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THE UNIQUENESS OF ENTIRE FUNCTIONS THAT SHARE A SMALL FUNCTION WITH ITS DIFFERENTIAL POLYNOMIALS*

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In this paper, we investigate the relationship between an entire function and its linear differential polynomial when they share one small function by applying the value distribution theory and complex oscillation theory. As a consequence of the main result, we generalize the study of Gundersen and Yang and the proof is different from theirs.

Key Words: Entire Functions; Differential Polynomial; Small Function; Uniqueness

1. INTRODUCTION AND MAIN RESULTS

In this paper, by meromorphic function we always mean a meromorphic function in the complex plane $\mathbb{C}$. We take for granted the usual notations in Nevanlinna theory[6]. Let f and g be two non-constant meromorphic functions. By $S(r,f)$ we denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of r with finite linear measure. Then the meromorphic function α is called a small function of f if $T(r,\alpha)=S(r,f)$. If $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities, then we say that f and g share the small function α CM. In addition, we will use $\sigma(f)$ and $\sigma_2(f)$ to denote the order of growth and hyper-order of f respectively. For a set $E\subset \mathbb{R}^+$, let mes(E) and lm(E) denote the linear and the logarithmic measure of E respectively. The upper and the lower logarithmic density of E are defined by

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$$\overline{\log\text{dens}}\, E = \overline{\lim_{r\to\infty}} \left(\int_1^r (\chi_E(t)/t)\, dt \right) / \log r,$$

$$\underline{\log\text{dens}}\, E = \underline{\lim_{r\to\infty}} \left(\int_1^r (\chi_E(t)/t)\, dt \right) / \log r$$

where χ_E is the characteristic function of E.

Consider the uniqueness of an entire function f and its derivative f' that share one value a. The problem has attracted many investigations. It is well known that R. Brück give a conjecture in 1995: Let f be a non-constant entire function with $\sigma_2(f)$ be finite and not a positive integer. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$ where c is a non-zero constant[1]. He proved that the conjecture holds for $a = 0$ or $N(r, 0, f') = S(r, f)$ in the same paper. In addition, he gave some counterexamples to show that the conjecture fails when f and f' share a ignoring the multiplicity and that the restriction on the growth of f is necessary. After his work Zhang extended the result by allowing a weaker restrictive condition on the zeros of f'[12]. In 1998, the conjecture is partly solved by Gundersen and Yang. They proved the case that f is of finite order[4]. Soon, Yang generalized it to kth derivative[11]. However, using the same method in [4] and [11], we can not get further works. Hence, there is no corresponding result about the uniqueness of an entire function that shares a small function CM with its kth derivative or its differential polynomials. In this paper, we have tried the problem by a new method. In the sequel, we get

$$L[f] = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_0 f + \beta, \quad (k > 0) \qquad \ldots (*)$$

where $a_j\, (j = 0, 1, \ldots, k)$ are constants with $a_k \neq 0$, β is a small function of f. In the present paper, we prove the following theorem which is an improvement and generalization of G. Gundersen and Yang's results. Indeed, we shall prove:

***Theorem* 1** — *Let f be a non-constant entire function of finite order and $\sigma(f) \neq 1$ and let α be a non-zero small function of f. If f and $L[f]$ share α CM, then*

$$\frac{L[f] - \alpha}{f - \alpha} = c, \qquad \ldots (1.1)$$

where c is a non-zero constant.

It is easy to see that the order restriction $\sigma(f) \neq 1$ in Theorem 1 is sharp by the following two examples shown for two cases $\beta \not\equiv 0$ and $\beta \equiv 0$ respectively.

Example 1 — Let $f(z) = e^{-z} + 1$ and $L[f] = \sum_{i=0}^{2n-1} f^{(i)} + c$, where $c \neq 0$ is a constant and n is a positive integer. Then $L[e^{-z} + 1] = c + 1$ and

$$\frac{L[f](z) - 1}{f(z) - 1} = ce^{z}.$$

Example 2 — Let $f(z) = e^{2z} + 1$ and $L[f] = f''' - f'' + f' - 6f$. Then

$$\frac{L[f](z) - 1}{f(z) - 1} = -7e^{-2z}.$$

2. Preliminary Lemmas

We need the following lemmas in the proof of Theorem 1.

Lemma 1([8]) — If g is an entire function of order σ, and let $\nu_g(r)$ be the central index of g, then

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\log \nu_g(r)}{\log r}.$$

Lemma 2 ([2]) — Suppose that w is a meromorphic function with $\sigma(w) = \eta < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_1 \subset (1, +\infty)$ that has finite linear measures and finite logirithmic measure, such that

$$|w(z)| \leq \exp\left\{r^{\eta + \varepsilon}\right\}$$

holds for

$$|z| = r \notin [0, 1] \cup E_1, r \to \infty.$$

Applying Lemma 2 to $\frac{1}{w}$, it is easy to see that for any given $\varepsilon > 0$, there is a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure, such that

$$\exp\left\{-r^{\eta + \varepsilon}\right\} \leq |w(z)| \leq \exp\left\{r^{\eta + \varepsilon}\right\},$$

holds for

$$|z| = r \notin [0, 1] \bigcup E_2, r \to \infty.$$

Lemma 3 — Let f be a transcendental entire function with $\sigma(f) = \rho < \infty$ and let a set $E_3 \subset [1, \infty)$ have upper logarithmic density 1. Then for any given $\varepsilon > 0$ there exists a point range $\left\{ z_n = r_n e^{i\theta_n} \right\}$ satisfying $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$ and $r_n \in E_3$ such that

$$\rho \geq \varlimsup_{r_n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \geq \varliminf_{r_n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \geq \frac{\rho}{1+\varepsilon}.$$

PROOF : By Lemma 1, we know that there exists a sequence $\left\{ r'_n \right\} \left(r'_n \to \infty \right)$ satisfying $\rho = \lim_{n \to \infty} \frac{\log \nu_f(r'_n)}{\log r'_n}$. Set $H = \bigcup_{n=1}^{\infty} \left[r'_n, (r'_n)^{1+\varepsilon} \right]$. It is easy to see $\overline{\log \text{dens}}\, H \geq \varepsilon$. Moreover,

$$\overline{\log \text{dens}}\, (E_3) + \overline{\log \text{dens}}\, H - \underline{\log \text{dens}}\, (E_3 \cap H) \leq \overline{\log \text{dens}}\, (H - (E_3 \cap H)) + \overline{\log \text{dens}}\, E_3 \leq 1.$$

Thus, $\underline{\log \text{dens}}\, (E_3 \cap H) \geq \varepsilon > 0$. Without loss of generality, we can take $r_n \in \left[r'_n, (r'_n)^{1+\varepsilon} \right] \cap E_3$. Consider r_n, we have

$$\frac{\log \nu_f(r_n)}{\log r_n} \geq \frac{\log \nu_f(r'_n)}{(1+\varepsilon) \log r'_n}$$

since $\nu_f(r)$ is increasing. From this, we can get

$$\varlimsup_{r_n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \geq \frac{\rho}{1+\varepsilon}, \quad \varliminf_{n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \geq \frac{\rho}{1+\varepsilon}.$$

On the other hand, clearly

$$\rho \geq \varlimsup_{n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \geq \varliminf_{n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n},$$

From the above argument, we can get the conclusion.

Lemma 4 ([5]) — Suppose that $T(r)$ is a continuous and non-decreasing positive function on $[r_0, \infty)$ $(r_0 \geq 1)$ whose values are positive and satisfies $T(r) \to \infty$ $(r \to \infty)$. If there exists an increasing sequence $\{r_n\}$, $r_n \uparrow \infty$ $(n \to \infty)$, such that $\lim_{n \to \infty} \frac{\log T(r_n)}{\log r_n} \leq \mu < +\infty$, then for any given τ_1 (> 1) and τ_2 (> 1), we have

$$\underline{\log\ \text{dens}}\ E_4 \geq 1 - \mu \frac{\log \tau_1}{\log \tau_2},$$

where

$$E_4 = \{r : T(\tau_1 r) \leq \tau_2 T(r)\}.$$

Lemma 5 — Let f be a non-constant entire function of order $\sigma(f) = \mu < +\infty$. Suppose that a is a non-zero small function of f. Then exists a set $E_5 \subset (1, \infty)$ satisfying $\underline{\log\ \text{dens}}\ E_5 = 1$, such that

$$\frac{\log^+ M(r, a)}{\log^+ M(r, f)} \to 0, \quad \frac{M(r, s)}{M(r, f)} \to 0$$

holds for

$$|z| = r \in E_5, r -. \infty.$$

PROOF : By Lemma 4, there exists a set $H_1 = \{r \mid T(4r, f) \leq 2^k T(r, f)\}$ with $\underline{\log\ \text{dens}}$ $H_1 \geq 1 - \frac{2\mu}{k}$ where k is an integer satisfying $k \geq \max(\{4, [\mu + 1]\})$. It is obvious that H_1 is a closed set. Set $r_1 = \min\{H_1 \cap [1 + \infty)\}$, $r_2 = \min\{H_1 \cap [2r_1, +\infty)\}$, ..., $r_\upsilon = \min\{H_1 \cap [2_{\upsilon-1}, +\infty)\}$. We can get a sequence $\{r_\upsilon\}$ $(r_\upsilon \to +\infty)$ and $H_1 \subset \bigcup_{\upsilon=1}^{\infty} [r_\upsilon, 2r_\upsilon]$, so

$$\underline{\log\ \text{dens}}\ H_1 \leq \underline{\log\ \text{dens}} \left\{ \bigcup_{\upsilon=1}^{\infty} [r_\upsilon, 2r_\upsilon] \right\}. \qquad \text{... (2.1)}$$

By Lemma 2, there exists a set $H_2 \subset (1, +\infty)$ with finite linear measure, such that a has finite and not zero moduli,

$$\frac{T(r, a)}{T(r, f)} \to 0 \qquad \text{... (2.2)}$$

for $|z| = r \notin [0, 1] \cup H_2, r \to \infty$. Set $H_3 = H_1 - H_2$, then $\underline{\log\ \text{dens}}\ H_3 = \underline{\log\ \text{dens}}\ H_1$.

Let a_s $(s = 1, 2, \ldots, n(3r_\upsilon, a))$ denote the poles of a in $|z| \le 3r_\upsilon$. By the Boutroux-Cartan theorem, we have

$$\prod_{s=1}^{n(3r_\upsilon, a)} |z - a_s| \ge \left(\frac{r_\upsilon}{2^{k-1} e} \right)^{n(3r_\upsilon, a)} \qquad \ldots (2.3)$$

except some z in the finite union γ of disks whose sum of the radii is at most $\frac{r_\upsilon}{2^{k-2}}$. Therefore, there exist circles of radius ρ that have no intersection with γ in $r_\upsilon \le |z| \le 2r_\upsilon$, and (2.3) holds for $|z| = \rho$. Let E_υ^* denote the set of those values of ρ, then mes $E_\upsilon^* \ge \left(1 - \frac{1}{2^{k-3}}\right) r_\upsilon$.

Applying the Poisson-Jensen formula (see [6]), we have

$$\log^+ |a(z)| \le \frac{3r_\upsilon + \rho}{3r_\upsilon - \rho} m(3r_\upsilon, a) + \sum_{|a_s| \le 3r_\upsilon} \log \left| \frac{(3r_\upsilon)^2 - \bar{a}_s z}{3r_\upsilon (z - a_s)} \right| \qquad \ldots (2.4)$$

where $|z| = \rho \in E_\upsilon^*$. Substituting (2.3) into (2.4), we obtain

$$\log^+ |a(z)| \le \frac{3r_\upsilon + \rho}{3r_\upsilon - \rho} m(3r_\upsilon, a) + n(3r_\upsilon, a) \log(3 \cdot 2^{k+1} e)$$

$$\le c_0 T(4r_\upsilon, a) \le c_0 T(4r_\upsilon, f) \le 2^k c_0 T(r_\upsilon, f) \le 2^k c_0 \log^+ M(\rho, f)$$

where c_0 is some positive constant and $|z| = \rho \in E_\upsilon^* - H_2$, $\upsilon \to \infty$. From the above inequality and (2.2), it is easy to see

$$\frac{\log^+ M(\rho, a)}{\log^+ M(\rho, f)} \le 2^k c_0 \frac{T(4r_\upsilon, a)}{T(4r_\upsilon, f)} \to 0, \qquad \ldots (2.5)$$

where $|z| = \rho \in E_\upsilon^* - H_2$, $\upsilon \to \infty$. Since f is a non-constant entire function, we have $M(r, f) \to \infty$ $(r \to \infty)$. Considering this and (2.5), we have

$$\frac{M(\rho, a)}{M(\rho, f)} \to 0, \qquad \ldots (2.6)$$

where

$$|z| = \rho \in E_\upsilon^* - H_2, \ \upsilon \to \infty.$$

Set $E_5 = \bigcup_{\upsilon=1}^{\infty} E_{\upsilon}^{*} - H_2$, then $\underline{\log\ \text{dens}}\ E_5 = \underline{\log\ \text{dens}} \bigcup_{\upsilon=1}^{\infty} E_{\upsilon}^{*}$. Moreover, there exists a sequence $\left\{ r_n' \right\}$, $r_n' \uparrow \infty\ (n \to \infty)$ such that

$$\underline{\log\ \text{dens}}\ E_5 = \lim_{n \to \infty} \left(\int_1^{r_n'} (\chi_{E_5}(t)/t)\, dt \right) / \log r_n'.$$

For every E_{υ}^{*}, we have

$$\int_{r_\upsilon}^{2r_\upsilon} (\chi_{E_\upsilon^*}(t)/t)\, dt \geq \log 2 - \int_{r_\upsilon}^{r_\upsilon + \frac{1}{2^{k-3}} r_\upsilon} 1/_t\, dt = \log \frac{2}{1 + 1/2^{k-3}}. \qquad \text{... (2.7)}$$

Now we discuss the following two cases.

Case 1 — Suppose that $r_n' \in \left[r_{\upsilon_n}, 2r_{\upsilon_n} \right]$ for some υ_n. Clearly we have

$$\frac{\int_1^{r_n'} (\chi_{E_5}(t)/t)\, dt}{\log r_n'} \geq \frac{\int_{r_1}^{r_{\upsilon_n}} (\chi_{E_5}(t)/t)\, dt}{\log 2r_{\upsilon_n}}. \qquad \text{... (2.8)}$$

Case 2 — Suppose that $r_n' \notin \bigcup_{\upsilon=1}^{\infty} \left[r_{\upsilon_n}, 2r_{\upsilon_n} \right]$. Set r_{υ_n} be the closest to r_n' of $\{r_\upsilon\}$ and $r_{\upsilon_n} \geq r_n'$, then

$$\frac{\int_1^{r_n'} (\chi_{E_5}(t)/t)\, dt}{\log r_n'} \geq \frac{\int_{r_1}^{r_{\upsilon_n}} (\chi_{E_5}(t)/t)\, dt}{\log 2r_{\upsilon_n}}. \qquad (2.9)$$

Set mes $H_2 = \delta$, from (2.7), we have

$$\int_{r_1}^{r_{\upsilon_n}} (\chi_{E_5}(t)/t)\, dt = \sum_{\upsilon=1}^{\upsilon_n - 1} \int_{r_\upsilon}^{2r_\upsilon} (\chi_{E_5}(t)/t)\, dt \geq \sum_{\upsilon=1}^{\upsilon_n - 1} \int_{r_\upsilon}^{2r_\upsilon} (\chi_{E_\upsilon^*}(t)/t)\, dt - \frac{\delta}{r_1}$$

$$\geq \frac{1}{\log 2}\log\frac{2}{1+1/2^{k-3}}\sum_{\upsilon=1}^{\upsilon_n-1}\int_{r_\upsilon}^{2r_\upsilon}(\chi_{E_\upsilon^*}(t)/t)\,dt-\frac{\delta}{r_1}.$$

Combining this with (2.8) and (2.9), we obtain

$$\underline{\log\ \text{dens}}\ E_5 \geq \left(1-\frac{2\mu}{k}\right)\frac{1}{\log 2}\log\frac{2}{1+1/2^{k-3}}.$$

We know that $\phi(x)=\frac{1}{\log 2}\log\frac{2}{1+1/2^{x-3}}\left(1-\frac{2\mu}{x}\right)$ is continuous on $[3,\infty)$ and tends to 1 $(x\to\infty)$. Hence, log dens $E_5 = 1$.

3. Proof of Theorem 1

Under the hypothesis of Theorem 1, it follows from the Hadamard factorization theorem that

$$\frac{L[f]-\alpha}{f-\alpha} = e^Q, \qquad \text{... (3.1)}$$

where Q is a polynomial. Clearly, Q is a constant when $\sigma(f)<1$. Hence, in the following we only consider the case that $\sigma(f)>1$. We assume that Q is a non-zero polynomial of deg $Q= m > 0$. Rewrite (3.1) as

$$a_k\frac{f^{(k)}}{f}+a_{k-1}\frac{f^{(k-1)}}{f}+\ldots+a_0+\frac{\beta-\alpha}{f}-e^Q\left(1-\frac{\alpha}{f}\right)=0. \qquad \text{... (3.2)}$$

First, we recall a basic property of polynomials (see [9]): Let $Q(z)=b_m z^m+b_{m-1}z^{m-1}+\ldots+b_0$ where $b_m=\alpha_m e^{i\theta_m}$ with $\alpha_m>0$ and $\theta_m\in[0,2\pi)$. For any given ε satisfying $0<\varepsilon<\frac{r}{4m}$, we introduce $2m$ closed angles

$$S_j: -\frac{\theta_m}{m}+(2j-1)\frac{\pi}{2m}+\varepsilon<\theta<-\frac{\theta_m}{m}$$

$$+(2j+1)\frac{\pi}{2m}-\varepsilon \quad (j=0,1,\ldots,2m-1).$$

Then there exists a positive number $R=R(\varepsilon)$ such that if $z\in S_j$ where j is even

$$\text{Re}\ \{Q(z)\} > \alpha_m(1-\varepsilon)\sin(m\varepsilon)r^m \qquad \text{... (3.3)}$$

for $|z| = r > R$; if $z \in S_j$ where j is odd

$$\text{Re } \{Q(z)\} > -\alpha_m (1-\varepsilon) \sin(m\varepsilon) r^m \quad \text{... (3.4)}$$

for $|z| = r > R$.

Next, in view of the Wiman-Valiron theory (see [7, 10]), we have the basic formulas

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1+o(1)) \quad (j = 1, 2) \quad \text{... (3.5)}$$

where $|f(z)| = M(r,f)$ and $|z| = r \notin E_6 \cup [0, 1]$ with $\text{lm } E_6 < \infty$. By Lemma 5, there exists two sets E_7 and E_8 satisfying log dens $E_i = 1 (i = 7, 8)$ such that

$$\frac{M(r, \alpha)}{M(r,f)} \to 0, \quad \frac{M(r, \beta - \alpha)}{M(r,f)} \to 0 \quad \text{... (3.6)}$$

hold for $|z| = r \in E_7 \cap E_8$ and $r \to \infty$. In addition,

$\underline{\text{log dens}}\ E_7 + \underline{\text{log dens}}\ E_8 - \overline{\text{log dens}}\ (E_7 \cap E_8) \le \underline{\text{log dens}}\ (E_7 - (E_7 \cap E_8)) + \underline{\text{log dens}}\ E_8 \le 1.$

Clearly, it leads to $\overline{\text{log dens}}\ (E_7 \cap E_8) = 1$. By Lemma 3, for any given $\varepsilon > 0$, we can take a point range $\left\{ z_n = r_n e^{i\theta_n} \right\}$ satisfying $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$ and $r_n \in E_7 \cap E_8 - E_6$ such that

$$\sigma(f) \ge \varlimsup_{r_n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \ge \lim_{r_n \to \infty} \frac{\log \nu_f(r_n)}{\log r_n} \ge \frac{\sigma(f)}{1+\varepsilon}. \quad \text{... (3.7)}$$

Moreover, there exists a subsequence $\left\{ \theta_{n_j} \right\}$ of $\{\theta_n\}$ such that $\lim_{j \to \infty} \theta_{n_j} = \theta_0$. Without loss of generality, we assume that it is $\{\theta_n\}$. Substituting (3.5) and (3.6) into (3.1), we get for $\left\{ z_n = r_n e^{i\theta_n} \right\}$

$$a_k (\nu_f(r_n))^k + a_{k-1} (\nu_f(r_n))^{k-1} z_n + \ldots + (a_0\, o(1))$$

$$- (1 + o(1))\, z_n^k\, e^{Q(z_n)} = 0. \quad \text{... (3.8)}$$

Now we consider the following three cases.

Case 1 — $\theta_0 \in S_j$ where j is even. Since S_j is an open set, so $\theta_n \in S_j$ for sufficiently large n. From (3.3), we have

$$|\exp\left\{Q(z_n)\right\}| \geq \exp\left\{\delta r_n^m\right\}, \qquad \text{... (3.9)}$$

for sufficiently large n and $\delta > 0$ is a constant. Combining (3.9) with (3.8), we have

$$\left(\sum_{i=0}^{k} 2a_i\right)(\nu_f(r_n))^k \geq \exp\left\{\delta r_n^m\right\},$$

which leads to $\sigma(f) = \infty$ from (3.7). It is a contradiction.

Case 2 — $\theta_0 \in S_j$ where j is odd. Similarly $\theta_n \in S_j$ for sufficiently large n. From (3.4), we have

$$|\exp\left\{Q(z_n)\right\}| \geq \exp\left\{-\delta r_n^m\right\}, \qquad \text{... (3.10)}$$

for sufficiently large n and $\delta > 0$ is a constant. Taking ε small enough such that

$$\frac{\sigma(f)}{1+\varepsilon} \geq 1+\varepsilon. \qquad \text{... (3.11)}$$

From the second estimate of (3.7), we know that

$$\nu_f(r_n) \geq r_n^{1+\varepsilon}, \quad (\nu_f(r_n))^{\kappa} = o(1)\,(\nu_f(r_n))^k \quad (0 < \kappa < k) \qquad \text{... (3.12)}$$

for sufficiently large n. Combining (3.10) and (3.12) with (3.8), we have

$$(a_k + o(1))\,(\nu_f(r_n))^k \leq r_n^k \exp\left\{-\delta r_n^m\right\}.$$

It follows from this and (3.7) and $\sigma(f) = 0$, which contradicts with $\sigma(f) > 1$.

Case 3 — $\theta_0 = \dfrac{\theta_m}{m} + (2j-1)\dfrac{\pi}{2m}$ for some $j \in \{0, 1, ..., 2m-1\}$. Using the same method as above, we can only deduce that Re $\left\{Q(r_n e^{\theta_0})\right\} = 0$. Since the straight line arg $z = \theta_0$ is an asymptotic line of $r_n e^{i\theta_n}$, there is a positive integer N such that when $n > N$ we have $-1 <$ Re $\left\{Q(r_n e^{\theta_0})\right\} < 1$. Thus, we know $\left| e^{Q(r_n e^{i\theta_n})} \right| = c_1$ where $c_1 > 0$ is a constant. Substituting this (3.7) and (3.11) into (3.8), we get

$$(|a_k| + o(1))\, r_n^{k(1+\varepsilon)} \leq c_1,$$

which is a contradiction.

This completes the proof of Theorem 1.

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