1 \neq e_2$. Then $e_2, e_1' \in M_1 \setminus M_2$, this contradicts with $M_1M_2 \in E(\mathcal{H})$.



G_1 $G_2, (k \geq 3)$

By the same reason, $e_2 = e_3$. Hence $M_1M_3 \in E(\mathcal{M}(G))$. The proof is completed. ■

The following result is an immediate consequence of Theorem 2.3 and Theorem 2.4.

Corollary 2.5 — No graph containing an induced subgraph isomorphic to any of the graphs G_1 and G_2 of Fig. 1 is a maximum matching graph.

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