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FIXED POINTS AND QUASI-EQUILIBRIUM PROBLEMS IN NONCOMPACT *L*-CONVEX SPACES*

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In this paper, some new fixed point theorems are first proved in noncompact *L*-convex spaces. As applications, some new existence theorems of general quasi-equilibrium problems are obtained in noncompact *L*-convex spaces. These theorems improve and generalize a number of important known results in recent literature.

Key Words: Fixed Point; General Quasi-equilibrium Problem; Compactly Local Intersection Property; *L*-Convex Space

1. INTRODUCTION

Let X and Y be nonempty sets and 2^X be the family of all subsets of X. Let $T: X \to Y$ be a single-valued mapping, $A: X \to 2^X$ be a set-valued mapping and $\phi: X \times Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The general quasi-equilibrium problem $GQEP$ (T, A, ϕ) is to find $\hat{x} \in X$ such that

$$\begin{cases} \hat{x} \in A(\hat{x}) \\ \phi(\hat{x}, T\hat{x}, y) \leq 0, \qquad y \in A(\hat{x}). \end{cases}$$

The $GQEP$ (T, A, ϕ) was introduced and studied by Ding[1] which includes various classes of equilibrium problems and various variational and quasi-variational inequality problems studied by many authors in topological vector spaces as special cases, see, e.g., References 2-7 and the references therein.

In this paper, we first prove some new fixed point theorems in noncompact L-convex spaces. As applications, some new existence theorems of equilibrium points for the $GQEP$ (T, A, ϕ) are obtained in noncompact L-convex spaces. These results include many known key results in the fields as special cases.

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2. Preliminaries

Let X and Y be two nonempty sets. We denote by 2^Y and $\mathcal{F}(X)$ the family of all subsets of Y and the family of all nonempty finite subsets of X respectively. If X is a topological space a subset A of X is said to be compactly open (resp., compactly closed) in X if for any nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. The following notions were introduced by Ding[8]. For any given nonempty subset A of X, we define the compact closure and the compact interior of A, denoted by ccl (A) and cint (A) as

$$\text{ccl}\,(A) = \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\},$$

$$\text{cint}\,(A) = \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}$$

respectively. it is easy to see that cint (A) (resp., ccl (A)) is compactly open (resp. compactly closed) in X and for each nonempty compact subset K of X with $A \cap K \neq \varnothing$, we have ccl (A) $\cap K = cl_K(A \cap K)$ and cint $(A) \cap K = \text{int}_K(A \cap K)$ where $\text{cl}_K(A \cap K)$ and int_K $(A \cap K)$ denote the closure and the interior of $A \cap K$ in K respectively. It is clear that a subset A of X is compactly open (resp., compactly closed) in X if and only if cint $(A) = A$ (resp., ccl $(A) = A$). If X and Y are two topological spaces and $G : X \to 2^Y$ is a set-valued mapping, G is said to be transfer compactly open-valued (resp., transfer compactly closed-valued) on X if for $x \in X$ and for each compact subset K of Y with $G(x) \cap K \neq \varnothing, y \in G(x) \cap K$ (resp. $y \notin G(x) \cap K$) implies that there exists a point $x' \in X$ such that $y \in \text{int}_K$ $(G(x') \cap K)$ (resp., $y \notin \text{cl}_K$ $(G(x') \cap K)$). Clearly, each open-valued (resp., closed-valued) mapping $G : X \to 2^Y$ is transfer open-valued (resp., transfer closed-valued) (see the Definitions 6 and 7 of Tian[9]) and is also compactly open-valued (resp., compactly closed-valued). Each transfer open-valued (resp., transfer closed-valued) mapping $G : X \to 2^Y$ is transfer compactly open-valued (resp., transfer compactly closed-valued) and the inverse is not true in general. A mapping $G : X \to 2^Y$ is said to have the local intersection property on X if for each $x \in X$ with $G(x) \neq \varnothing$, there exists an open neighbourhood $N(x)$ of x in X such that $\bigcap_{z \in N(x)} G(z) \neq \varnothing$ (see Wu and Shen[10]). The Example 10 (page 65) shows that a set-valued mapping with the local intersection property may not have the property of open inverse values. Now, we introduce the following new notion for set-valued mappings. $G : X \to 2^Y$ is said to have the compactly local intersection property on X if for each nonempty compact subset K of X and for each $x \in K$ with $G(x) \neq \varnothing$, there exists a open neighbourhood $N(x)$ of x in X such that $\bigcap_{z \in N(x) \cap K} G(z) \neq \varnothing$. Clearly, if G has the compactly local intersection property, then for any compact subset K of X, the restriction $G|_K : K \to 2^Y$ of G on K has the local intersection property. It is also clear that each set-valued mapping with the local intersection property has the compactly local intersection property and the inverse is not true in general.

The notion of a L-convex space was introduced by Ben-El-Mechaiekh *et al.*[11]. An L-convexity structure on a topological space X is given by a nonempty set-valued mapping $\Gamma : \mathcal{F}(X) \to 2^X$ satisfying the following condition.

(1) for each $A \in \mathcal{F}(X)$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{F}(A)$ with $|B| = J + 1$, implies $\phi_A(\Delta_J) \subset \Gamma(B)$, where Δ_J denotes the face of Δ_n corresponding $B \in \mathcal{F}(A)$. The pair (X, Γ) is then called an L-convex space. A set $D \subset X$ is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset D$. A function $f : X \to \mathbf{R} \cup \{\pm\infty\}$ is said to be L-quasiconcave (resp., L-quasiconvex) if for any $\lambda \in \mathbf{R}$, the set $\{x \in X : f(x) > \lambda\}$ (resp., $\{x \in X : f(x) > \lambda\}$) is L-convex.

If an L-convex space (X, Γ) satisfies the additional condition:

(2) for each $A, B \in \mathcal{F}(X)$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$, then the pair (X, Γ) is called by Park and Kim[12] a generalized convex (or G-convex) space.

If an L-convex space (X, Γ) satisfies the additional condition:

(3) for each $A, B \in \mathcal{F}(X)$, there exists $A_1 \subset A$ such that $A_1 \subset B$ implies $\Gamma(A_1) \subset \Gamma(B)$, then (X, Γ) is called by Verma[13] a generalized H-space (or G-H-space).

It is clear that the notion of L-convex space includes the G-convex spaces, G-H-spaces, H-spaces and many topological spaces with various convexity structure as special cases and the converse are not true in general. (see Park and Kim[12]).

In order to prove main theorems, we need the following result which is Lemma 1.1 of Ding[14]

Lemma 2.1 — Let X and Y be topological spaces and $G : X \to 2^Y$ a set-valued mapping with nonempty values. Then the following conditions are equivalent:

(*i*) G has the compactly local intersection property,

(ii) for each compact subset K of X and for each $y \in Y$, there exists a open subset O_y of X (which may be empty) such that $O_y \cap K \subset G^{-1}(y)$ and $K = \bigcup_{y \in Y} (O_y \cap K)$,

(iii) for any compact subset K of X, there exists a set-valued mapping $F : X \to 2^Y$ such that $F(x) \subset G(x)$ for each $x \in X$, $F^{-1}(y)$ is open in X and $F^{-1}(y) \cap K \subset G^{-1}(y)$ for each $y \in Y$ and $K = \bigcup_{y \in Y} (F^{-1}(y) \cap K)$.

(iv) for each compact subset K of X and for each $x \in K$, there exists $y \in Y$ such that $x \in$ int $(G^{-1}(y)) \cap K$ and

$$K = \bigcup_{y \in Y} \text{int}_K (G^{-1}(y) \cap K),$$

(v) $G^{-1} Y \to 2^X$ is transfer compactly open-valued on X.

Lemma 2.2 — Let X and Y be topological spaces, D be a nonempty compactly closed subset of X and $\Phi, \Psi : X \to 2^Y$ be two set-valued mappings with nonempty values such that $\Phi(x) \subset \Psi(x)$ for each $x \in X$. Suppose that $\Phi^{-1}, \Psi^{-1} : Y \to 2^X$ are both transfer compactly open-valued on Y. Then the mapping $G : X \to 2^Y$ defined by

$$G(x) = \begin{cases} \Phi(x), & \text{if } x \in D \\ \Psi(x), & \text{if } x \in X \setminus D \end{cases}$$

is such that $G^{-1} : Y \to 2^X$ is also transfer compactly open-valued on Y.

PROOF : For any given $y \in Y$, we have $\Phi^{-1}(y) \subset \Psi^{-1}(y)$ since $\Phi(x) \subset \Psi(x)$ for each $x \in X$. It follows that

$$\begin{aligned} G^{-1}(y) &= \{x \in X : y \in G(x)\} \\ &= \{x \in D : y \in \Phi(x)\} \cup \{x \in X \setminus D : y \in \Psi(x)\} \\ &= (D \cap \Phi^{-1}(y)) \cup [(X \setminus D) \cap \Psi^{-1}(y)] \\ &= [(D \cap \Phi^{-1}(y)) \cup (X \setminus D)] \cap [(D \cap \Phi^{-1}(y)) \cup \Psi^{-1}(y)] \\ &= [X \cap (\Phi^{-1}(y) \cup (X \setminus D))] \cap [(D \cup \Psi^{-1}(y)) \cap (\Phi^{-1}(y) \cup \Psi^{-1}(y))] \\ &= [\Phi^{-1}(y) \cup (X \setminus D)] \cap \Psi^{-1}(y) \end{aligned}$$

$$= \Phi^{-1}(y) \cup [(X \setminus D) \cap \Psi^{-1}(y)].$$

Hence for each nonempty compact subset K of X, we have

$$G^{-1}(y) \cap K = (\Phi^{-1}(y) \cap K) \cup [(X \setminus D) \cap \Psi^{-1}(y) \cap K].$$

If $x \in G^{-1}(y) \cap K$, then we have that either $x \in \Phi^{-1}(y) \cap K$ or $x \in (X \setminus D) \cap \Psi^{-1}(y) \cap K$. If $x \in \Phi^{-1}(y) \cap K$, noting that Φ^{-1} is transfer compactly open-valued on Y, there exists $y' \in Y$ such that

$$x \in \text{int}_K(\Phi^{-1}(y') \cap K) \subset \text{int}_K(G^{-1}(y') \cap K).$$

If $x \in (X \setminus D) \cap \Psi^{-1}(y) \cap K$, noting that $X \setminus D$ is an open set and Ψ^{-1} is transfer compactly open-valued on Y, by using similar argument, we can show that there exists $y' \in Y$ such that $x \in \text{int}_K(G^{-1}(y') \cap K)$. This proves that G^{-1} is transfer compactly open-valued on Y.

Remark 2.2 : Lemma 2.2 improves and generalized Lemma 1.4 of Ding[4] and Proposition 4.1 of Cubiotti.

3.3. Fixed Point Theorems

In this section, we shall show some new fixed point theorems in noncompact L-convex spaces which are new generalizations of the famous Fan-Browder type fixed point theorems.

Theorem 3.1 — *Let* (X, Γ) *be a* L-convex space, K be a nonempty compact subset of X and $G : X \to 2^X$ be a set-valued mapping such that

(*i*) G satisfies one of the conditions (*i*)-(*v*) in Lemma 2.1,

(*ii*) for each $x \in X$, $G(x)$ is nonempty L-convex,

(*iii*) for each $N \in \mathcal{F}(X)$, there exists nonempty compact L-convex subset L_N of X containing N such that

$$L_N \setminus K \subset \bigcup_{y \in L_N} \text{cint}\, G^{-1}(y).$$

Then G has a fixed point in X.

PROOF : By (*ii*), we have $X = \bigcup_{y \in X} G^{-1}(y)$. By (*i*) and Lemma 2.1 (*iv*) with $X = Y$, we have

$$K = \bigcup_{y \in X} (\text{cint}\, G^{-1}(y) \cap K) = \bigcup_{y \in X} \text{int}_K (G^{-1}(y) \cap K).$$

Since K is compact, there exists a finite set $N = \{y_0, y_1, ..., y_n\} \in \mathcal{F}(X)$ such that

$$K = \bigcup_{i=0}^{n} (\text{cint}\, G^{-1}(y_i) \cap K) = \bigcup_{i=0}^{n} \text{cint}\, G^{-1}(y_i). \qquad ... (3.1)$$

Let L_N be the compact L-convex subset of X in the condition (*iii*). Then, by the condition (*iii*) and (3.1), we have $N \subset L_N$ and

$$L_N \bigcup_{y \in L_N} (\text{cint}\, G^{-1}(y) \cap L_N) = \bigcup_{y \in L_N} \text{int}_{L_N} (G^{-1}(y) \cap L_N).$$

Since L_N is compact, there exists $M = \{z_0, z_1, ..., z_m\}$ $\mathcal{F}(L_N)$ such that

$$L_N = \bigcup_{j=0}^{m} (\text{cint}\, G^{-1}(z_j) \cap L_N) = \bigcup_{j=0}^{m} \text{int}_{L_N} (G^{-1}(z_j) \cap L_N). \qquad ... (3.2)$$

Since L_N is also a L-convex space, there exists a continuous mapping $\phi_M : \Delta_m \to \Gamma(M)$ such that

$$\phi_M(\Delta_J) \subset \Gamma(B), \quad B \in \mathcal{F}(M), \ |B| = J + 1. \qquad ... (3.3)$$

Since L_N is compact and $\left\{ \text{int}_{L_N} (G^{-1}(z_j) \cap L_N) \right\}_{j=0}^{m}$ is a open cover of L_N, let $\{\psi_j\}_{j=0}^{m}$ is the continuous partition of unity subordinated to the open cover, then we have that for each $j \in \{0, 1, ..., m\}$ and $x \in L_N$,

$$\psi_j(x) \neq 0 \Leftrightarrow y \in \text{int}_{L_N} (G^{-1}(z_j) \cap L_N) \subset G^{-1}(z_j). \qquad ... (3.4)$$

Define a mapping $\psi : L_N \to \Delta_m$ by

$$\psi(x) = \sum_{j=0}^{m} \psi_j(x)\, e_i, \quad \forall\, x \in L_N, \qquad ... (3.5)$$

Note that $M \subset L_N$ and L_N is L-convex, we have $\Gamma(M) \subset L_N$ and hence, by (3.3) and (3.5), the function $\psi \circ \phi_M : \Delta_m \to \Delta_m$ is continuous. By Browaer fixed point theorem, there exists $z \in \Delta_m$ such that $z = \psi \circ \phi_M(z)$. Let $u = \phi_M(z)$, then we have $u = \phi \circ \psi_M(u)$. It follows from (3.5) that

$$\psi(u) = \sum_{j \in J(u)} \psi_j(u)\, e_j \in \Delta_{J(u)},$$

where $J(u) = \{j \in \{0, 1, ..., m\} : \psi_j(u) \neq 0\}$ and hence, by (3.3), we have

$$u = \phi_M \circ \psi(u) \in \phi_M(\Delta_{J(u)}) \subset \Gamma(\{z_j : j \in J(u)\}). \qquad \text{... (3.6)}$$

It follows from (3.4) that $u \in G^{-1}(z_j)$ for all $j \in J(u)$ and so $\{z_j : j \in J(u)\} \subset G(u)$. Since $G(u)$ is L-convex, by (3.6), we obtain

$$u \in \Gamma(\{z_j : j \in J(u)\}) \subset G(u),$$

i.e., u is fixed point of G.

Remark 3.1 : Theorem 3.1 is a variant of Theorem 2.3 of Ding[6] in noncompact L-convex spaces. If X is compact, by letting $X = K = L_N$ for each $N \in \mathcal{F}(X)$, the condition (*iii*) is satisfied automatically. Hence Theorem 3.1 also generalizes Theorem 2 of Lin and Park[5] from compact setting of G-convex spaces to noncompact setting of L-convex spaces.

Corollary 3.1 — Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X and $G : X \to 2^X$ be such that

(*i*) for each $y \in X, G^{-1}(y)$ is compact open in X,

(*ii*) for each $x \in X, G(x)$ is nonempty L-convex,

(*iii*) for each $N \in \mathcal{F}(X)$, there exists a compact L-convex subset L_N of X containing N such that

$$L_N \setminus K \subset \bigcup_{y \in L_N} G^{-1}(y).$$

Then G has a fixed point in X.

PROOF : The condition (i) implies that G^{-1} is transfer compactly open-valued and cint $G^{-1}(v) = G^{-1}(y)$. So the conclusion of Corollary 3.1 follows from Theorem 3.1.

4. EXISTENCE OF $GQEP\ (T, A, \phi)$

We first prove the following equilibrium existence theorem of $GQEP\ (T, A, \phi)$.

***Theorem* 4.1** — *Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X and Y be a nonempty set. Let $T : X \to Y, A : X \to 2^X$ and $\phi : X \times Y \times X \to \mathbf{R} \cup \{\pm \infty\}$ be such as:*

(i) A has nonempty L-convex values and satisfies one of the conditions (i)-(v) in Lemma 2.1.

(ii) the set $D = \{x \in X : x \in A(x)\}$ is compactly closed in X.

(iii) the mappings P, $B : X \to 2^X$ defined by

$$P(x) = \{y \in X : \phi(x, Tx, y) > 0\},$$

$$B(x) = \{y \in A(x) : \phi(x, Tx, y) > 0\}$$

both have the compactly local intersection property,

(iv) for each $x \in X, y \mapsto \phi(x, Tx, y)$ is L-quasiconcave and $\phi(x, Tx, x) \leq 0$,

(v) for each $N \in \mathcal{F}(X)$, there exists a nonempty compact L-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, if $x \notin D$, then there exists $y \in L_N$ such that $x \in cint\, A^{-1}(y)$; if $x \in D$, then there is $y \in L_N$ such that $x \in cint\, (\{x \in A^{-1}(y) : \phi(x, Tx, y) > 0\})$.

Then there exists $\hat{x} \in X$ such that

$$\left.\begin{array}{ll} \hat{x} \in A(\hat{x}) & \\ \phi(\hat{x}, T\hat{x}, y) \leq 0, & y \in A(\hat{x}), \end{array}\right\}$$

i.e., $\hat{x}$ is a equilibrium point of the GQEP (T, A, ϕ).

PROOF : Define a mapping $G : X \to 2^X$ by

$$G(x) = \begin{cases} B(x), & \text{if } x \in D \\ A(x), & \text{if } x \in X \setminus D. \end{cases}$$

From the condition (iii) and Lemma 2.1, it follows that the mappings $B^{-1}, A^{-1} : X \to 2^X$ are both transfer compactly open-valued on X. Note that $B(x) \subset A(x)$ for each $x \in X$, by Lemma 2.2,

$G^{-1}: X \to 2^X$ is also transfer compactly open-valued on X. By the assumptions (*i*) and (*iv*), for each $x \in X, G(x)$ is L-convex. Now assume that for each $x \in D, B(x) = A(x) \cap P(x) \neq \emptyset$, then for each $x \in X, G(x) \neq \emptyset$ by (*i*). It is easy to see that the condition (*v*) implies that the condition (*iii*) of Theorem 3.1 holds. Hence all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$. By the definition of D and G, we must have $\{x \in X : x \in G(x)\} \subset D$. It follows that $\hat{x} \in A(\hat{x}) \cap P(\hat{x}) \cap D$. In particular, we obtain $\phi(\hat{x}, T\hat{x}, \hat{x}) > 0$ which contradicts the assumption $\phi(x, Tx, x) \leq 0$ for each X in (*iv*). Therefore there exists $\hat{x} \in D$ such that $A(\hat{x}) \cap P(\hat{x}) = \emptyset$, that is $\hat{x} \in A(\hat{x})$ and $\phi(x, T\hat{x}\, y) \leq 0$ for an $y \in A(\hat{x})$. This completes the proof.

Remark 4.1 : Theorem 4.1 is a variant of Theorem 2.1 of Ding[14] in noncompact L-convex spaces.

Corollary 4.1 — Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X and Y be a nonempty set. Let $T: X \to Y, A: X \to 2^X$ and $f: X \times Y \to \mathbf{R} \cup \{\pm\infty\}$ be such that

(*i*) A has nonempty L-convex values and satisfies one of the conditions (*i*)-(*v*) in Lemma 2.1.

(*ii*) the set $D = \{x \in X : x \in A(x)\}$ is compactly closed.

(*iii*) the mapping $P, B: X \to 2^X$ defined by

$$P(x) = \{y \in X : f(x, Tx) - f(y, Tx) > 0\},$$

$$B(x) = \{y \in A(x) : f(x, Tx - f(y, Tx) > 0\}$$

both have the compactly locally intersection property,

(*iv*) for each $x \in X, y \mapsto f(x, Tx)$ is L-quasiconvex,

(*v*) for each $N \in \mathcal{F}(X)$, there exists a nonempty compact L-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, if $x \notin D$, then there exists $y \in L_N$ such that $x \in \operatorname{cint} A^{-1}(y)$; if $x \in D$, then there is $y \in L_N$ such that $x \in \operatorname{cint}(\{x \in A^{-1}(y) : f(x, Tx) - f(y, Tx) > 0\})$.

Then there exists $\hat{x} \in X$ such that

$$\left.\begin{array}{l} \hat{x} \in A(\hat{x}) \\ f(\hat{x}, T\hat{x}) \leq f(y, T\hat{x}) \quad y \in A(\hat{x}). \end{array}\right\}$$

PROOF : Define a mapping $\phi: X \times Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\phi(x, z, y) = f(x, z) - f(y, z), \quad \forall (x, z, y) \in X \times Y \times X.$$

It is easy to see that all conditions of Theorem 4.1 are satisfied. The conclusion of Corollary 4.1 follows from Theorem 4.1.

Remark 4.2 : Corollary 4.1 improves and generalized Theorem 2.1 of Ding[4], Theorem 4.2 of Cubiotti[3] and Theorem 3.1 of Ding[16] from topological vector spaces and G-convex space to very general L-convex spaces without linear structure under much weaker assumptions. Hence Theorem 4.1 further generalizes the above results.

Theorem 4.2 — *Let* (X, Γ) *be a L-convex space, K be a nonempty compact subset of X and Y be a topological space. Let* $T : X \to Y, A : X \to 2^X$ *and* $\phi : X \times Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ *be such that*

(i) A has a nonempty L-convex values such that $A^{-1} : X \to 2^X$ *has transfer compactly open values,*

(ii) the set $D = \{x \in X : x \in A(x)\}$ *is compactly closed in X,*

(iii) for each $y \in X, y \mapsto \phi(x, Tx, y)$ *is L-quasiconcave and* $\phi(x, Tx, x) \leq 0$,

(iv) for each $N \in \mathcal{F}(X)$, *there exists a nonempty compact L-convex subset* L_N *of X containing N such that for each* $x \in L_N \setminus K$, *if* $x \notin D$, *then* $A(x) \cap L_N \neq \phi$; *if* $x \in D$, *then there is* $y \in A(x) \cap L_N$ *satisfying* $\phi(x, Tx, y) > 0$.

Then the GQEP (T, A, ϕ) *has an equilibrium point* $\hat{x} \in X$.

PROOF : Define $P : X \to 2^X$ by

$$P(x) = \{y \in X : \phi(x, Tx, y) > 0\}, \qquad \forall x \in X.$$

By the condition (*iii*), we have that for each $y \in X, P^{-1}(y) = \{x \in X : \phi(x, Tx, y) > 0\}$ is compactly open on X and hence $P^{-1} : X \to 2^X$ has compactly open values. By the assumption, A^{-1} has compactly open values and hence the mapping $B^{-1} = (A \cap P)^{-1} = A^{-1} \cap P^{-1}$ also have compactly open values. By Lemma 2.1, the condition (*iii*) of Theorem 4.1 is satisfied. By the condition (*iv*), for each $x \in L_N \setminus K$, if $x \notin D$, we have $A(x) \cap L_N \neq \phi$ and hence there exists $y \in L_N$ such that $x \in A^{-1}(y) = \text{cint}\, A^{-1}(y)$ since $A^{-1}(y)$ is compactly open; if $x \in D$, we have $y \in L_N$ and $x \in A^{-1}(y) \cap P^{-1}(y) = \text{cint}\,(A^{-1}(y) \cap P^{-1}(y)) = \text{cint}\,(\{x \in A^{-1}(y) : f(x, Tx) - f(y, > Tx)\ 0\})$, since the set $A^{-1}(y) \cap P^{-1}(y)$ is compactly oρen. Hence the condition (v) of Theorem 4.1 is satisfied. It is easy to see that all conditions of Theorem 4.1 are satisfied. By Theorem 4.1, the Conclusion of Theorem 4.2 holds.

***Theorem* 4.3** — *Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X and Y be a topological space. Let $T: X \to Y$, $A: X \to 2^X$ and $\phi: X \times Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ be such that*

(i) A has a nonempty L-convex values such that $A^{-1}: X \to 2^X$ has compactly open values and the mapping $cl\,A: X \to 2^X$, defined by $(cl\,A)(x) = cl\,A(x)$, is upper semicontinuous,

(ii) for each $y \in Y, y \mapsto \phi(x, Tx, y)$ is lower semicontinuous on each compact subset of X and for each $y \in X, y \mapsto \phi(x, Tx, y)$ is L-quasiconcave and $\phi(x, Tx, x) \le 0$,

(iii) for each $N \in \mathcal{F}(X)$, there exists a nonempty compact L-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, if $x \notin \overline{D}, = \{x \in X : x \in clA(x)\}$, then $A(x) \cap L_N \neq \emptyset$; if $x \notin \overline{D}$, then there is $y \in A(x) \cap L_N$ satisfying $\phi(x, Tx, x) \ge 0$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in clA(\hat{x})$ and $\phi(\hat{x}, T\hat{x}, y) \le 0, \quad \forall\, y \in A(\hat{x})$.

PROOF : Since $clA: X \to 2^X$ is upper semicontinuous with closed values, the set $\overline{D} = \{x \in X,\ x \in clA(x)\}$ must be closed in X. By using $\overline{D}$ in place of D in Theorem 4.2, it easy to see that all conditions of Theorem 4.2 are satisfied. By Theorem 4.2, there exists $\hat{x} \in X$ such that $\hat{x} \in clA(\hat{x})$ and $\phi(\hat{x}, T\hat{x}, y) \le 0, \quad \forall\, y \in A(\hat{x})$.

Corollary 4.2 — Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X. Let $A: X \to 2^X$ and $f: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ be such that

(*i*) A satisfies the condition (*i*) of Theorem 4.3,

(*ii*) f is continuous on each compact subset of X such that for each $y \in X, y \mapsto f(y, x)$ is L-quasiconvex,

(*iii*) for each $N \in \mathcal{F}(X)$, there exists a nonempty compact L-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, if $x \notin \overline{D}, = \{x \in X : x \in clA(x)\}$, then $A(x) \cap L_N \neq \emptyset$; if $x \notin \overline{D}$, then there is a $y \in A(x) \cap L_N$ such that $f(y, x) < f(x, x)$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in clA(\hat{x})$ and $f(\hat{x}, \hat{x}) \le f(y, \hat{x})$, $\forall\, y \in A(\hat{x})$. If we further assume that $f(x, x) \ge 0 \ \forall\, x \in X$, then we have that $\hat{x} \in clA(\hat{x})$ and $f(y, \hat{x}) \ge 0, \ \forall\, y \in A(\hat{x})$.

PROOF : Define a mapping $\phi(x, z, y) = f(x, z) - f(y, z)$ for all $(x, z, y) \in X \times Y \times X$. By letting $X = Y$ and T being the identity mapping, it is easy to see that all conditions of Theorem 4.3 are satisfied. By Theorem 4.3, there exists $\hat{x} \in X$ such that $\hat{x} \in \mathrm{cl}A(\hat{x})$ and $f(\hat{x},\hat{x})-f(y,\hat{x}) = \phi(\hat{x},\hat{x},y) \leq 0$, $\forall\, y \in A(\hat{x})$, i.e., $f(\hat{x},\hat{x}) \leq f(y,\hat{x})$, $\forall\, y \in A(\hat{x})$. If $f(x,x) \geq 0$, $\forall\, x \in X$, we must have $\hat{x} \in clA(\hat{x})$ and $f(y,\hat{x}) \geq 0$, $\forall\, y \in A(\hat{x})$.

Remark 4.2 : If (X, Γ) is a compact *L*-convex space, by letting $X = K = L_N$ for each $N \in \mathcal{F}(X)$, then the condition (*iii*) of Theorem 4.3 and Corollary 4.2 are satisfied trivially. Theorem 4.3 and Corollary 4.2 generalize Theorem 3.2 of Ding[15] from *G*-convex space to *L*-convex spaces under weaker assumptions and in turn, generalize Theorem 4 of Lin and Park[5] from the compact setting of *G*-convex spaces to noncompact setting of *L*-convex spaces.

From Corollary 4.2, we obtain the following existence result for generalized quasi-equilibrium problems.

***Theorem* 4.4** — *Let (X, Γ) be a L-convex space, K be a nonempty compact subset of X and Y be a topological space. Let $T: X \to 2^Y$ have a continuous selection $g: X \to Y$. Let $A: X \to 2^X$ and $\phi: X \times Y \times X \to \mathbf{R} \cup \{\pm\infty\}$ be such that*

(i) A satisfies the conditions (i) of Theorem 4.3,

(ii) ϕ is a continuous function such that for each $(x,y) \in X \times Y, z \mapsto \phi(x,y,z)$ is L-quasiconvex,

(iii) for each $N \in \mathcal{F}(X)$, there exists a nonempty compact L-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, if $x \notin \overline{D}$, $= \{X \in X : x \in \mathrm{cl}A(x)\}$, then $A(x) \cap L_N \neq \emptyset$; if $x \notin \overline{D}$, then there is a $y \in A(x) \cap L_N$ satisfying $\phi(x, g(x), y) < \phi(x, g(x), x)$.

Then there exists $\hat{x} \in X$ and $\hat{y} = g(\hat{x}) \in T(\hat{x})$ such that

$$\begin{cases} \hat{x} \in \mathrm{cl}A(\hat{x}) \\ \phi(\hat{x},\hat{y},\hat{x}) \leq 0, \quad \phi(\hat{x},\hat{y},\hat{z}) \qquad z \in A(\hat{x}). \end{cases}$$

If further assume $\phi(x, g(x), x) \geq 0$ for all $x \in X$, then we obtain

$$\begin{cases} \hat{x} \in \mathrm{cl}A(\hat{x}) \\ \phi(\hat{x},\hat{y},z) \geq 0, \qquad z \in A(\hat{x}). \end{cases}$$

PROOF : Define $f: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f(z,x) = \phi(x, g(x), z), \quad \forall (z,x) \in X \times X.$$

Then the conclusion of Theorem 4.4 holds from Corollary 4.2.

Remark 3.3 : Theorem 4.4 generalizes Theorem 3.4 of Ding[15] and Corollary 5 of Lin and Park[5] to noncompact setting of *L*-convex spaces and Theorem 3.1 of Chang *et al.*[16] in several aspects. Some applications of our fixed point theorems and existence theorems for quasi-equilibrium problems in this papers will be given in other papers.

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