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CHARACTERIZATION OF GENERALIZED *B*-SCROLLS AND CYLINDERS OVER FINITE TYPE CURVES

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In this article, we construct a generalized B-scroll in a Minkowski space and we completely characterize it with Gauss map. We then give a characterization of ruled surfaces in Minkowski space with finite type Gauss map.

Key Words : Minkowski Space; Gauss Map; Finite Type; Null Scroll; Generalized *B*-Scroll

1. INTRODUCTION

In late 1970's the notion of finite type immersion into a Euclidean space was introduced by Chen. Quote a lot of works were done in this field of study since then. He also extended the notion of finite type immersion of submanifolds into a pseudo-Euclidean space in 1980's. A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ with signature $(m - s)$ s, is said to be of *finite type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, then M is said to be of *k-type*. Similarly, a smooth map ϕ on an n-dimensional pseudo-Riemannian submanifold M of $\mathbb{E}_s^m$ is said to be of *finite type* if ϕ is a finite sum of $\mathbb{E}_s^m$-valued eigenfunctions of Δ. We also similarly define a smooth map of k-type on M. A very typical interesting smooth map on the submanifold M is the Gauss map which will be discussed in the next section 2.

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Several authors studied submanifolds with finite type Gauss map ([2, 3, 8, 9] etc.). There are many examples of submanifolds in pseudo-Euclidean space $\mathbb{E}_1^m$ with finite type Gauss map, for example, B-scrolls in $\mathbb{E}_1^3$, several kinds of cylinders and extended B-scrolls in $\mathbb{E}_1^4$ are those with 1-type Gauss map[7,9]. Recently, Kim and Yoon[8] showed that cylinders over curves of finite type are the only ruled surface with space-like or time-like base curve in Minkowski m-space ($m \geq 3$) with finite type Gauss map, but they didn't succeed to figure out the ruled surface with null base curve. However, the authors[9] characterized ruled surfaces with 1-type Gauss map in Minkowski space. Therefore, we wonder what ruled surfaces in Minkowski space are those with finite type Gauss map if the base curve is null. Relating with such matters, we raise the following natural question: "*Can we completely classify the family of ruled surfaces with finite type Gauss map in Minkowski m-space?*"

In this article, we study ruled surfaces in Minkowski m-space $\mathbb{E}_1^m$, and we give the complete classification theorem of ruled surfaces with finite type Gauss map, which is a generalization of the author's results in their former work[9].

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned.

2. Preliminaries

Let $\mathbb{E}_s^m$ be an m-dimensional pseudo-Euclidean space of signature $(m - s, s)$. In particular, for $m \geq 2$, $\mathbb{E}_1^m$ is called a *Minkowski* m-space. A curve in $\mathbb{E}_s^m$ is called *null* if its tangent vector field is null along it.

Let $x : M \to \mathbb{E}_s^m$ be an isometric immersion of an oriented n-dimensional pseudo-Riemannian submanifold M into $\mathbb{E}_s^m$. From now on, a submanifold in $\mathbb{E}_s^m$ always means pseudo-Riemannian.

Let $e_1, e_2, ..., e_m$ be an adapted local orthonormal frame in $\mathbb{E}_s^m$ such that $e_1, e_2, ..., e_n$ are tangent to M and $e_{n+1}, e_{n+2}, ..., e_m$ normal to M. Then, the Gauss map $G : M \to G(n, m) \subset \mathbb{E}^N$, $G(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p)$, of x is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}_s^m$ which is obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}_s^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}_s^m$, $N = \binom{m}{n}$.

An indefinite inner product $\langle , \rangle$ on $G(n, m) \subset \mathbb{E}^N$ is defined by

$$\langle e_{i_1} \wedge \ldots \wedge e_{i_n}, e_{j_1} \wedge \ldots \wedge e_{j_n} \rangle = \det(\langle e_{i_l}, e_{j_k} \rangle).$$

Then, $\{e_{i_1} \wedge \ldots \wedge e_{i_n} \mid 1 \leq i_1 < \ldots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N_k$ for some positive integer k. Let $\{x_1, x_2, \ldots, x_n\}$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the induced pseudo-Riemannian metric, $\langle\, , \rangle$ on M from that of $\mathbb{E}^m_s$ we denote by (g^{ij}) (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|}\, g^{ij} \frac{\partial}{\partial x^j}\right). \qquad \ldots (2.1)$$

Now, we define a ruled surface M in $\mathbb{E}^m_1$. Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve of J into $\mathbb{E}^m_1$ and $\beta = \beta(s)$ a vector field along α with $\alpha'(s) \wedge \beta(s) \neq 0$ for every $s \in J$. Then, a ruled surface M is defined by the parameterization given as follows :

$$x = x(s, t) = \alpha(s), + t\,\beta(s), \quad s \in J, \quad t \in I.$$

For a such a ruled surface, α and β are called the *base curve* and the *director curve* respectively. In particular, if β is constant, the ruled surface is said to be *cylindrical*, and if it is not so, it is called *non-cylindrical*. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows : If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M^1_+ or M^2_+ if β' is non-null or null, respectively. When β is time-like, β' is space-like because of the causal character. In this case, M is said to be of type M^3_+. On the other hand, for the ruled surface of type M_-, it is also said to be of type M^1_- or M^2_- if β' is non-null or null, respectively.

The second and third named authors found a class of flat ruled surfaces of type M^2_+ with finite type Gauss map in[8], which is the only class of non-cylindrical ruled surfaces with finite type Gauss map in $\mathbb{E}^m_1$. We call such ruled surfaces as ruled surfaces of type *FNC*-M^2_+.

If the base curve α is a null curve and the vector field β along α is a null vector field, then the ruled surface M is called a *null scroll*. In particular, a null scroll with Cartan frame in $\mathbb{E}^3_1$ is said to be a *B-scroll*[1, 6]. The authors[9] defined an *extended B-scroll* in Minkowski m-space $\mathbb{E}^m_1$.

For the ruled surfaces with non-null base curves in $\mathbb{E}^m_1$ with finite type Gauss map, the second and third named authors obtained the following theorem :

***Theorem* 2.1**[8] — *Cylinders over curves of finite type and ruled surfaces of type FNC-*M_+^2 *are the only ruled surfaces over non-null base curve with finite type Gauss map in* $\mathbb{E}_1^m$ ($m \geq 4$).

3. GENERALIZED B-SCROLLS

Let M be a null scroll generated by a null curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^m$ $(m \geq 4)$ and $\beta = \beta(s)$ a null vector field along α, which is up to congruences parameterized by

$$x = x(s, t) = \alpha(s) + t\,\beta(s), \quad s \in J, \ t \in I$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle \beta, \beta \rangle = \langle \alpha', \beta \rangle = -1$, where I and J are some open intervals. Furthermore, we may assume $\langle \alpha', \beta' \rangle = 0$, which is equivalent to choose α as a null geodesic of M.

Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}_1^m$ and let $A(s), B(s), C_1(s), \ldots, C_{m-2}(s)$ be a null frame along α satisfying

$$\langle A, A \rangle = \langle B, B \rangle = \langle A, C_i \rangle = \langle B, C_i \rangle = 0,$$

$$\langle A, B \rangle = -1, \quad \langle C_i\, C_j \rangle = \delta_{ij}, \quad \alpha'(s) = A(s)$$

for $1 \leq i, j \leq m-2$. Let $X(s)$ be the matrix $(A(s)\,B(s), C_1(s), \ldots, C_{m-2}(s))$ consisting of column vectors of $A(s)\,B(s), C_1(s), \ldots, C_{m-2}(s)$ with respect to the standard coordinate system in $\mathbb{E}_1^m$. We then have

$$X^t(s)\,EX(s) = T,$$

where E = diag $(-1, 1, \ldots, 1, 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & O & \\ -1 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ & O & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where $X^t(s)$ denotes the transpose of $X(s)$.

Consider a system of ordinary differential equations

$$X'(s) = X(s)\,M(s), \qquad \ldots (3.1)$$

where

$$M(s) = \begin{pmatrix} 0 & 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & -k_1(s) & -k_2(s) & \cdots & -k_{m-2}(s) \\ -k_1(s) & -a & 0 & -w_2(s) & \cdots & -w_{m-2}(s) \\ -k_2(s) & 0 & w_2(s) & 0 & \cdots & -z_{2,m-2}(s) \\ -k_3(s) & 0 & w_3(s) & z_{23}(s) & \cdots & -z_{3,m-2}(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ -k_{m-2}(s) & 0 & w_{m-2}(s) & z_{2,m-2}(s) & \cdots & O_t \end{pmatrix}$$

a is a constant, $k_1, k_2, ..., k_{m-2}, w_j\,(2 \le j \le m-2), z_{ij}\,(2 \le i \le m-l-2, 3 \le j \le m-2)$ some smooth functions and the last right corner O_l is the zero matrix of degree $l\,(1 \le l \le m-3)$.

For a given initial condition $X(0) = (A(0)\ B(0)\ C_1(0) \ldots C_{m-2}(0))$ satisfying $X^t(0)\,EX(0) = T$, there is a unique solution to eq. (3.1). Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^t(s)\,EX(s)) = 0$ and hence we have

$$X^t(s)\,EX(s) = T.$$

Therefore, $A(s), B(s), C_1(s), ..., C_{m-2}(s)$ form a null frame along a null curve α in $\mathbb{E}_1^m$. Let $x(s,t) = \alpha(s) + tB(s)$. Then, it defines a time-like surface M in $\mathbb{E}_1^m$, which is called the *generalized B-scroll* in $\mathbb{E}_1^m$. In particular, if all of w_i and z_{ij} are identically zero, then it is just an extended B-scroll defined in[9].

Now, we show that the functions $z_{ij}(s)$ can be taken as zero. Let $V(s)$ = Span $\{A(s), B(s), C_1(s)\}$ be the 3-dimensional Lorentz distribution along α. According to Lemma 4.1 in Section 4, we can choose an orthonormal basis $\{\overline{C}_2(s), \ldots \overline{C}_{m-2}(s)\}$ for the orthogonal complement $V(s)$ satisfying $\overline{C}'_i(s) \in V(s)$. Then, $\overline{X}(s) = (A(s)\,B(s)\,C_1(s)\,\overline{C}_2(s) \ldots \overline{C}_{m-2}(s))$ satisfies

$$\overline{X}'(s) = \overline{X}(s)\,\overline{M}(s)$$

where

$$M(s) = \begin{pmatrix} 0 & 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & -k_1(s) & -\bar{k}_2(s) & \cdots & -\bar{k}_{m-2}(s) \\ -k_1(s) & -a & 0 & -\bar{w}_3(s) & \cdots & -\bar{w}_{m-2}(s) \\ -\bar{k}_2(s) & 0 & \bar{w}_2(s) & 0 & \cdots & 0 \\ -\bar{k}_3(s) & 0 & \bar{w}_3(s) & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\bar{k}_{m-2}(s) & 0 & \bar{w}_{m-2}(s) & \cdots & \cdots & 0 \end{pmatrix}$$

from which, we can define a null scroll $\bar{x}(s, t) = \alpha(s) + tB(s)$ that is the same generalized B-scroll we defined above.

If we use (2.1), the Laplacian Δ of M is given by

$$\Delta f = 2f_{st} + 2ta^2 f_t + t^2 a^2 f_{tt}$$

for a function f defined on M. Let H be the mean curvature vector field of M defined by $H = \frac{1}{2} trh$, where h is the second fundamental form of M. The Beltrami eq. $\Delta x = -2H$ gives

$$H(s, t) = -ta^2 B(s) + aC_1(s),$$

which implies

$$\Delta H = 2a^2 H.$$

And the shape operator A_H associated with the mean curvature vector field H has the form

$$\begin{pmatrix} a^2 & 0 \\ ak_1 & a^2 \end{pmatrix} \qquad \text{... (3.2)}$$

in the coordinate frame $\{x_s, x_t\}$. The minimal polynomial of A_H is $(t - a^2)^2$ if $ak_1 \neq 0$.

The Gauss map $G = x_s \wedge x_t$ of M satisfies

$$\Delta G = 2a^2 - 2a \sum_{j=2}^{m-2} w_j C_j \wedge B,$$

$$\Delta^2 G = 2a^2 \Delta G.$$

4. Characterization of Ruled Surfaces in Minkowski Space

Using similar idea to prove Lemma 4.1 in[9] and slightly modifying it, we can obtain

Lemma 4.1 — Let $V(s)$ be a smooth $l\,(\geq 2)$-dimensional non-degenerate distribution of index 1 in a Minkowski $m\,(\geq 3)$-space $\mathbb{E}_1^m$ along a curve $\alpha = \alpha(s)$. Then, we can choose orthonormal vector fields $C_1(s), \ldots, C_{m-l}(s)$ along α which generate the orthogonal component $V(s)$ satisfying $C_i'(s) \in V(s)$ for $1 \leq i \leq m - l$.

***Theorem* 4.2** — *Let M be a null scroll in Minkowski m-space $\mathbb{E}_1^m$. If M has finite type Gauss map G, then G is of 1-type or null 2-type.*

Proof : Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}_1^m$ and $\beta = \beta(s)$ a null vector field along α. Then, the null scroll M is up to congruences parametrized by

$$x = x(s, t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I \qquad \text{... (4.1)}$$

such that $\langle \alpha', \alpha' \rangle = 0 \langle \beta, \beta \rangle = 0, \langle \alpha', \beta \rangle = -1$, where I and J are some open intervals. Furthermore, without loss of generality we may choose α as a null geodesic of M. We then we have $\langle \alpha'(s), \beta'(s) \rangle = 0$ for all s. Therefore, we have the natural fram $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta', \quad x_t = \beta. \qquad \text{... (4.2)}$$

Using the induced Lorentz metric on M and formula (2.1), we have the Laplacian operator Δ on M given by

$$\Delta = 2\frac{\partial^2}{\partial s\, \partial t} + \frac{\partial q}{\partial t}\frac{\partial}{\partial t} + q\frac{\partial^2}{\partial t^2}, \qquad \text{... (4.3)}$$

where $\quad q = \langle x_s, x_s \rangle.$

On the other hand, the Gauss map G on M is defined by

$$G = x_s \wedge x_t = \alpha' \wedge \beta + t\beta' \wedge \beta \qquad \text{... (4.4)}$$

since $x_s \wedge x_t$ has length 1.

We now suppose that the Gauss map G of M is of finite type. Then, there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ and a constant vector C in $I\!E^N_{m-1} = \left(N = \binom{m}{2}\right)$ such that

$$\Delta^k G + \lambda_k \Delta^{k-1} G + \ldots + \lambda_2 \Delta G + \lambda_1 G + C = 0 \qquad \text{... (4.5)}$$

for some positive integer k. By a direct computation, eq. (4.5) holds if and only if

$$-2^k (v^{k-1} \beta' \wedge \beta)' - \sum_{i=1}^{k-1} \lambda_{i+1} 2^i (v^{i-1} \beta' \wedge \beta)' + \lambda_1 \alpha' \wedge \beta + C = 0, \qquad \text{... (4.6)}$$

$$\left(2^k v^k + \sum_{i=1}^{k-1} \lambda_{i+1} 2^i v^i + \lambda_1 \right) \beta' \wedge \beta = 0 \qquad \text{... (4.7)}$$

where $\quad v(s) = \langle \beta'(s), \beta'(s) \rangle.$

Suppose $\beta'(s) \wedge \beta(s) \equiv 0$ for all s. Then, the causal character of β implies that β is a null constant vector field. Thus, $G = \alpha' \wedge \beta$ and thus $\Delta G = 0$, that is, G is of null 1-type.

Let us consider an open subset $\mathcal{U} = \{p \in M \mid (\beta \wedge \beta)(p) \neq 0\}$ of M. Suppose $\mathcal{U} \neq 0$. We then have from (4.7)

$$\lambda_1 + 2\lambda_2 v + \ldots + 2^{k-1} \lambda_k v^{k-1} + 2^k v^k = 0 \qquad \text{... (4.8)}$$

on $\mathcal{U}$. v is a solution of an algebraic equation $\lambda_1 + 2\lambda_2 z + \ldots + 2^{k-1}\lambda_k z^{k-1} + 2^k z^k = 0$ with real coefficients and thus v is a real constant. By the causal character of β, v is a positive constant on each component of $\mathcal{U}$. Since υ vanishes on $M - \mathcal{U}$, by continuity. $\mathcal{U} = M$. It is easily derived that $\Delta^2 G = 2\,\upsilon\,\Delta G$ on M, i.e., G is either of non-null 1-type or of null 2-type. It completes the proof. □

***Theorem* 4.3** — *Let M be a null scroll in Minkowski m-space $\mathbb{E}_1^m$. Then, M has finite type Gauss map if and only if M is an open portion of an extended B-scroll or a generalized B-scroll in $\mathbb{E}_1^m$.*

PROOF : As is described in Theorem 4.2, we assume that the null scroll M is parametrized by

$$x = x(s, t) = \alpha(s) + t\,\beta(s), \quad s \in J, \quad t \in I \qquad \text{... (4.9)}$$

such that $\langle \alpha' \alpha' \rangle = 0, \langle \beta, \beta \rangle = 0 \langle \alpha', \beta \rangle = -1$, where I and J are some open intervals. Furthermore, we assume that α is chosen as a null geodesic of M, i.e., $\langle \alpha'(s), \beta'(s) \rangle = 0$ for all s.

Suppose that M has finite type Gauss map G.

Case 1 — Suppose $\beta'(s) \wedge \beta(s) \equiv 0$ for all s. In this case, β is a null constant vector field and the Gauss map G is of null 1-type.

If $\lambda_1 \neq 0$, then (4.6) gives the base curve α is a plane curve spanned by the constant vector field β and a constant vector field D which are linearly independent in $\mathbb{E}_1^m$. Therefore, M is an open portion of a Minkowski plane.

If $\lambda_1 = 0$ (4.6) implies $C = 0$. let $V(s) = \{A(s), B(s)\}$ be a smooth distribution of index 1 along α such that $A(s) = \alpha'(s), B(s) = \beta(s)$ satisfying $\langle A(s), A(s) \rangle = \langle B(s), B(s) \rangle = 0$, $\langle A(s), B(s) \rangle = -1$ for all s. Then, by Lemma 4.1 we can choose an orthonormal basis $\{C_1(s), C_2(s), C_3(s) ..., C_{m-2}(s)\}$ for the orthogonal complement $V^{\perp}(s)$ satisfying $C_j'(s) \in V(s)$ for all $j = 1, ..., m-2$. Thus, we may put

$$A'(s) = -\sum_{i=1}^{m-2} k_i(s)\,C_i(s), \quad B'(s) = 0, \quad C_i'(s) = -k_i(s)B(s) \qquad \text{... (4.10)}$$

for $i = 1, 2, ..., m-2$. For a certain initial condition it has a unique solution to (4.10) that is a part of a flat extended B-scroll (see[9]).

Case 2 — Suppose $\beta'(s) \wedge \beta(s) \not\equiv 0$. In the proof of Theorem 4.2 we see that υ is a positive constant a^2 $(a > 0)$ on an open subset $\mathcal{U}$ on which $\beta'(s) \wedge \beta(s) \neq 0$. By continuity, $\mathcal{U} = M$. Let $A(s) = \alpha'(s), B(s) = \beta(s), C_1(s) = -\frac{1}{a}\beta'(s)$. Let $V(s)$ be the vector space spanned by $A(s), B(s), C_1(s)$ along α. According to Lemma 4.1, we have an orthonormal vector fields $\{C_2(s), ..., C_{m-2}(s)\}$ generating the orthogonal complement $V(s)$ satisfying $C_j'(s) \in V(s)$ for $j = 2, ..., m-2$. Let

$k_j(s) = \langle C_j'(s), A(s)\rangle = -\langle A'(s), C_j(s)\rangle$ ($j = 1, \ldots, m-2$) and $w_j(s) = \langle C_1'(s), C_j(s)\rangle$ ($j = 2, \ldots, m-2$). Then, we have

$$C_1'(s) = -aA(s) - k_1(s)B(s) + \sum_{j=2}^{m-2} w_j(s)C_j(s),$$

$$A'(s) = -\sum_{j=1}^{m-2} k_j(s)C_j(s), \quad C_i'(s) = -k_i(s)B(s) - w_j(s)C_1(s)$$

for $i = 2, \ldots, m-2$. The converse is very straightforward. Hence, the proof is completed. □

Combining Theorem 2.1 and Theorem 4.3, we have

***Theorem* 4.4** (*Characterization*) — *Let M be a ruled surface with finite type Gauss map in $\mathbb{E}_1^m$. Then, M is an open part of cylinders over non-null base curve of finite type, ruled surfaces of type FNC-M_+^2 or one of generalized B-scrolls.*

5. The Generalized B-Scrolls and the Minimal Polynomial of A_H

In Section 3, we discussed the minimal polynomial of the shape operator A_H of the generalized B-scroll. In fact, if $ak_1 \neq 0$, it is given by $(t-a^2)^2$ induced from (3.2).

In this section, we show the converse also holds. Let M be a null scroll parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I \qquad \ldots (5.1)$$

where α is a null geodesic and $\langle \beta, \beta\rangle = 0 \langle \alpha', \beta\rangle = -1, \langle \alpha', \beta'\rangle = 0$ for some open intervals I and J. Then, we have the mean curvature vector field H

$$H(s,t) = -\beta'(s) - t\,\upsilon(s)\beta(s),$$

where $\upsilon(s) = \langle \beta'(s), \beta'(s)\rangle \geq 0$. The straightforward computation gives the shape operator A_H associated with the mean curvature vector field H of the form

$$A_H = \begin{pmatrix} \upsilon(s) & 0 \\ \varphi(s,t) & \upsilon(s) \end{pmatrix} \qquad \ldots (5.2)$$

in the coordinate frame $\{x_s, x_t\}$ where $\varphi(s,t) = -\langle \alpha'(s), \beta''(s)\rangle + \frac{1}{2}\upsilon'(s)\,t$.

We now prove the following theorem :

***Theorem* 5.1** — *Let M be a null scroll in an m-dimensional Minkowski space $\mathbb{E}_1^m$. If the shape operator A_H of M in the direction of the mean curvature vector field H has the minimal polynomial of the form $(t-\mu)^2$ for some constant μ, then M is a generalized B-scroll.*

PROOF : Suppose that the shape operator A_H of M in the direction of the mean curvature vector field H has the minimal polynomial of the form $(t-\mu)^2$ for some constant μ. From (5.2), the function $\upsilon = \langle \beta', \beta' \rangle \geq 0$ is the constant μ. If $\upsilon = 0$, then H vanishes. It contradicts that the minimal polynomial of A_H is $(t-\mu)^2$. Thus, υ is positive. If we follow the proof of Case 2 in Theorem 4.3, we see that the null scroll M is a generalized B-scroll. It completes the proof. □

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