Indian J. pure appl. Math., **35**(9) : 1089-1092, September 2004

A NOTE ON SIMPLE INJECTIVE AND MININJECTIVE RINGS*

DINESH KHURANA

Centre for Advanced Study in Mathematics, Punjab University, Chandigarh 160 014, *India*

(*Received* 8 *September* 2000; *after final revision* 3 *May* 2001; *accepted* 26 *August* 2004)

Answering an open queston of Nicholson and Yousif [3, page 5312] in affirmative, Xue [5, Theorem 1] proved that a right principally injective, right Kasch, semilocal ring has finite left Goldie dimension. We prove the analogous result for right simple injective rings. Let R be a semilocal, right mininjective ring with right socle essential as left ideal. It is proved that left Goldie dimension of R is finite and equals the composition length of left R module $R/J(R)$ and that left socle of R coincides with the right socle. This generalizes [5, Proposition 4]. Improving upon [5, Proposition 5], we prove that a semilocal, right mininjective ring R, for which the composition length of $Soc(R_R)$ as left R-module coincides with the composition length of $R/J(R)$ as left R-module, is right Kasch.

Key Words: Simple injective Ring; Principally Injective Ring; Mininjective Ring; Kasch Ring.

Harada[1] calls a ring R right simple injective if every R-homomorphism with simple image from a right ideal of R to R is given by left multiplication by an element of R. A ring R is called right principally injective (resp. right mininjective) if every R-homomorphism from a principal (minimal) right ideal of R into R is given by left multiplication by an element of R. These rings have been studied, recently[2,4]. Clearly, right simple injective ring is right mininjective, but the converse is not true even for commutative local rings (see Example 8). If every simple right R-module embeds in R, then R is said to be right Kasch.

Throughout, R is an associative ring with identity and modules are unitary. The right and left annihilators of a subset X of a ring R are denoted by $r_R(X)$ and $l_R(X)$ respectively. We write $J = J(R)$ for the Jacobson radical of ring R and $Soc(M)$, $Rad(M)$ for the socle and radical of a module M respectively. The Goldie dimension and the composition length of a module M are denoted by $G.dim(M)$ and $c(M)$ respectively. By N M we shall mean that N is an essential submodule of M. We shall denote R/J by $\overline{R}$.

*Research supported by CSIR, India

Lemma 1 — (Nicholson and Yousif [4, Theorem 1.14]). For a right mininjective ring R, $Soc\,(R_R) \subseteq Soc\,({}_RR)$.

The following result improves upon [3, Theorem 1.3 (1)], where it is proved that a right principally injective, right Kasch ring with finite $G.dim\,({}_RR)$ is semilocal.

Proposition 2 — Let R be a right principally injective, right Kasch ring. If $Soc\,(R_R)$ is finitely generated as left R-module, then R is semilocal.

PROOF : Let $Soc\,(R_R) = \oplus_{i=1}^{n} S_i$, where each S_i is a minimal left ideal of R (see Lemma 1). As R is right Kasch, $J = r_R\,(Soc\,(R_R)) = \bigcap_{i=1}^{n} r_R\,(S_i)$. As, each $r_R\,(S_i)$ is maximal right ideal (see [3, Theorem 1.2]), R is semilocal.

Lemma 3 — (Nicholson and Yousif [4, Proposition 2.2]). A ring R isa right mininjective if and only if $Hom_R\,(S, R)$ is zero or simple left R-module for every simple right R-module S.

Lemma 4 — For a right simple injective, right Kasch ring R, $l_R\,(J) \leq {}_RR$.

PROOF Let $0 \neq b \in R$ and K be a maximal submodule of right R module bR. As R is right Kasch, there exists an embedding $\sigma : bR/K \to R_R$. Let $n : bR \to bR/K$ be the cononical map. As R is right simple injective and $\sigma n\,(bR)$ is simple, σn is given by left multiplication by some element c of R. Thus $cb = \sigma n(b) \neq 0$ and $cbJ = \sigma n(bJ) = \sigma n(bRJ) \subseteq \sigma n(Rad(bR)) \subseteq \sigma n(K) = 0$. This completes the proof.

The part *(iii)* below generalizes a result of Xue [5, Proposition 4], who proved it for principally injective rings.

Proposition 5 — Let R be a semilocal right mininjective ring with $\overline{R} = A \oplus B$, where A is the sum of all those minimal right ideals of $\overline{R}$ which imbed in R_R. Then the following hold:

(*i*) $G.dim\,({}_RR) \geq c\,(A_R)$.

(*ii*) If $Soc\,(R_R) \leq {}_RR$, then

$$Soc\,({}_RR) = Cos\,(R_R) \text{ and } G.dim\,({}_RR) = c\,({}_RSoc({}_RR)) = c(A_R).$$

(*iii*) If R is right Kasch and $Soc\,(R_R) \leq {}_RR$, then $G.dim\,({}_RR) = c\,(\overline{R}_R)$.

(*iv*) If R is right Kasch and right simple injective, then $G.dim\,({}_RR) = c\,(\overline{R}_R)$.

PROOF : (*i*) As R is semilocal, $Soc(R_R) = l_R(J)$ and as R is right mininjective, by Lemma 1, $Soc(R_R) \subseteq Soc({}_RR)$. Thus $Soc({}_RR) \supseteq Soc(R_R) = l_R\,(J) \cong Hom_R(\overline{R}_R, R)$

$$\cong Hom_R(A_R, R) \cong \oplus_{i=1}^{n} Hom_R(S_i, R),$$

where each S_i is simple right R-module and $A = \oplus_{i=1}^{n} S_i$. As $Hom_R(S_i, R) \neq 0$, by Lemma 3, $Hom_R(S_i, R)$ is a simple left R-module for each i. Thus $c({}_RSoc\,({}_RR)) \geq c\,(A_R)$ and so $G.dim\,({}_RR) \geq c(A_R)$.

(*ii*) As $Soc(R_R) \leq {}_RR$, $Soc\,({}_RR) \subseteq Soc(R_R)$ and so, by Lemma 1, $Soc({}_RR) = Soc(R_R)$. Now, by (*i*) above, $c({}_RSoc({}_RR)) = c(A_R)$. And so $Soc({}_RR) \leq {}_RR$, $G.dim({}_RR)) = c({}_RSoc({}_RR)) = c(A_R)$.

(*iii*) If R is right Kasch, then $\overline{R}_R = A_R$ and so, by (*ii*), $G.dim({}_RR) = c(\overline{R}_R)$.

(*iv*) By Lemma 4, $l_R\,(J) = Soc\,(R_R) \leq {}_RR$. Thus, by (*iii*), $G.dim({}_RR) = c(\overline{R}_R)$. This completes the proof.

The following result generalizes [5, Proposition 5], where it is proved for right principally injective rings and the proof runs on the similar lines.

Proposition 6 — Let R be a semilocal, right mininjective ring with $c\,({}_R\,Soc(R_R)) = c(\overline{R}_R)$. Then R is right Kasch.

PROOF : Let S be any simple right R-module. Then there exists a finitely generated semisimple right R-module T such that $\overline{R}_R = S \oplus T$. Then

$$\begin{aligned} c(\overline{R}_R) &= c({}_RSoc(R_R)) = c(l_R\,(J)) \\ &\cong c(Hom_R(S+T, R)) \\ &= c(Hom_R\,(S, R)) + c(Hom_R\,(T, R)) \\ &\leq 1 + c(T) \quad \text{see Lemma 3} \\ &= c\,(S+T) = c(\overline{R}_R). \end{aligned}$$

This, in view of Lemma 3, implies that $c({}_RHom_R\,(S, R)) = 1$. This completes the proof.

From Proposition 5 and Proposition 6 we have

Corollary 7 — Let R be a semilocal right simple injective ring. Then R is right Kasch if and only if $c({}_R Soc(R_R)) = c(\overline{R}_R)$.

The following is an example of commutative local, Kasch, mininjective ring which is not simple injective. This example, in particular, shows that our generalization in Proposition 6 of Xue's result is nontrivial. let ${}_R M_R$ be a bimodule then

$$S = R \dot{+} M = \{(r, m) : r \in R \text{ and } m \in M\}$$

is a ring with componentwise addition and multiplication defined as follows:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Example 8 — Let R be any commutative local ring which is not a field (e.g., **Z** localized at 2) and $S = R \dot{+} R/J$. Then S is a commutative local ring with $J(S) = J \dot{+} R/J$ and $Soc(S) = l_s(J(S)) = 0 \dot{+} R/J$. Obviously, $Soc(S)$ is simple and hence by [4, Theorem 3.2], S is mininjective. Also as $Soc(S)$ is non-zero and S is local, S is Kasch ring. But $Soc(S)$ is not large as $Soc(S) \cap J \dot{+} 0 = 0$ and $J \dot{+} 0$ is an ideal of S. Thus, by Lemma 4, S is not simple injective. Also, by [2, lemma 2.3], R is not principally injective.

The above example also shows that we cannot drop "$Soc(R_R) \leq {}_R R$" from the hypothesis of Proposition 5 (*iii*).

ACKNOWLEDGEMENT

The author is indebted to the referee for many useful suggestions which lead to the current version of Proposition 5 and also to the inclusion of Example 8.

REFERENCES

1. M. Harada, Self mini-injective rings, *Osaka J. math.*, **19** (1982), 587-97.
2. W. K. Nicholson and M. F. Yousif, Principally injective rings, *J. Alg.*, **174** (1995), 77-93.
3. W. K. Nicholson and M. F. Yousif, On a theorem of Camillo, *Comm. Alg.*, **23** (1995), 5309-5314.
4. W. K. Nicholson and M. F. Yousif, Mininjective rings, *J. Alg.*, **187** (1997), 548-78.
5. Weimin Xue, A note on principally injective rings, *Comm. Alg.*, **26** (1998), 4187-4190.