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EFFECT OF TRANSVERSE SHEAR AND ROTATORY INERTIA ON THE FREE AXISYMMETRIC VIBRATION OF A STEPPED CIRCULAR PLATE

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Free axisymmetric vibration of a ciruclar plate whose thickness, density and elastic properties vary in any number of steps along the radial direction, is analyzed using shear theory of plates. Numerical results for natural frequencies and normalized mode shapes computed for first four normal modes of vibration for clamped, simply-supported and free boundary conditions are compared with those of classical theory.

Key Words: Shear Theory of Plates; Stepped Variation in Thickness; Natural Frequency

1. INTRODUCTION

In the theory concerned with the behaviour of plate elements, the Poisson-Kirchoff classical equation[1,2] formed the basis for a long time. In 1951, Mindlin[3] presented an improved theory of plate by including the effects of transverse shear and rotatory inertia on the vibration of an isotropic plate. The basic equations of the theory, which are analogous to the Timoshenko equation of beam[4], were found extremely useful in predicting the high-frequency flexural vibrations of a circular disc used as a component in many engineering structures. Wittrick[5] provided the value for the shear deformation factor for a class of eigenvalue problems, in addition to the fundamental contribution by Mindlin.

Many practical structures exhibit geometric steps. On several occasions fabriacation and assembly considerations dictate such necessities. Attempts are continuously made to study the vibration characteristics of structures with such specified geometries. Gallego-Juarez[6] has analyzed the axisymmetric vibration of circular plates of single circular step using the classical uniform plate solution to each zone of different thickness. SanEmeterio *et al.*[7] have developed an approximate procedure, based on the Rayleigh method, for the study of an axisymmetric vibration mode of circular plate having any number of stepped variation in thickness. But they have imposed a restriction that the number of steps has to be equal to the number of nodal circles. Gutierrez and Laura[8] have given approximate approaches for stepped plate applied only to the first mode of vibration. Natural frequencies corresponding to axisymmetric and antisymmetric modes of vibration of a circular plate with stepped thickness over a concentric circular region have been studied by Avalos *et al.*[9] using the optimized Rayleigh-Ritz approach. Gu and Wang[10] have used differential quadrature element method for the free vibration analysis of circular plate with stepped thickness over a concentric region. Avalos *et al.*[11] have also analyzed the annular plates of stepped thickness.

In the present paper, the effect of transverse shear and rotatory inertia on the free axisymmetric vibration of a stepped circular plate is analyzed. The plate is assumed to be made up of n concentric circular plate elements joined end to end. The plate elements are having, in general, different constant thicknesses, densities and elastic properties. The arbitrary constants arising in the solution are solved by boundary and continuity conditions. Numerical results for natural frequencies and normalized mode shapes for first four normal modes, computed for a plate made up of three plate elements, are compared with those of classical theory[12] for various types of boundary conditions. The variations in radii, thicknesses and densities of the plate elements are taken in such a way that radius, average thickness and average density of the plate remain constant.

2. Equations of Motion

An isotropic circular plate of radius a, whose thickness, density and elastic properties along the radial direction vary in steps is considered using classical theory of plates. The plate is referred to

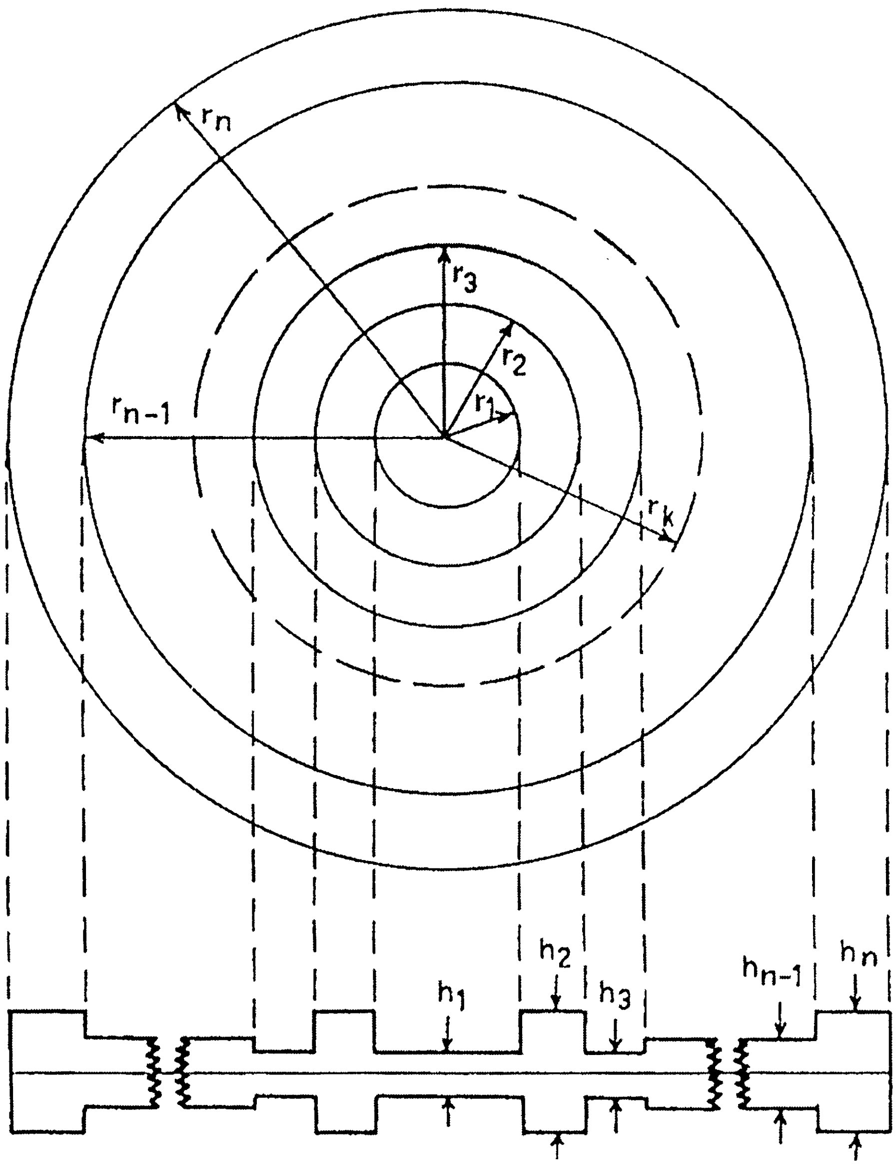

FIG. 1. Thickness profile of the plate

Cylindrical co-ordinates by taking origin at centre, z-axis along the thickness and r-axis along the radius of the plate. The middle plane and the circular boundary of the plate are $z = 0$ and $r = a$ respectively. The plate is assumed to be made up of n concentric plate elements joined end to end with their middle planes lying in the plane $z = 0$. The innermost plate element is a solid circular plate and the other are annular plates. Their Young's moduli, densitites, Poisson ratio, constant thicknesses, inner radii and outer radii are taken as $E_k, \rho_k, \nu_k, h_k, r_{k-1}$ and r_k ; $k = 1, 2, ..., n$; where $r_o = 0$ and $r_n = a$ and $r_k - r_{k-1} = a_k$. A thickness profile of the plate is shown in Fig. 1.

The equations of motion of kth plate element of constant thickness according to shear theory of plates are

$$\left\{\begin{array}{l} \dfrac{E_k h_k^3}{12(1-\nu_k^2)}\left[\psi_{k,rr} + \dfrac{1}{r}\psi_{k,r} - \dfrac{1}{r^2}\psi_k\right] - \dfrac{E_k h_k k_s}{2(1+\nu_k)}(\psi_k + w_{k,r}) - \dfrac{\rho_k h_k^3}{12}\psi_{k,tt} = 0 \\ \dfrac{E_k k_s h_k}{2(1+\nu_k)}\left[\psi_{k,r} + \dfrac{1}{r}\psi_k + w_{k,rr} + \dfrac{1}{r}w_{k,r}\right] - \rho_k h_k \psi_{k,rr} = 0, \\ r_{k-1} \le r \le r_k ; \quad k = 1, 2, ..., n. \end{array}\right\} \quad ... (1)$$

where w_k, ψ_k, t and k_s are the transverse deflections of the kth plate element, rotations of the normal to the middle plane of the kth plate element, time and shear constant. A comma followed by variable suffix denotes differentiation with respect to that variable. Making eqs. (1) non-dimensional, one gets

$$\left\{\begin{array}{l} L_k\left[\overline{\psi}_{k,RR} + \dfrac{1}{R}\overline{\psi}_{k,R} - \dfrac{1}{R^2}\overline{\psi}_k\right] - N_k\left[\overline{\psi}_k + W_{k,R}\right] - \dfrac{\gamma_k H_k^3}{12}\overline{\psi}_{k,TT} = 0, \\ N_k\left[\overline{\psi}_{k,R} + \dfrac{1}{R}\overline{\psi}_k + W_{k,RR} + \dfrac{1}{R}W_{k,R}\right] - \gamma_k H_k W_{k,TT} = 0, \\ R_{k-1} \le R \le R_k ; \quad k = 1, 2, ..., n, \end{array}\right\} \quad ... (2)$$

where

$$W_k = w_k/a, \; R = r/a, \; H_k = h_k/a, \; R_k = r_k/a, \; e_k = E_k/E,$$

$$\gamma_k = \rho_k/\rho_a, \; L_k = e_k H_k^3/12\left(1 - \nu_k^2\right),$$

$$T = t\sqrt{E/a^2\rho_a}, \; N_k = k_s e_k H_k/2(1+\nu_k).$$

ρ_a is the average density of the plate and E is the Young's modulus of some standard material.

3. Solution

For harmonic solution, $\overline{W}_k(R,T)$ and $\overline{\psi}_k(R,T)$ are taken as

$$\left\{\begin{array}{l} \overline{W}_k(R,T) = W_k(R)\, e^{i\Omega T} \\ \text{and} \\ \overline{\psi}_k(R,T) = \psi_{k,R}(R)\, e^{i\Omega T} \end{array}\right\}, \quad \dots (3)$$

where Ω is the circular frequency of vibration.

Substitution of solutions (3) in eq. (2) gives

$$H_k^2\left[\psi_{k,RRR} + \frac{1}{R}\psi_{k,RR} - \frac{1}{R^2}\psi_{k,R}\right] - 6K_k\left[\psi_{k,R} + W_{k,r}\right] + \omega_k^2 H_k^2 \psi_{k,R} = 0, \quad \dots (4)$$

$$K_k\left[\psi_{k,RR} + \frac{1}{R}\psi_{k,RR} + \frac{1}{R}W_{k,R}\right] + 2\omega_k^2 W_k = 0, \quad \dots (5)$$

where

$$K_k = k_s(1 - \nu_k) \text{ and } \omega_k^2 = \frac{\gamma_k \Omega^2 (1 - \nu_k^2)}{e_k}.$$

Integration of eq. (4) w.r.t. R and taking $\nabla^2 \equiv \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}$, we get

$$\left\{\begin{array}{l} H_k^2\left(\nabla^2 + \omega_k^2\right)\psi_k - 6k_s(\psi_k + W_k) = 0 \\ K_k \nabla^2(\psi_k + W_k) + 2\omega_k^2 W_k = 0 \end{array}\right\}. \quad \dots (6)$$

Eliminating ψ_k from the above equations, we get

$$\left[H_k^2 K_k \nabla^4 + H_k \omega_k^2 (2 + K_k)\nabla^2 + 2\omega_k^2\left(H_k^2 \omega_k^2 - 6K_k\right)\right] W_k = 0. \quad \dots (7)$$

A solution of the Bessel equation $\left(\nabla^2 + \lambda^2\right)W_k = 0$ will also be a solution of the above equations provided

$$H_k^2 K_k \lambda^4 + H_k^2 \omega_k^2 (2 + K_k)\lambda^2 + 2\omega_k^2\left(H_k^2 \omega_k^2 - 6K_k\right) = 0.$$

If the roots of this equation are $\pm\lambda_k$ and $\pm i\mu_k$, where

$$\begin{matrix} +\lambda_k^2 \\ -\mu_k^2 \end{matrix} = \omega_k^2 \left[\frac{2+K_k}{2K_k} \pm \sqrt{\left(\frac{2+K_k}{2K_k}\right)^2 + \frac{2}{K_k}\left(\frac{6K_k}{H_k^2\,\omega_k^2 - 1}\right)} \right],$$

the solutions for W_k and ψ_k can be taken as

$$W_k(R) = G_k(R)\,D_k, \qquad \text{... (8)}$$

$$\psi_k(R) = G_k(R)\,C_k, \qquad \text{... (9)}$$

where

$$G_1(R) = [J_o(\lambda_1 R) \quad I_o(\mu_1 R)]$$

$$D_1 = [d_{11} \quad d_{21}]', \;\; C_1 = [c_{11} \quad c_{21}]'$$

$$G_\ell(R) = [J_o(\lambda_\ell R) \;\; I_o(\mu_\ell R) \;\; Y_o(R_\ell R) \;\; K_o(\mu_\ell R)]$$

$$D_\ell = [d_{1\ell}\; d_{2\ell} \;\; d_{3\ell}\; d_{4\ell}]', \;\; C_\ell = [c_{1\ell} \;\; c_{2\ell} \;\; c_{3\ell} \;\; c_{4\ell}]'$$

$$\ell = k+1; \;\; k = 1, 2, ..., n-1,$$

and

J_o, I_o, Y_o and K_o are Bessel's functions of zero order.

Substitution of above solutions in eq. (5) yields

$$c_{1k} = \frac{b_k\,\omega_k^2 - \lambda_k^2}{\lambda_k^2}\,d_{1k}, \quad c_{2k} = \frac{b_k\,\omega_k^2 + \mu_k^2}{\mu_k^2}\,d_{2k},$$

$$c_{3k} = \frac{b_k\,\omega_k^2 - \lambda_k^2}{\lambda_k^2}\,d_{3k}, \quad c_{4k} = \frac{b_k\,\omega_k^2 + \mu_k^2}{\mu_k^2}\,d_{4k},$$

where

$$b_k = 2/K_k, \;\; d_{31} = d_{41} = 0.$$

Thus,

$$\psi_{k,R}(R) = S_k(R)\,D_k, \qquad \text{... (10)}$$

where

$$S_k(R) = [f_{1k}\,J_1(\lambda_k R) - f_{2k}\,I_1(\mu_k R) \;\; f_{1k}\,Y_1(\lambda_k R) \;\; f_{2k}\,K_1(\mu_k R)],$$

$$f_{1k} = (\lambda_k^2 - b_k\, \omega_k^2)/\lambda_k,\; f_{2k} = (\mu_k^2 + b_k\, \omega_k^2)/\mu_k$$

and

J_1, I_1, Y_1 and K_1 are the Bessel's functions of first order.

3.1 Continuity Conditions

The continuity conditions between the plate elements at $R = R_k$; $k = 1, 2, \ldots, n-1$ can be taken as

$$\left\{\begin{array}{l} W_\ell(R_k) = W_k(R_k), \\ \psi_{\ell,R}(R_k) = \psi_{k,R}(R_k), \\ L_\ell\left[S_{\ell,R}(R_k) + \dfrac{\nu_\ell}{R_k} S_\ell(R_k)\right] = D_\ell = L_k\left[S_{k,R}(R_k) + \dfrac{\nu_k}{R_k} S_k(R_k)\right] D_k, \\ N_\ell\left[S_\ell(R_k) + G_{\ell,R}(R_k)\right] D_\ell = N_k\left[S_k(R_k) + G_{k,R}(R_k)\right] D_k, \end{array}\right\} \quad \ldots (11)$$

where

$$\ell = k + 1.$$

From eqs. (8), (10) and (11), one gets

$$D_\ell = B^{(\ell)} D_k,\;\; B^{(\ell)} = A_\ell^{-1}(R_k)\, A_k(R_k), \quad \ldots (12)$$

where

$$A_1(R_1) = \left\{\begin{array}{ll} J_o(\lambda_1 R_1) & I_o(\mu_1 R_1) \\ f_{11} J_1(\lambda_1 R_1) & -f_{21} I_1(\mu_1 R_1) \\ L_1 f_{11}\left\{\lambda_1 J_o(\lambda_1 R_1) - \dfrac{(1-\nu_1)}{R_1} J_1(\lambda_1 R_1)\right\} & -L_1 f_{21}\left\{\mu_1 I_o(\mu_1 R_1) - \dfrac{(1-\nu_1)}{R_1} I_1(\mu_1 R_1)\right\} \\ N_1(f_{11} - \lambda_1) J_1(\lambda_1 R_1) & -N_1(f_{21} - \mu_1) I_1(\mu_1 R_1) \end{array}\right\} \quad \ldots (13)$$

$$A_\ell(R_k) = \left\{ \begin{array}{llll} J_o(\lambda_\ell R_k) & I_o(\mu_\ell R_k) & Y_o(\lambda_\ell R_k) & K_o(\mu_\ell R_k) \\ f_{1\ell} J_1(\lambda_\ell R_k) & -f_{2\ell} I_1(\mu_\ell R_k) & f_{1\ell} Y_1(\lambda_\ell R_k) & f_{2\ell} K_1(\mu_\ell R_k) \\ L_\ell f_{1\ell}(\lambda_\ell J_o(\lambda_\ell R_k) & -L_\ell f_{2\ell}(\mu_\ell I_o(\mu_\ell R_k) & L_\ell f_{1\ell}(\lambda_\ell Y_o(\lambda_\ell R_k) & -L_\ell f_{2\ell}(\mu_\ell K_o(\mu_\ell R_k) \\ -\dfrac{(1-\nu_\ell)}{R_k} J_1(\lambda_\ell(R_k)) & -\dfrac{(1-\nu_\ell)}{R_k} I_1(\mu_\ell R_k)) & -\dfrac{(1-\nu_\ell)}{R_k} Y_1(\lambda_\ell R_k)) & +\dfrac{(1-\nu_\ell)}{R_k} K_1(\mu_\ell R_k)) \\ N_\ell(f_{1\ell}-\lambda_\ell) J_1(\lambda_\ell R_k) & -N_\ell(f_{2\ell}-\mu_\ell) I_1(\mu_\ell R_k) & N_\ell(f_{1\ell}-\lambda_\ell) Y_i(\lambda_1 R_k) & N_\ell(f_{2\ell}-\mu_\ell) K_1(\mu_\ell R_k) \end{array} \right\} \quad \text{... (14)}$$

Matrices $A_k(R_k)$ can be obtained by replacing ℓ by k in eq. (14).

From eq. (12), we have

$$D_\ell = V^{(\ell)} D_1, \; V^{(\ell)} = B^{(\ell)} B^{(\ell-1)} \ldots B^{(2)} = \left[v_{pq}^{(\ell)} \right]_{4\times 2} = [V_1 \; V_2] \text{ (say).} \quad \text{... (15)}$$

In this way the $4n - 2$ constants arising in solutions (8) and (10) are reduced to 2.

3.2 Boundary Conditions

The plate is subjected to following types of boundary conditions:

3.2.1 Clamped plate (C-plate)

For a plate clamped at the boundary, we have

$$W_n = \psi_{n,R} = 0 \quad \text{at} \quad R = 1. \quad \text{... (16)}$$

3.2.2 Simply-supported plate (S-plate)

For a plate simply supported at the boundary, we have

$$W_n = L_n \left\{ \psi_{n,RR} + \frac{\nu_n}{R} \psi_{n,R} \right\} = 0 \quad \text{at} \quad R = 1. \quad \text{... (17)}$$

3.2.3 Free plate (F-plate)

For a plate free at the boundary, we have

$$L_n = L_n \left\{ \psi_{n,RR} + \frac{\nu_n}{R} \psi_{n,R} \right\} = N_n \left\{ \psi_{n,R} + W_{n,R} \right\} = 0 \quad \text{at} \quad R = 1. \quad \text{... (18)}$$

Using relation (15) in solutions (8) and (10) and then putting them in condition (16) to (18) as the case may be, we get

$$\left\{\begin{array}{l} s_{11}\,d_{11} + s_{12}\,d_{21} = 0 \\ s_{21}\,d_{11} + s_{22}\,d_{21} = 0 \end{array}\right\}, \qquad \text{... (19)}$$

where

$$s_{1i} = \left\{\begin{array}{ll} G_n(1)\,V_i & \text{for } C\text{–plate and } S\text{–plate} \\ [S_{n,R}(1) + \nu_n S_n(1)]\,V_i & \text{for } F\text{–plate} \end{array}\right\}$$

and

$$s_{2i} = \left\{\begin{array}{ll} S_{n,R}(1)\,V_i & \text{for } C\text{–plate} \\ [S_{n,R}(1) + \nu_n S_n(1)]\,V_i & \text{for } S\text{–plate} \\ [S_n(1) + G_{n,R}(1)]\,V_i & \text{for } F\text{–plate.} \end{array}\right\}$$

3.3 Frequency Equation

For non-trivial solution of homogeneous eq. (19),

$$s_{11}\,s_{22} - s_{12}\,s_{21} = 0. \qquad \text{... (20)}$$

The countably infinite roots of eq. (20) are the natural frequencies Ω for various modes of vibration.

4. Results and Discussion

The variations in radii, thicknesses and densities of the plate elements are taken in such a way that radius, average thickness and average density of the whole plate remain constant. For that

$$\alpha_k = a_k/a_1,\ \ \beta_k = h_k/h_1,\ \ \delta_k = \rho_k/\rho_1.$$

If h_a is the average thickness of the plate and $H_a = h_a/a$, then

$$H_k = \frac{\beta_k H_a}{\sum_{i=1}^{n} \beta_i (R_i^2 - R_{i-1}^2)},\quad \gamma_k = \frac{\delta_k H_a}{\sum_{i=1}^{n} \beta_i\,\delta_i (R_i^2 - R_{i-1}^2)}$$

and

$$R_k = \frac{\sum_{i=1}^{k} \alpha_i}{\sum_{i=1}^{n} \alpha_i}.$$

Numerical results for frequencies and normalized mode shapes for first four normal modes of vibration for a plate made up of three plate elements whose first and third elements are identical, i.e., for $\alpha_3 = \beta_3 = \delta_3 = 1.0$ by taking $v_1 = v_2 = v_3 = 0.3$, $e_2 = e_3 = 1.0$, $k_s = \pi^2/12$ and $H_a = 0.15$ are plotted in graphs.

Ω vs β_2 for C-plates for first four modes of vibration are plotted in Fig. 2 for various values of α_2, while Ω vs δ_2 are plotted in Fig. 3 for various values of β_2 for shear theory as well as classical theory. Same type of graphs are plotted for S-plates and F-plates in Figs. 4-6 respectively. It is clear in all these figures that frequencies are lower for the case of shear theory. The difference in frequencies increases as one goes to higher modes.

Normalized mode shapes for first four normal modes for shear theory as well as for classical theory are plotted in Figs. 8, 9 and 10 for C-plates, S-plates and F-plates reqpectively. Figure 8 shows that in first mode of vibration the deflections at the peaks are slightly higher in shear theory. This difference is more visible when the thickness and density of the middle plate element are higher than other two elements (Fig. 8(b) and 8(d)). In second, third and fourth modes, deflection at the peaks is higher in classical theory except in second mode of Fig. 8(b).

Figure 9 shows a very small difference in deflections of classical and shear theories for first mode of vibration. This difference is slightly visible when we change the densities of the middle plate element. In second mode, deflections at the peaks are higher in classical theory except when the density of the middle element is higher than other two elements (Fig. 9(d)). In third and fourth mode deflection at the peaks are higher in case of classical theory except when the thickness of the middle element is higher than other two elements (Fig. 9(b)).

Figure 10 shows that in first mode the deflection are higher in shear theory except when $\delta_2 = 1.6$ (Fig. 10(d)), while in second mode deflection at peaks is higher in classical theory except when $\delta_2 = 1.6$. In third and fourth mode, deflection at peaks is higher in classical theory.

In case of uniform plate, deflection at the peaks is expected to be higher in shear theory than in classical theory in all the modes of vibration due to the fact that the transverse deflection of the plate in shear theory is due to bending as well as transverse shear. In the present case, the plate is made up of plate elements of different thicknesses and densities. At the same time, in different modes of vibration of present plate, the position of nodal circles also plays an important

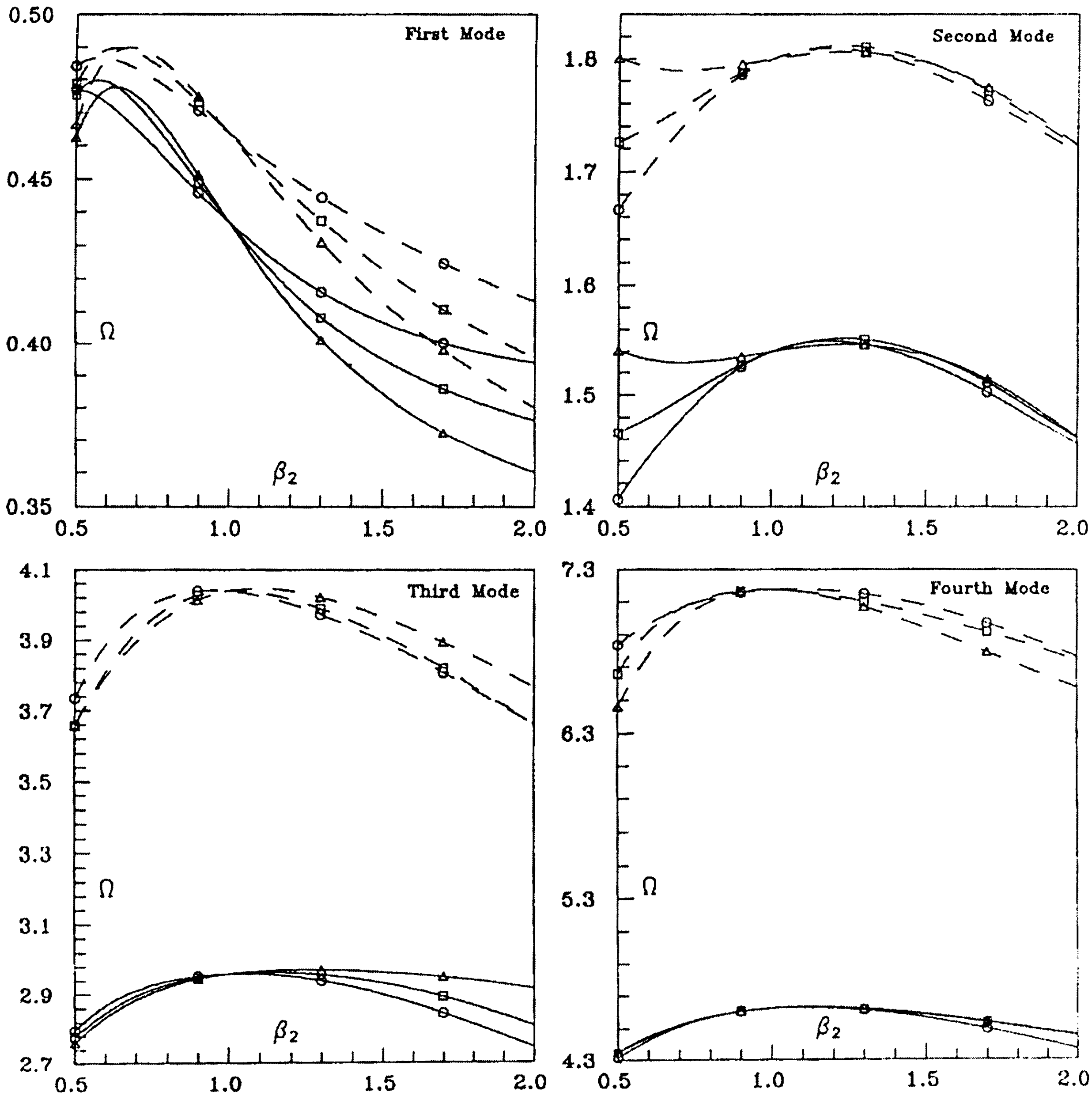

FIG. 2. Ω vs β_2 for C-plate for various values of α_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, δ_2 = 1.0, ——— Shear Theory, ------ Classical Theory

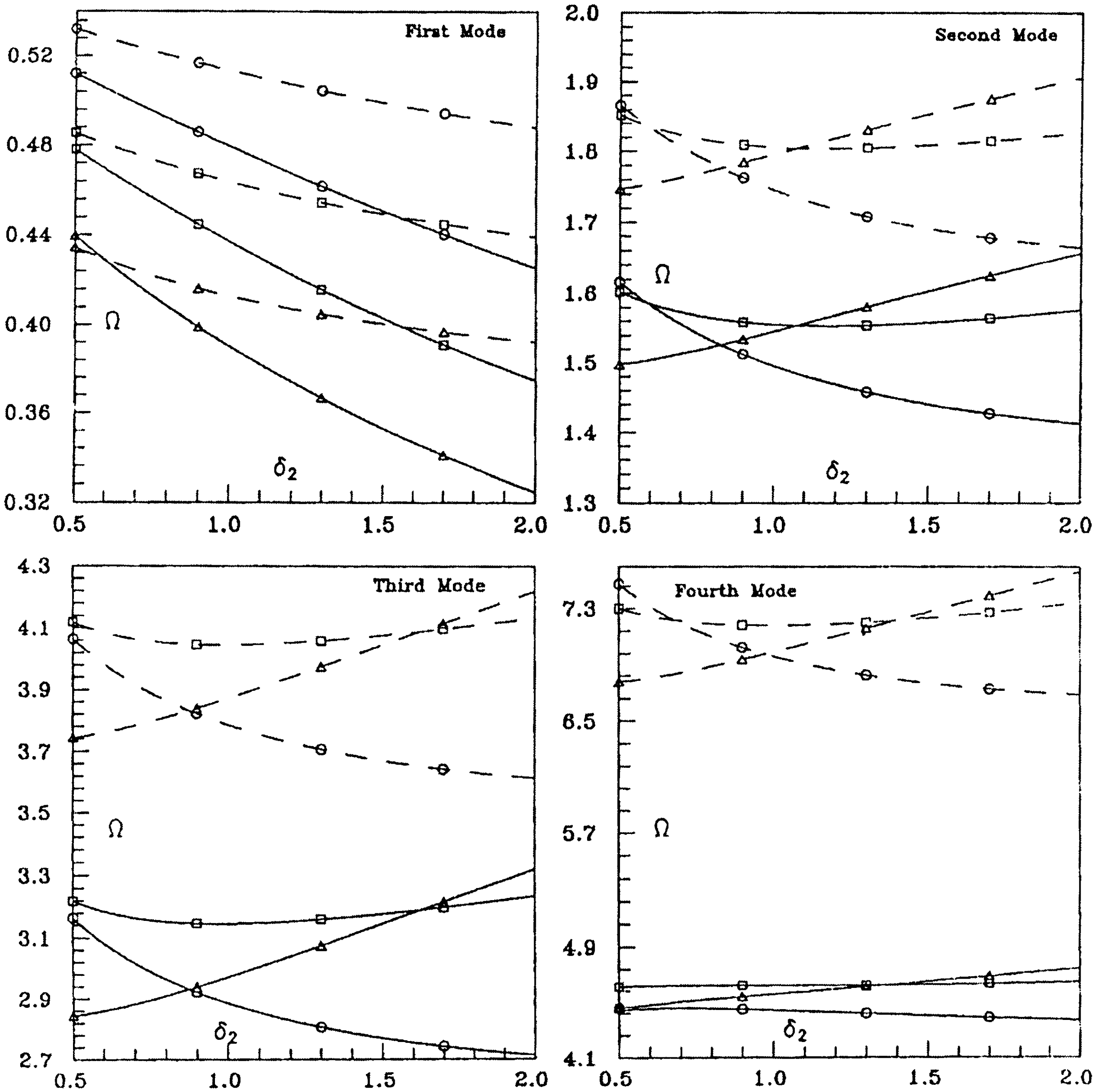

FIG. 3. Ω vs δ_2 for C-plate for various values of β_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, α_2 = 1.0, ——— Shear Theory, ------ Classical Theory

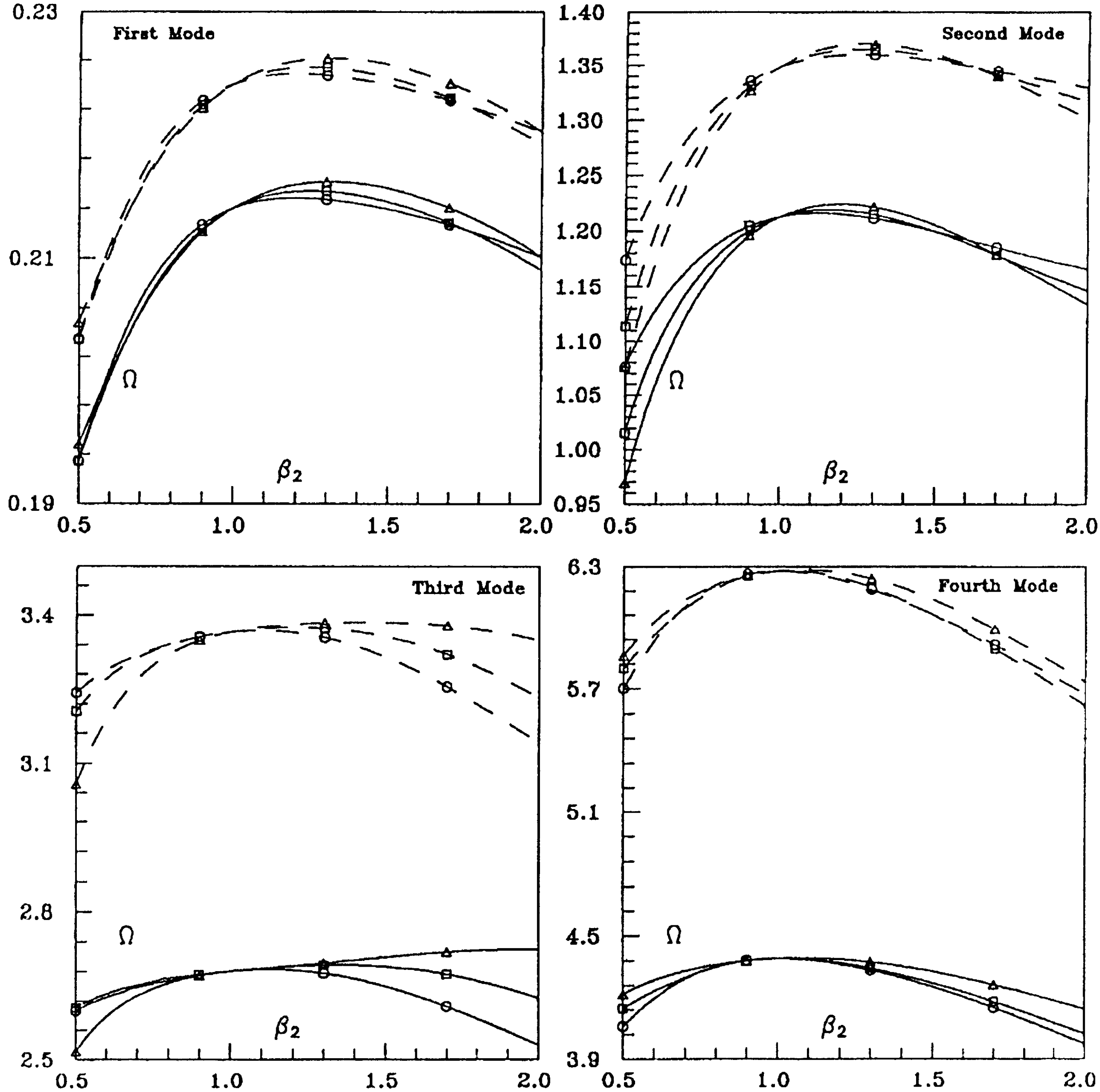

FIG. 4. Ω vs β_2 for S-plate for various values of α_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, δ_2 = 1.0, ——— Shear Theory, ------ Classical Theory

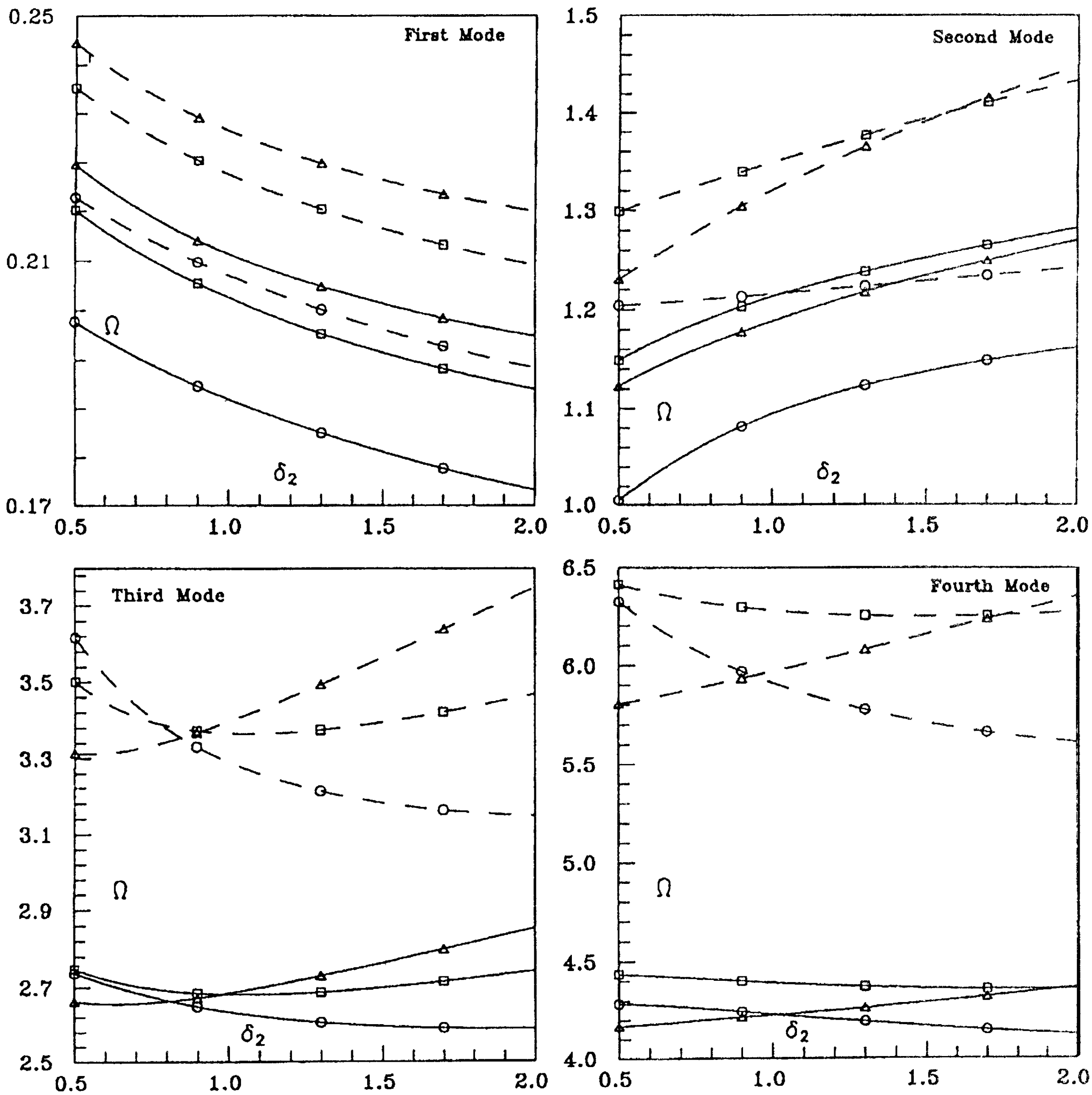

FIG. 5. Ω vs δ_2 for S-plate for various values of β_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, α_2 = 1.0, ——— Shear Theory, ------ Classical Theory

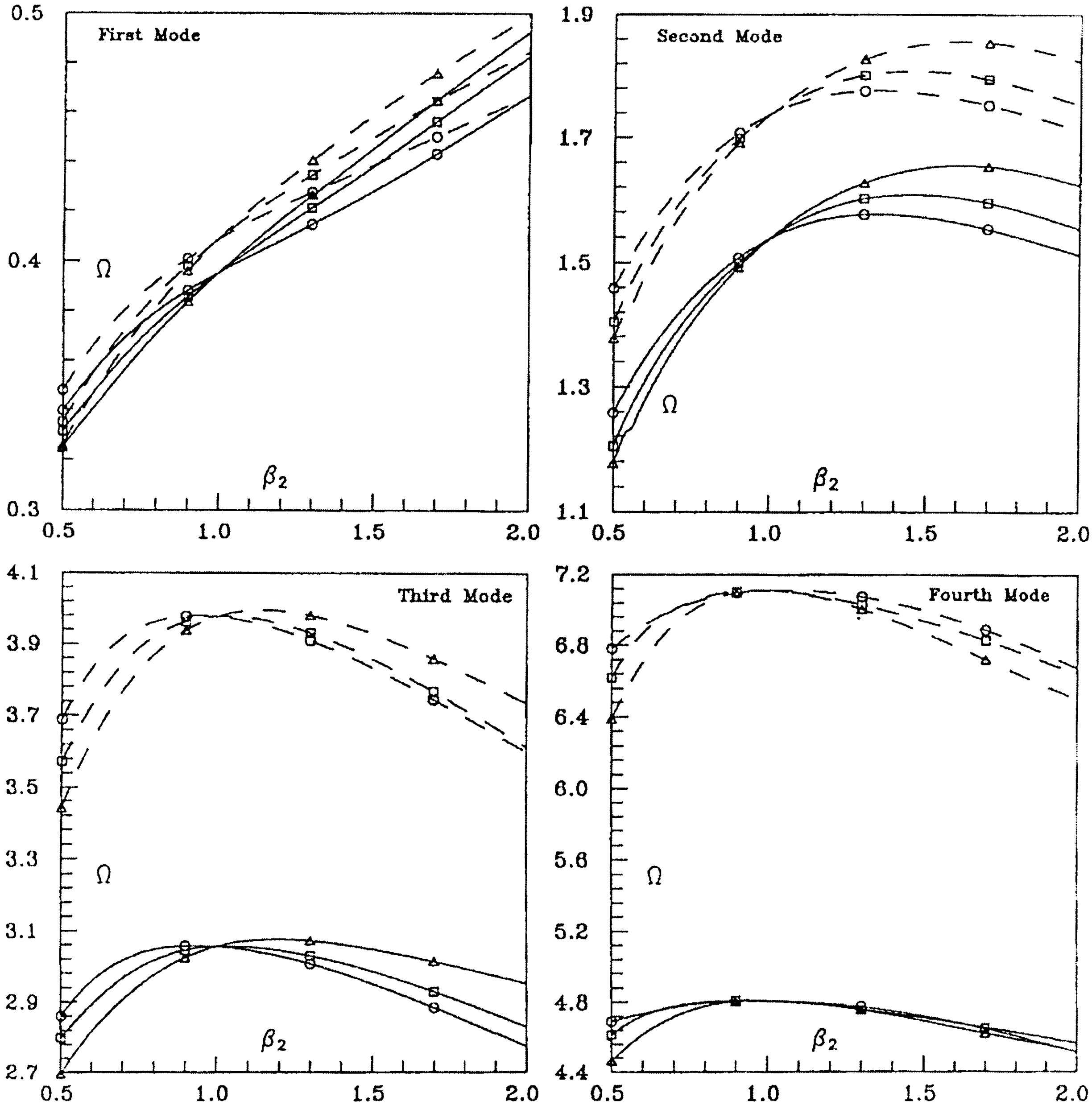

FIG. 6. Ω vs β_2 for F-plate for various values of α_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, $\delta_2 = 1.0$, ———Shear Theory, ------ Classical Theory

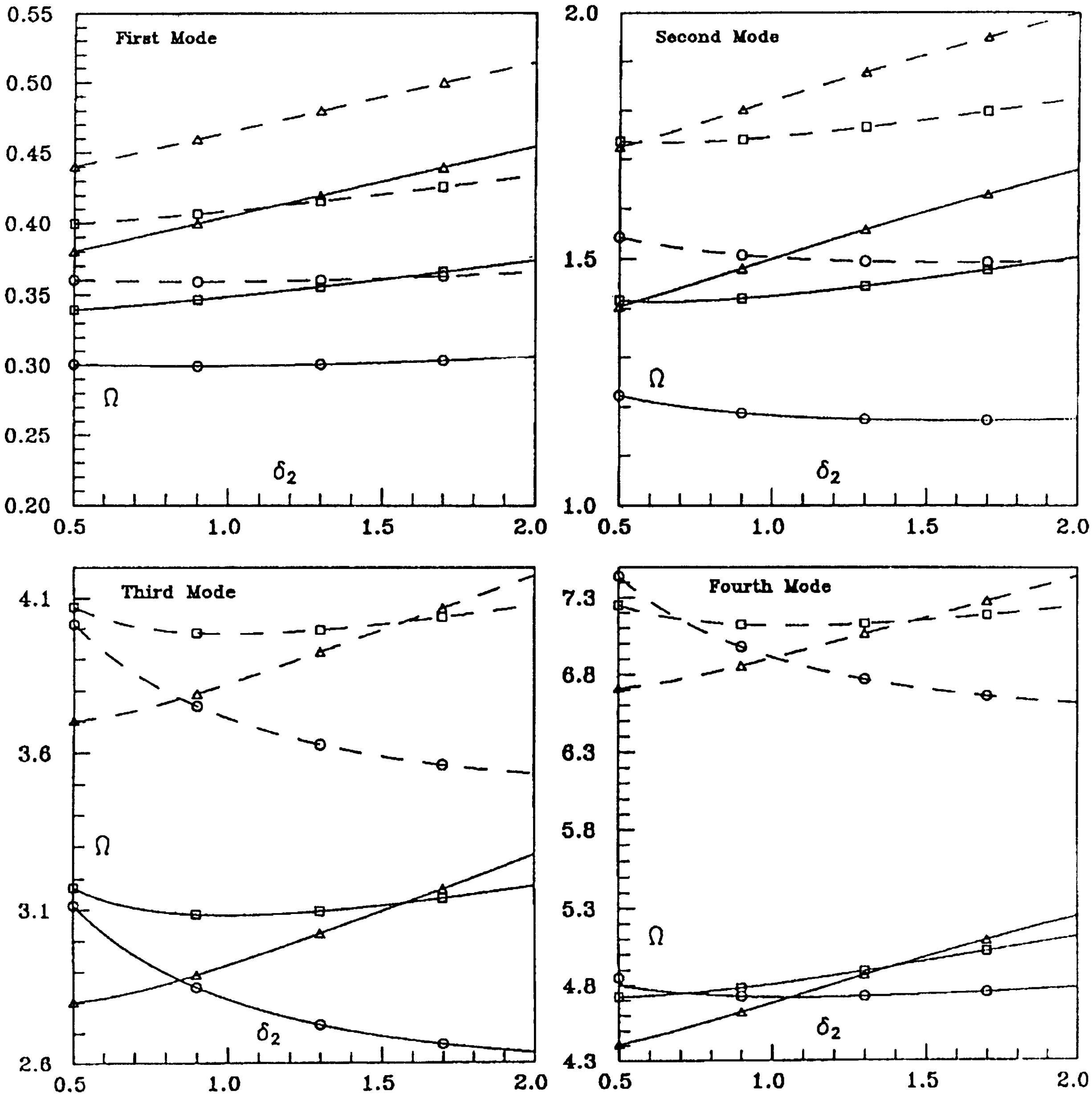

FIG. 7. Ω vs δ_2 for F-plate for various values of β_2:

—○—, 0.6; —□—, 1.0; —△—, 1.6, α_2 = 1.0, ——— Shear Theory, ------ Classical Theory

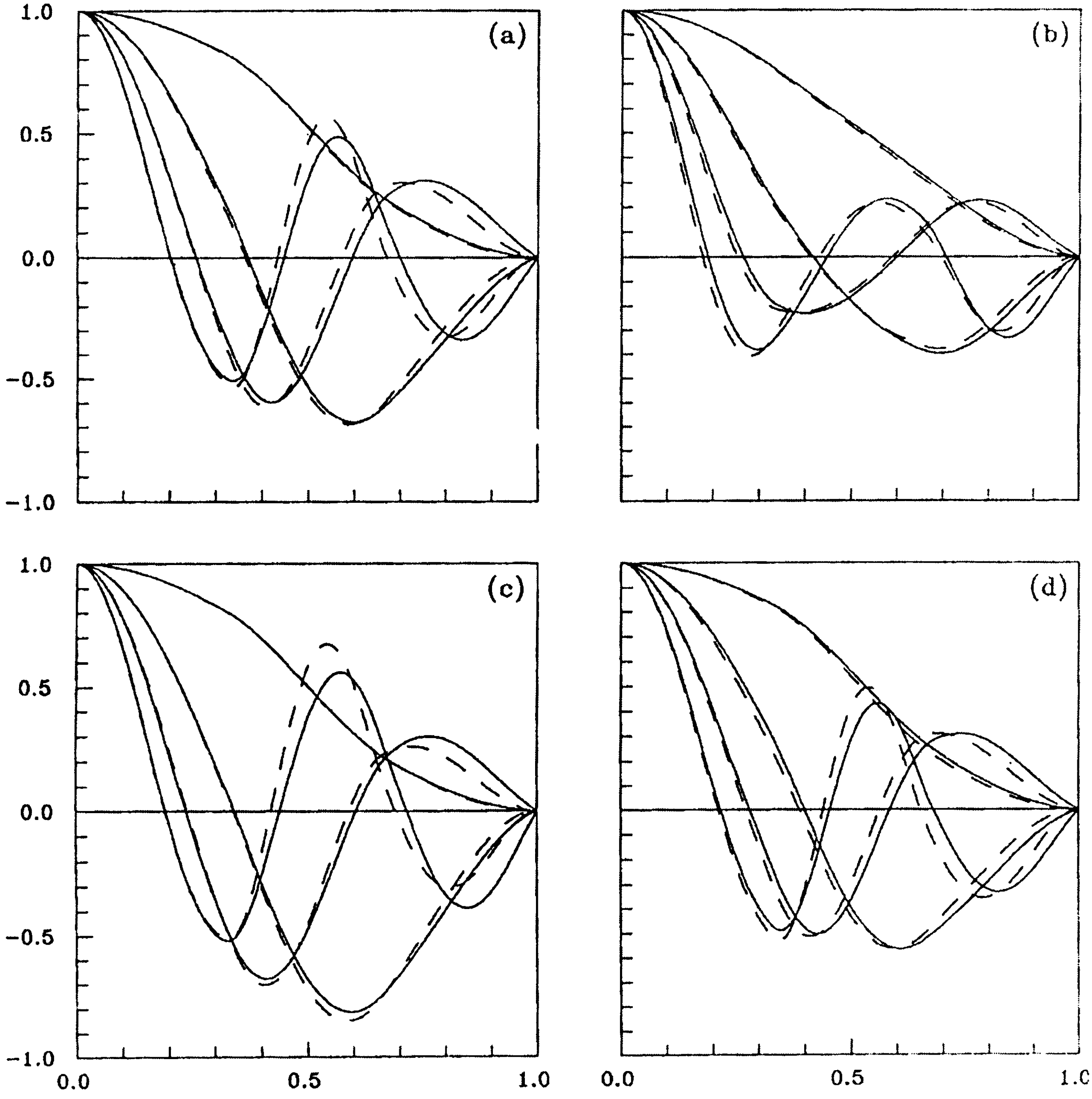

FIG. 8. Normalized transverse deflection in the first four modes of vibration for C-plate (a)δ_2 = 1,0; β_2 = 0.6; (b) δ_2 = 1.0, β_2 = 1.6; (c) δ_2 = 0.6, β_2 = 0.6; (d) δ_2 = 1.6, β_2 = 0.6 for α_2 = 1.0, ------- Shear Theory, - - - - Classical Theory:

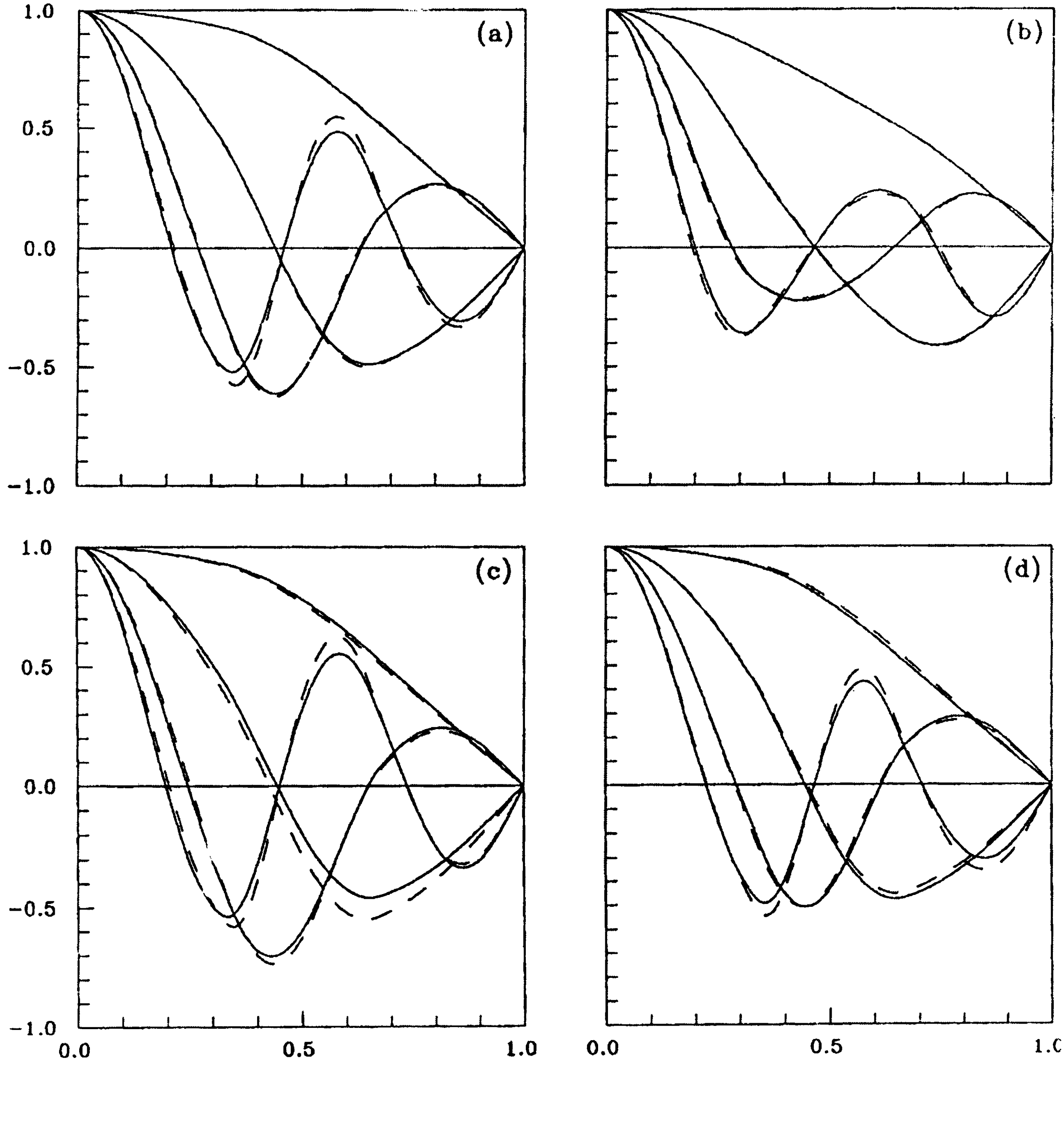

FIG. 9. Normalized transverse deflection in the first four modes of vibration for S-plate (a)δ_2 = 1,0; β_2 = 0.6; (b) δ_2 = 1.0, β_2 = 1.6; (c) δ_2 = 0.6, β_2 = 0.6; (d) δ_2 = 1.6, β_2 = 0.6 for α_2 = 1.0, ------- Shear Theory, - - - - Classical Theory:

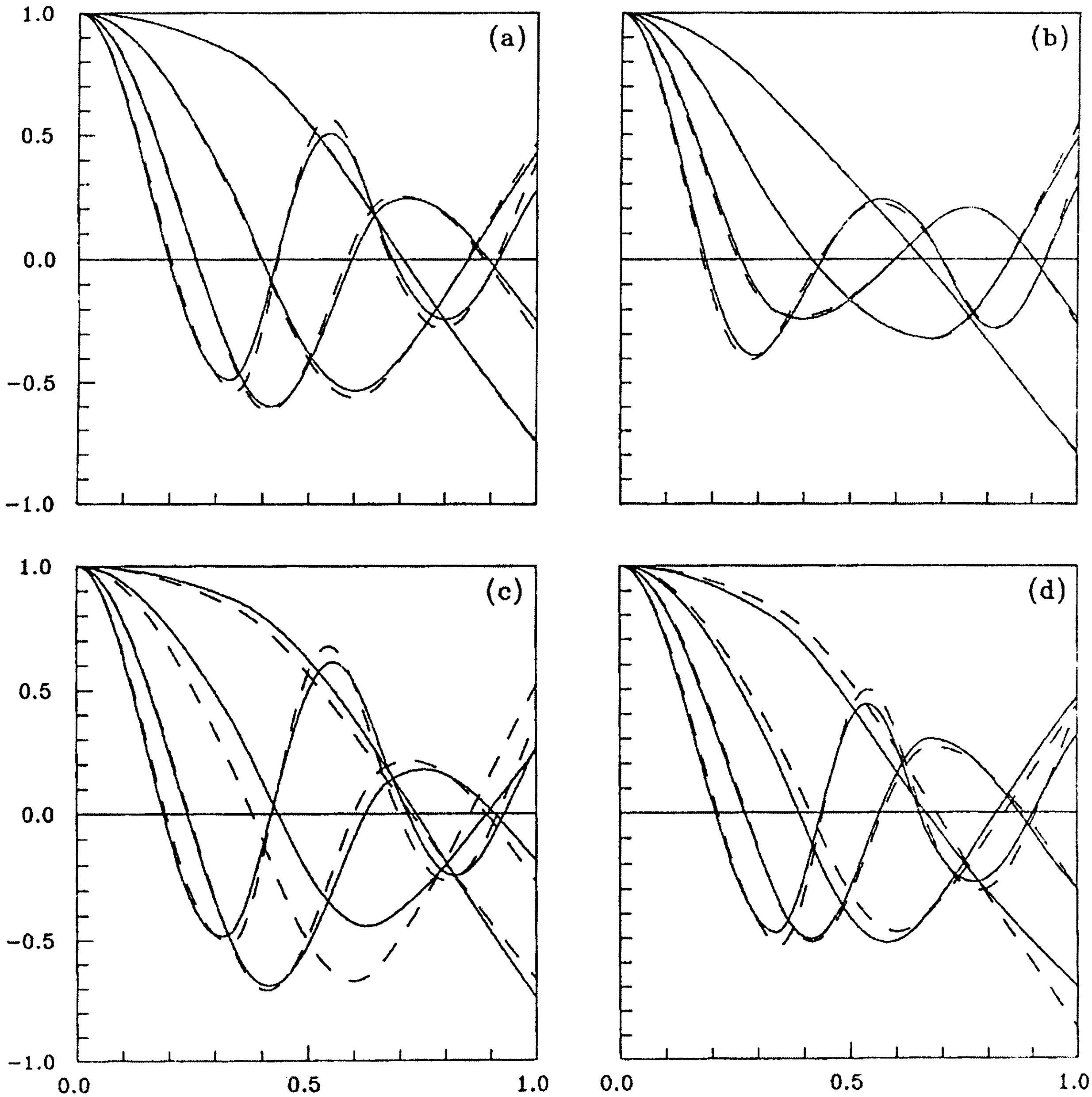

FIG. 10. Normalized transverse deflection in the first four modes of vibration for F-plate (a)δ_2 = 1,0; β_2 = 0.6; (b) δ_2 = 1.0, β_2 = 1.6; (c) δ_2 = 0.6, β_2 = 0.6; (d) δ_2 = 1.6, β_2 = 0.6 for α_2 = 1.0, ------- Shear Theory, - - - - Classical Theory:

role for the deflection at the peaks. Therefore, it is difficult to explain why in some cases deflection at the peak is less in shear theory as it can not be attributed to one factor only.

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