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ON THE RELATIONSHIP BETWEEN THE RECURRENCES IN NILPOTENT GROUPS AND THE BINOMIAL FORMULA

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In this paper, we constitute 2-step Fibonacci sequences by the two generating elements of a group of exponent p (p is prime) and nilpotency class n. Then, we examine the relationship between the number of recurrence sums involved in the jth term of the last component of the Fibonacci sequences and the coefficients of the binomial formula $(a+b)^{n-2}$, for $n \geq 2$.

Key Words: Fibonacci Numbers; Fibonacci Sequence; Binomial Formula; Nilpotent Groups

1. INTRODUCTION

The study of Fibonacci sequences in groups began with the earlier work of Wall[15] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties Wilcox extended the problem to abelian group [16]. Profile co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups[6]. Aydin and Smith proved[4] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized[7,8] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2, 3 and exponent p. Then it is shown[2] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent p and nilpotency class 2.

Let (s_i) denote the ordinary Fibonacci sequence in $GF(p)$ defined by the recurrence

$$S_{i+2} = S_i + S_{i+1}$$

with initial data $s_0 = 0, s_1 = 1$. This sequence or loop must be periodic. One can define seqences by a linear recurrence in any group. Let G be a finite group. We define a family of recurrences, parameterized by some integer n, via equations

$$x_i = x_{i-1}\, x_{i-2} \cdots x_{i-n} \qquad \ldots (R_n)$$

We fix $n > 1$. We may start the recurrence with any initial data $g_0, g_1, ..., g_{n-1}$ and then the recurrence will define a periodic bi-infinite sequence (g_i) indexed by the integers. The term loop is used to describe a bi-infinite sequence satisfying the recurrence. Notice that if $l \in Z$, then putting $h_i = g_{i+l}$ for $0 \leq i < n$ we obtain another set of initial data. The recurrence (R_n) then gives us another loop (h_i) of G. Notice that (h_i) is just (g_i) shifted through l steps. We say that (h_i) is a rotation of (g_i).

The fundamental period of a loop is clearly a significant number and sometimes is called the Wall's number. Following Wall[15], the letter k is used (suitably adorned) to describe this period.

Wall investigated the recurrence R_2 in finite cyclic groups. This subject immediately reduces to the study of cyclic groups of prime power order. This paradigm is exploited elsewhere[1,4,5] to study recurrences in finite nilpotent groups, since they are the cartesian product of their Sylow p-subgroups. Another early contributor to this field was Vinson[14], who was particularly interested in ranks of apparition, mistranslated from the French. More interest has come in the works[6,9,12,13&16]

Wall distinguishes the special loop $s = (s_i)$ with initial data $s_0 = 0, s_1 = 1$ in $Z/p^n Z$.

Let $k(s, p^n)$ denote the fundamental period of s.

Let us consider the more general recurrences written multiplicatively as

$$x_i = x_{i-1}^{a_1} \, x_{i-2}^{a_2} \cdots x_{i-n}^{a_n}$$

or additively as

$$x_i = \sum_{r=1}^{n} a_r x_{i-r} \qquad \ldots (A_n)$$

The exponents (coefficients) are deemed to be integers. Such a recurrence may not be so well behaved as R_n, for it may be singular in particular group. This means that initial data may not define a unique bi-infinite periodic sequence; the problem arises when attempting to extend the sequence through negative integer subscripts.

An important problem in Fibonacci numbers theory is to obtain Fibonacci sequences in nilpotent groups with higher class. As a new approach for the solution of this problem we introduce a new formula. For this purpose we used the properties of the finite groups and special grpup law.

2. TWO STEP RECURRENCES

The theory of general two-step recurrences is very well understood (See 10, and Chapter 2 of [11]). The following result of this sectionis well-known[5-15].

Theorem 2.1 — *Let H be the additive group of a finite field $k = GF(p^t)$. Consider the standard sequence $s = (s_i)$ for the recurrence A_2. Let $k(s)$ denote the Wall number of s. Suppose*

$b = (b_i)$ is any loop satisfying A_2 in H, then $k(b)$ divides $k(s)$. If $k(b) < k(s)$ then b must either be the trivial loop (all entries 0) or must be a geometric sequence3.

A result of this theorem is that any short loop must be trivial, or contain no entry which is zero. Suppose that b is of the latter form. Choose $i \in Z$ and let b^{+i} be the rotation of b through i steps, so $b_0^{+i} = b_i$ and $b_1^{+i} = b_{i+1}$. Put $c = b_0 b_i^{-1} b$. Thus $k(c) = k(b)$. Now $k(c-b)$ must divide $k(b)$ and so $d = c - b$ is a short loop. However, $d_0 = 0$ so d is a trivial loop. Thus $b_1 = b_0 b_i^{-1} b_{i+1}$. We conclude that $\mu = b_i^{-1} b_{i+1}$ is independent of i so b is a geometric progression with common ratio μ. The ratio μ must be a root of $x^2 + a_1 x + a_2$.

Definition 2.2 — A group G is torsion-free if every nonidentity element of G has infinite order.

Definition 2.3 — Let $H \rhd G$, $K \rhd G$ and $K \leq H$. If H/K is contained in the center of G/K, then H/K is called a central factor of G. A group G is called nilpotent if it has finite series of normal subgroups.

$$G = G_0 \geq G_1 \geq \ldots \geq G_r = 1$$

such that G_{i-1}/G_i is a central factor of G for each $i = 1, 2, \ldots, r$. The smallest possible r is called the nilpotency class of G.

Consider a group $H = H^Z$ with the following presentation:

$$H^Z = \langle x, y, z, t, u : [y, x] = z, [z, x] = t, [t, x] = u \rangle$$

where those paris of generators with unspecified commutators are implicitly deemed to commute.This is a torsion-free nilpotent group of nilpotency class 4. In fact, the group is generated by x and y. Each element of the group will have a unique representation as $x^a y^b z^c t^d u^e$ where a, b, c, d, $e \in Z$. In fact, we may as well think of this group as being a rather strange group structure on Z^5. The group multiplication law will be given by rational polynomials in 10 variables and inversion by rational polynomials in 5 variables. These polynomials must, of course, have the property that, regarded as maps, they assume integral values when supplied with integral arguments.

Let p be a prime and work in $(Z/pZ)^5$ instead. There are no difficulties associated with this reduction modulo p, because the polynomials involved do not have primes other than 2 or 3 involved in denominators of co-efficients. We regard the variables a, b, c, d, $e \in Z$ as being in Z/pZ and

obtain a group structure on $(Z/pZ)^5$. We call this group $H^{Z/pZ}$ and for $p > 3$ this will be a group of exponent p. We shall demonstrate this shortly. The natural ring epimorphism $Z \to Z/pZ$ induces a natural epimorphism of groups $\Phi : H^Z \to H^{Z/pZ}$. We abuse the letters x, y, z, t, u to denote the images of the corresponding elements of H^Z under Φ.

The relatively free two generator nilpotent groups of class 4 with exponent laws 2 and 3 are respectively, $C_2 \times C_2$ and the (class 2) extra-special group of order 27. These are, in a sense, degenerate, since for all other primes p, the relatively free two generator exponent p class 4 groups are all of genuine class 4 and have order p^8. It is on these primes which we shall focus.

We examine the details of the multiplication law of the groups $H^{Z/pZ}$. The element u is central and the group $V = \langle y, z, t\, u \rangle$ is abelian. A concrete representation of this group is to think of V as a four dimensional $GF(p)$-vector space. This space admits an automorphism β (really x) given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The group $H^{Z/pZ}$ is then realized as the semi-dirct product of V with $\langle \beta \rangle$.

Notice that the matrix of β^n is

$$\begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} \\ 0 & 0 & 1 & \binom{n}{1} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so for primes p other than 2 and 3 we have $\beta^p = 1$. We wish to show that for primes $p > 3$, the group $H^{Z/pZ}$ has exponent p.

Suppose $v \in V$ and that $n \in N$, then

$$(\beta^n \cdot v)^p = \beta^{np} \cdot v\,(1 + \beta^n + \ldots + \beta^{n(p-1)})$$

by the definition of semi-direct products. Assuming that p is a prime other than 2 or 3, we have that $\beta^p = 1$, so $\beta^{np} = 1$. It remains to show that $\eta = 1 + \beta^n + \ldots + \beta^{n(p-1)} = 0$.

Let $m_n \in GF(p)[X]$ denote the minimum polynomial of β^n, so m_n is irreducible and divides

$$1 - X^p = (1 - X)(1 + X + X^2 + \ldots + X^{p-1}).$$

The ring $GF(p)[X]$ is a unique factorization domain. If m_n divides $(1 - X)$ then $\beta^n = 1$ so $\eta = 0$.

Conversely, if m_n fails to divide $(1 - X)$ then it must divide $1 + X + X^2 + \ldots + X^{p-1}$ so $\eta = 0$.

Thus for primes p greater than 3 the group $H^{Z/pZ}$ has exponent p.

We now investigate the group law of $H^{Z/pZ}$ in some detail, reverting to the description of $H^{Z/pZ}$ as being $(Z/pZ)^5$ with a particular group law. In the obvious notation

$$(a, b, c, d, e)(a', b', c', d', e') = (a'', b'', c'', d'', e''),$$

the entries of the 2-step Fibonacci sequence constructed by two elements of G have the following form

$$a'' = a + a'$$

$$b'' = b + b'$$

$$c'' = c + c' + a'b$$

$$d'' = d + d' + a'c + \binom{a'}{2} b$$

$$e'' = e + e' + a'd + \binom{a'}{2} c + \binom{a'}{3} b^{[1]}.$$

3. Main Result

Now we want to generalize the formulas in the above lemma for a group of exponent p and nilpotency class n.

Theorem 3.1 — *If G is a group of exponent p, nilpotency class n and with the presentation*

$$G = \langle x_1, x_2, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots [x_n, x_1] = x_{n+1} \rangle$$

then entries of the two-step Fibonacci sequences formed by two elements of G have the following form.

$$c_1 = a_1 + b_1, \; c_2 = a_2 + b_2, \; c_3 = a_3 + b_3 + b_1 a_2$$

$$c_4 = a_4 + b_4 + b_1 a_3 + \binom{b_1}{2} a_2$$

$$c_5 = a_5 + b_5 + b_1 a_4 + \binom{b_1}{2} a_3 + \binom{b_1}{3} a_2$$

$$c_6 = a_6 + b_6 + b_1 a_5 + \binom{b_1}{2} a_4 + \binom{b_1}{3} a_3 + \binom{b_1}{4} a_2$$

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$$c_{n+1} = a_{n+1} + b_{n+1} + b_1 a_n + \binom{b_1}{2} a_{n-1} + \binom{b_1}{3} a_{n-2} + \ldots + \binom{b_1}{n-1} a_2.$$

PROOF : Let G be the group which has the following presentation

$$G = \langle x_1, x_2, x_3, \ldots, x_n, x_{n+1} : [x_2, x_1] = x_3, [x_3, x_1] = x_4, \ldots, [x_n, x_1] = x_{n+1} \rangle$$

where those pairs of generators with unspecified commutators are implicitly deemed to commute. This is a group of nilpotency class n and exponent p. In fact this group is generated by $x = x_1$ and $y = x_2$. Each element of the group will have a unique representation as $x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n+1}^{a_{n+1}}$ where $a_1, a_2, a_3 \ldots, a_{n+1} \in Z/pZ$. The element x_{n+1} is central and the group $H = \langle x_2, x_3, \ldots, x_n, x_{n+1} \rangle$ is Abelian. Let $x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n+1}^{a_{n+1}}$, $x_1^{b_1} x_2^{b_2} x_3^{b_3} \ldots x_{n+1}^{b_{n+1}}$ be two elements of G. Then, suppose that

$$\left(x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n+1}^{a_{n+1}} \right) \left(x_1^{b_1} x_2^{b_2} x_3^{b_3} \ldots x_{n+1}^{b_{n+1}} \right) = \left(x_1^{c_1} x_2^{c_2} x_3^{c_3} \ldots x_{n+1}^{c_{n+1}} \right).$$

Under the group law, we can calculate the entries of two-step Fibonacci sequences as follows:

$$\left(x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n+1}^{a_{n+1}} \right) \left(x_1^{b_1} x_2^{b_2} x_3^{b_3} \ldots x_{n+1}^{b_{n+1}} \right) = \left(x_1^{c_1} x_2^{c_2} x_3^{c_3} \ldots x_{n+1}^{c_{n+1}} \right)$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_1^{b_1} x_2^{b_2} x_3^{b_3} \ldots x_{n+1}^{a_{n+1} + b_{n+1}}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n-1}^{a_{n-1} - 1} \; x_{n-1} x_1 x_1^{b_1 - 1} \; x_2^{b_2} x_3^{b_3} \ldots x_n^{a_n + b_n} \; x_{n+1}^{a_{n+1} + b_{n+1} + a_n b_1}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_{n-1}^{a_{n-1}} \; x_1 x_{n-1} x_1 x_1^{b_1 - 1} \; x_2^{b_2} x_3^{b_3} \ldots x_n^{a_n + b_n} \; x_{n+1}^{a_{n+1} + b_{n+1} + a_n b_1}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{a_{n-1}-1} x_1 x_{n-1} x_1^{b_1-1} x_n x_{n+1}^{b_1-1} x_2^{b_2} x_3^{b_3} \cdots x_n^{a_n+b_n} x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{a_{n-1}-1} x_1 x_{n-1} x_1 x_1^{b_1-2} x_n x_{n+1}^{b_1-1} x_2^{b_2} x_3^{b_3} \cdots x_n^{a_n+b_n} x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{a_{n-1}-1} x_1^2 x_{n-1} x_n x_1^{b_1-2} x_n x_{n+1}^{b_1-1} x_2^{b_2} x_3^{b_3} \cdots x_n^{a_n+b_n} x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1}$$

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{a_{n-1}-1} x_1^2 x_{n-1} x_1^{b_1-2} x_i^2 x_{n+1}^{b_1-1+b_1-2} x_2^{b_2} x_3^{b_3} \cdots x_n^{a_n+b_n} x_{n+1}^{a_{n+1}+b_{n+1}+a_n b_1}$$

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$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{a_{n-1}-1} x_1^{b_1} x_{n-1} x_n^{b_1} x_{n+1}^{b_1-1+\cdots+1} x_x^{b_2} x_3^{b_3} \cdots x_n^{a_n+b_n} x_{x+1}^{a_{n+1}b_{n+1}+a_n b_1}$$

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$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-2}^{a_{n-2}} x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots x_{n-2}^{a_{n-2}+b_{n-2}} x_n^{a_n+b_n+b_1 a_{n-1}} x_{n+1}^{a_{n+1}b_{n+1}+a_n b_1+\binom{b_1}{2}a_{n-1}}$$

We can do the same operations between $x_{n-2}^{a_{n-2}}$ and $x_1^{b_1}$. Then the last equation is

$$= x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-3}^{a_{n-3}} x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots x_{n-2}^{a_{n-2}+b_{n-2}} x_{n-1}^{a_{n-1}+b_{n-1}+b_1 a_{n-2}} x_n^{A} x_{n+1}^{B}$$

where

$$A = a_n + b_n + b_1 a_{n-1} + \binom{b_1}{2} a_{n-2},$$

$$B = a_{n+1} + b_{n+1} + a_n b_1 + \binom{b_1}{2} a_{n-1} + \binom{b_1}{3} a_{n-2}.$$

So, we can do the same operations between $x_{n-3}^{a_{n-3}}$, $x_{n-4}^{a_{n-4}}$, ..., $x_2^{a_2}$ and $x_1^{b_1}$. Then the last equation is

$$= x_1^{a_1+b_1} x_2^{a_2+b_2} x_3^{a_3+b_3+b_1 a_2} \cdots x_n^{C} x_{n+1}^{D}$$

where

$$C = a_n + b_n + b_1 a_{n-1} + \binom{b_1}{2} a_{n-2} + \ldots + \binom{b_1}{n-2} a_2,$$

$$D = a_{n+1} + b_{n+1} + a_n b_1 + \binom{b_1}{2} a_{n-1} + \binom{b_1}{3} a_{n-2} + \ldots + \binom{b_1}{n-1} a_2.$$

We are done.

Now we examine the relationship between a number of sums involved in the jth term of last component of Fibonacci sequence constructed by the formulas in Theorem 3.1 and the coefficients of the binomial formula $(a+b)^{n-2}$ for $n \geq 2$.

Theorem 3.2 — *If G has the group structure in the Theorem 3.1 then a number of sums included in the jth term of last component of Fibonacci sequence constructed by the formulas in Theorem 3.1 are equal to coefficients of the binomial formula $(a+b)^{n-2}$ for $n \geq 2$, respectively.*

PROOF : We prove this by induction on n. Let $l(x, y, ...)$ be the Fibonacci loop in G and the jth entry of l be $(s_{j-1}, s_j, z_{j1}, z_{j2}, ..., z_{j(n-1)})$. If $n = 2$, then the group G will be a group of exponent p and nilpotency class 2. Thus, the jth entry of l will be (s_{j-1}, s_j, z_{j1}). Using the formula for the multiplication, a simple induction shows that for positive integer j, we have

$$z_{j1} = \sum_{r=0}^{j-1} S_{j-r-1} S_r^2 \qquad \ldots (1)$$

Hence, our argument is obvious for $n = 2$. For $n = 3, 4$ the jth entry of l will be $(S_{j-1}, S_j, Z_{j1}, Z_{j2})$, $(S_{j-1}, S_j, Z_{j1}, Z_{j2}, Z_{j3})$, respectively. Using the formula for the multiplication, a simple induction shows that for positive integer j, we have

$$Z_{j2} = \sum_{r=0}^{j-1} S_{j-r-1} S_r Z_{j1} + \sum_{m=0}^{j-1} S_{j-m-1} S_m \binom{S_m}{2} \qquad \ldots (2)$$

and

$$Z_{j3} = \sum_{r=0}^{j-1} S_{j-r-1} S_r Z_{j2} + \sum_{m=0}^{j-1} S_{j-m-1} Z_{j1} \binom{S_m}{2} + \sum_{i=0}^{j-1} S_{j-i-1} S_i \binom{S_i}{3} \qquad \ldots (3)$$

Notice that if we write (1) in (2), we have

$$Z_{j2} = \sum_{r=0}^{j-1} \sum_{m=0}^{j-1} S_{j-r-1} S_r S_{r-m-1} S_m^2 + \sum_{m=0}^{j-1} S_{j-m-1} S_m \binom{S_m}{2} \quad \ldots (4)$$

and if we write (4) in (3), we have

$$Z_{j3} = \sum_{r=0}^{j-1} \sum_{m=0}^{r-1} \sum_{i=0}^{m-1} S_{j-r-1} S_r S_{r-m-1} S_m S_{n-m-1} S_i^2$$

$$+ \sum_{m=0}^{j-1} \sum_{i=0}^{m-1} S_{j-m-1} S_m \binom{S_m}{2} S_{m-i-1} S_i$$

$$+ \sum_{m=0}^{j-1} \sum_{i=0}^{m-1} S_{j-m-1} S_i \binom{S_m}{2} S_{i-i-1} S_i^2 + \sum_{m=0}^{j-1} S_{j-i-1} S_i \binom{S_i}{3}.$$

Notice that 1-tuples recurrence $\sum$ sum occurs in z_{j1}, one 2-tuples recurrence $\sum\sum$ sums and one 1-tuples recurrence $\sum$ sum occur in z_{j2}, one 3-tuples recurrence $\sum\sum\sum$ sums, two 2-tuples recurrence $\sum\sum$ sums and one 1-tuple recurrence $\sum$ sum occur in z_{j3}. Hence, a number of these recurrence sums are equal to the coefficients of $(a+b)^0$, $(a+b)^1$ and $(a+b)^2$ respectively.

Suppose that our argument is true for nilpotency class n.

When G is the group of exponent p and nilpotency class $(n+1)$, the jth term of last component of our sequence constructed by formulas in the Theorem 3.1 can be written as

$$\sum A(z_{j(n-1)}) + \sum B(z_{j(n-2)}) + \ldots + \sum C(z_{j1}) + \sum D \quad \ldots (5)$$

where $A, B, C, \ldots, D$ depend on s_i's and $z_{j(i)}$ is the last component of Fibnacci sequence in nilpotent group of exponent p and nilpotency class $(i+1)$; $i = 1, 2, \ldots (n-1)$. Notice that in (5),

$$\sum B(z_{j(n-2)}) + \ldots + \sum C(z_{j1}) + \sum D \quad \ldots (6)$$

is $z_{j(n-1)}$ (the last component of Fibonacci sequence in nilpotent group of exponent p and nilpotncy class n). Therefore by induction hypotehsis, eq. (5) can be written as

$$\sum \left(\sum_{i=0}^{n-2} \binom{n-2}{i} a^{n-2-i} b^i \right) + \left(\sum_{i=0}^{n-2} \binom{n-2}{i} a^{n-2-i} b^i \right) \quad \ldots (5')$$

By induction hypothesis when G is the group of expoentn p and nilpotency class n, a number

of recurrence sums included in the jth term of last component of our sequence constructed by formulas in the Theorem 3.1 are as shown :

A number of $(n-1)$-tuple recurrence sums are $\binom{n-2}{0}$, a number of $(n-2)$-tuple recurrence sums are $\binom{n-2}{1}$, a number of $(n-3)$-tuple recurrence sum are $\binom{n-2}{2}$, ..., a number of 1-tuple recurrence sums are $\binom{n-2}{n-2}$. Hence, in

$$\sum \left(\sum_{i=0}^{n-2} \binom{n-2}{i} a^{n-2-i} b^i \right) \quad \text{... (7)}$$

$\left(\sum_{i=0}^{n-2} \binom{n-2}{i} a^{n-2-i} b^i \right)$ includes on $(n-1)$-tuples recurrence sum. If $\sum$ is added the one $(n-1)$-tuples recurrence sum in (7), we have one n-tuples recurrence sum in (6). So, from (6), a number of $(n-1)$-tuples recurrence sums is $\binom{n-2}{1}$ and from $\sum_{i=0}^{n-2} \binom{n-2}{i} a^{n-2-i} b^i$, a number of $(n-1)$-tuples recurrence sums is $\binom{n-2}{0}$.

Hence, a number of $(n-1)$-tuples recurrence sums are

$$\binom{n-2}{1} + \binom{n-2}{0} = \binom{n-1}{1}.$$

So from (6), we have

$$\binom{n-2}{1} + \binom{n-2}{0} = \binom{n-1}{1}$$

which is a number of $(n-2)$-tuples recurrence sums

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$$\binom{n-2}{n-1} + \binom{n-2}{n-2} = \binom{n-1}{n-1}$$

which is a number of 1-tuple recurrence sum.

Notice that a number of these sums are equal to coefficients of the binomial formula $(a+b)^{n-1}$, respectively.

This completes the proof.

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