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LIFTING OF THE GENERATORS OF AN IDEAL AND NORI'S HOMOTOPY CONJECTURE

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Let A be a commutative Noetherian ring and I an ideal of the polynomial ring A $[T]$ that contains a monic polynomial. Suppose I/I^2 is generated by r elements, where $r \geq dim\,(A\,[T]/I) + 2$ and $I(0) = \{f(0) \mid f \in I\}$ is generated by $a_1, ..., a_r$ over A. Then we find conditions for the existence of a set of generators $f_1, ..., f_r$ of I such that $f_i\,(0) = a_i$, for all $i = 1, ..., r$.

We give an example of an ideal where the above problem has a negative answer. We also show by an example that the generators of I need not contain a monic polynomial. Besides the above we give an another proof of the Laurent Plynomial analogue of the Nori's homotopy conjecture (Theorem (0.1).

Key Words : Commutative Noetherian Ring; Projective Module; Dimension; Laurent Polynomial Ring

1. INTRODUCTION

Let A be a ring and I an ideal of the polynomial ring $A[T]$. Let $I(0) : = \{f(0) \mid f \in I\}$. For a module M over a ring, $\mu\,(M)$ will denote the minimal number of generators of M.

In[1] Mandal proved that if A is a commutative Noetherian ring and I is an ideal of $A[T]$ containing a monic polynomial and I/I^2 is generated by r elements, where $r \geq dim\,(A\,[T]/I) + 2$, then I is also generated by r elements.

It is a natural question whether we can lift a set of r generators of $I(0)$ to a set of r generators of I. Moreover, can we find a set of r generators of I with one of them monic? In[7] we proved that both questions have a positive answer when A is semilocal. In this paper we further investigate the problem and we get positive results for the first question assuming :

(1) $\mu\,(I/I^2) = 1$ and A is an integral domain

(2) $I\,(0) = I\,(0)^2$

(3) I is a zero dimensional ideal and T is not a divisor of zero in $A[T]/I$.

But the question for the one dimensional case has the negative answer. We will show that, in general, the answer to both the questions in the problem is in negative. We also give an another proof of the following Laurent Polynomial analogue of the Nori's homotopy conjecture. This proof is based on induction on the order of pole of the Laurent polynomial.

***Theorem* 1** — *Let A be a commutative Noetherian ring and I an ideal of the Laurent polynomial ring $A[T, T^{-1}]$ that contains a doubly monic polynomial. Let P be a projective A-module of rank $r \geq dim(A[T, T^{-1}]/I) + 2$. Write $I(1) = \{f(1) \mid f \in I\}$. Let $S : P \to I(1)$ and $\phi : P[T, T^{-1}] \to I/I^2$ be two surjective homomorphisms such that $\phi(1) \equiv S \bmod (T-1, I(1)^2)$. Then there exists a surjective homomorphism $\psi : P[T, T^{-1}] \to I$ such that $\psi(1) = S$ and ψ lifts ϕ.*

1. Preliminaries

In this section we define some of the terms used in the paper and state some results for later use.

- An ideal I of a ring A is called zero dimensional if every prime ideal of A containing I is a maximal ideal, i.e. $dim(A/I) = 0$. Note that if I is the intersection of finitely many maximal ideals, then $dim(A/I) = 0$ and the converse is also true if I is a reduced ideal.

- A polynomial f in $A[T]$ is said to be monic if the coefficient of the highest degree term of f is equal to one. A polynomial f in Laurent polynomial ring $A[T, T^{-1}]$ is said to be doubly monic if the coefficient of the highest degree term and the lowest degree term is equal to one.

- $F^n K_0 A$ denotes the subgroup of the Grothendieck group $K_0 A$ generated by all the residue fields of all smooth maximal ideals of height n.

- Let X be a subset of Spec (A) and let $\mathcal{N} = \{0, 1, 2, 3, ...\}$. Let $d : X \to \mathcal{N}$ be a function. Then we define a partial ordering on X by defining $p_1 \leq p_2$ if either $p_1 = p_2$ or $p_1 \subset p_2$ and $d(p_1) > d(p_2)$.

A function $d : X \to \mathcal{N}$ is said to be a generalised dimension function if, for an ideal I of A, $V(I) \cap X$ has only finitely many minimal elements with respect to the partial ordering $\leq$, where $V(I)$ denotes the set of prime ideals containing I.

We need the following results to prove the results. We now state a theorem of Satya Mandal, which we have used in the proof of the proposition 2.2, theorem 2.6, theorem 4.3 and in the example 2.

***Theorem* 1.1** — [1] *Suppose A is a ring and I is an ideal of $A[T]$ containing a monic polynomial. Suppose that P is a projective A-module of rank $r \geq dim(A[T]/2) + I$ and $S : P \to I(0)$ and $\phi : P[T] \to I/I^2$ are surjective homomorphisms such that $\phi(0) \equiv S \bmod I(0)^2$. Then there is a surjective homomorphism $\psi : P[T] \to I$ such that $\psi(0) = S$ and ψ lifts ϕ.*

We say that the ideal I is efficiently generated if $\mu(I) = \mu(I/I^2)$. We now state the following theorem of Satya Mandal, which we have used in the proofs of the most of the results.

***Theorem* 1.2** — *[2] Let I be an ideal of $A[T]$ over a commutative Noetherian ring A. Suppose that I contains a monic polynomial and $\mu(I/I^2) \geq dim\,(A[T]/I) + 2$. Then $\mu(I) = \mu(I/I^2)$.*

The following theorem due to Eisenbud-Evans[9] is used in the proof of theorem 2.6.

***Theorem* 1.3** — *[9] Let R be a commutative Noetherian ring and M be a finitely generated R-module. Let X be a subset of spec (R) and $d : X \rightarrow \mathcal{N}$ be a generalised dimension function, where $\mathcal{N}$ is a set of non negative integers. Let $x_1, \ldots, x_r$ be elements of M and N a submodule of M such that $(Rx_1 + \ldots + Rx_r + N)_p = M_p$, for all $p \in X$. Assume $r \geq \sup\,\{\mu(M_p) + d(p) \mid p \in X\}$. Then there exists $x_i' = x_i + n_i$, where $n_i \in N$ for $i \in \{1, \ldots, r\}$ such that $M' = Rx_1' + \ldots + Rx_r'$ is fully basic in M at all $p \in X$ i.e. $(M/M')_p = 0$, for all $p \in X$.*

We now state the following theorem of Quillen-Suslin, which we will use in the proof of proposition 2.1.

***Theorem* 1.4** *[10] If P is a finitely generated projective $A[T]$-module and $f \in A[T]$ is a monic polynomial such that P_f is a free $A[T]_f$-module, then P is a free $A[T]$-module.*

2. Results About the Lifting of Generators of an Ideal

In this section we prove the results where the problem has affirmative answer. Through out this section A will denote a commutative Noetherian ring with identity and I denote an ideal of the polynomial ring $A[T]$ that contains a monic polynomial.

Proposition 2.1 — Let A be an integral domain. Suppose that $\mu(I/I^2) = 1$. Then $\mu(I(0)) = 1$ over A and for any generator a of $I(0)$, we can find a polynomial $f \in I$ such that $I = \langle f \rangle$ and $f(0) = a$.

Proof : Since A is an integral domain and $\mu(I/I^2) = 1$, then $\mu(I_p/I_p^2) = 1$, for all $p \in$ Spec $(A[T])$. Since I_p contains a monic polynomial, I_p is free over $A[T]_p$, for all $p \in$ Spec $(A[T])$. This implies that I is a projective ideal of rank 1. Therefore I_f is free over $A[T]_f$, where f is a monic polynomial in I. Then by Quillen-Suslin theorem 1.4, I must be a principal ideal. This implies that $\mu(I) = 1$ and $\mu(I(0)) = 1$. Suppose $I(0) = \langle a \rangle$ and $I = \langle g \rangle$ such that $g(0) = b$. Then $b = ca$, for some unit $c \in A$. Take $f = c^{-1} g$. Then $I = \langle f \rangle$ and $f(0) = a$.

Proposition 2.2 — Suppose that I/I^2 is generated by r elements, where $r \geq dim\,(A[T]/I) + 2$ and $I(0) = I(0)^2$. Then for every generating set $\{a_1, \ldots, a_r\}$ of $I(0)$ there exist $f_1, \ldots, f_r \in I$ such that (*i*) $I = \langle f_1, \ldots, f_r \rangle$

(*ii*) $f_i(0) = a_i$ for $i \in \{1, \ldots, r\}$.

PROOF : Since I contains a monic polynomial and $\mu(I/I^2) \geq dim(A[T]/I) + 2$, by Theorem 1.2, I is efficiently generated by r elements. Suppose that $I(0) = \langle a_1, \ldots, a_r \rangle$ over A and $I = \langle g_1, g_2, \ldots, g_r \rangle$ over $A[T]$. Define $S : A^r \to I(0)$ by $S(e_i) = a_i$, where $\{e_1, e_2, \ldots, e_r\}$ is a usual basis of A^r, for $i \in \{1, \ldots, r\}$, and $\phi : A[T]^r \to I/I^2$ defined by $\phi(e_i) = \bar{g}_i$ for each $i \in \{1, \ldots, r\}$, where $\bar{g}_i$ denotes the image of g_i in I/I^2. Then both S and ϕ are clearly surjective homomorphisms and $\phi(0) = S \bmod I(0)^2$ because $I(0) = I(0)^2$. Then by theorem 1.1, there exists a surjective homomorphism (lift of ϕ) $\chi : A[T]^r \to I$ such that $\chi(0) = S$ and $\chi = \phi \bmod I^2$ with $\nu \circ \chi = \phi$. This implies that there exist $f_1, \ldots, f_r \in I$ such that

(*i*) $I = \langle f_1, \ldots, f_r \rangle$

(*ii*) $f_i(0) = a_i$ for $i \in \{1, \ldots, r\}$

(*iii*) $f_i = g_i \bmod I^2$. □

Corollary 2.3 — Let $\mu(I/I^2) := r \geq A[T]/I + 2$ and $I(0) = \langle a_1, \ldots, a_r \rangle = A$. Then we can find $f_1, \ldots, f_r \in I$ such that

(*i*) $I = \langle f_1, \ldots, f_r \rangle$

(*ii*) $f_i(0) = a_i$ for $i \in \{1, \ldots, r\}$.

PROOF : Since $I(0) = A$, $I(0) = I(0)^2$. Therefore the result follows from the proposition 2.2 □

Lemma 2.4 — Suppose T is not a divisor of zero in $A[T]/I$. Then $I/TI \cong I(0)$ as A-module

PROOF : Define $\phi : I/TI \to I(0)$ by $\phi(\bar{f}) = f(0)$. Then it is clear that ϕ is an A-module isomorphism. □

Lemma 2.5 — Let I be an ideal of $A[T]$ generated by r elements and $I(0) = \langle a_1, \ldots, a_r \rangle$ over A. Suppose $g_1, g_2, \ldots, g_r$ in I are such that $g_i(0) = a_i$, for $i \in \{1, 2, \ldots, r\}$. and T is not a divisor of zero in $A[T]/I$. Then $I = \langle g_1, g_2, \ldots, g_r \rangle + TI$.

PROOF : Since $\langle g_1, g_2, \ldots, g_r \rangle$ and TI are contained in I, $\langle g_1, g_2, \ldots, g_r \rangle + TI \subseteq I$. Now suppose $f \in I$. Then $f(0) \in I(0)$. We can write $f(0) = \sum_{i=1}^{i=r} \alpha_i a_i$ for some $\alpha_i \in A$. This means that

$f-\sum_{i='1}^{i=r}\alpha_i g_i$ has no constant term and $f-\sum_{i='1}^{i=r}\alpha_i g_i = Th$, for some $h\in A[T]$. Since T is not a divisor of zero in $A[T]/I$, $h\in I$. Thus $f \sum_{i=1}^{i=r}\alpha_i g_i + Th \in \langle g_1, g_2, ..., g_r\rangle + TI$. This completes the proof of the lemma. □

***Theorem* 2.6** — *Let I be a zero dimensional ideal. Assume that T is not a divisor of zero in $A[T]/I$. Suppose $\mu(I/I^2) := r \geq 2$ and $I(0) = \langle a_1, ..., a_r\rangle$ over A. Then we can find $f_1, ..., f_r$ in I such that*

(i) $(I)_p = \langle f_1, ..., f_r\rangle_p$ *for all* $p\in V(I)$.

(ii) $f_i(0) = a_i$, *for* $i\in\{1, ..., r\}$.

PROOF : Since I is a zero dimensional ideal, *dim* $(A[T]/I) = 0$. Also $\mu(I/I^2)\geq 2$ and I contains a monic polynomial. Therefore by theorem 1.2, $\mu(I) = \mu(I/I^2) = r$. Let $I(0) = \langle a_1, ..., a_r\rangle$ over A, then by lemma 2.5 there exist $g_1, g_2, ..., g_r$ in I such that, $I = \langle g_1, ..., g_r\rangle + TI$. Thus $I_p = \langle g_1, ..., g_r\rangle_p + TI_p$, for all $p\in$ Spec $(A[T])$. Now consider a function $d : V(I)\to\mathcal{N}$ defined by $d(p)$ = dim $(A[T]/p)$. Then d is a generalised dimension function. If $I\subseteq p$, then dim $(A[T]/p) = 0$ and $\mu(I_p)\leq\mu(I) = r$.

Also *sup* $\{\mu(I_p) + d(p) \mid p\in V(I)\}$ $\leq\mu(I) = r$. Therefore by Theorem 1.3 there exist $f_1, ..., f_r\in I$ and $f_i = g_i + Th_i$, $h_i\in I$, $1\leq i\leq r$ and satisfy the required properties. □

Corollary 2.7 — Let M be a maximal ideal of $A[T]$ containing a monic polynomial. Suppose that $\mu(M/M^2) = r\geq 2$ and $M(0) := \{f(0) \mid f\in M\} = \langle a_1, ..., a_r\rangle$. Also assume that T is not a divisor of zero in $A[T]/M$. Then there exist $f_1, ..., f_r\in M$ such that

(i) $M = \langle f_1, ..., f_r\rangle$

(ii) $f_i(0) = a_i$ for $i\in\{1,..., r\}$.

PROOF : Since M is a maximal ideal, *dim* $(A[T]/M) = 0$ i.e. M is a zero dimensional ideal. Therefore, by theorem 2.6, we have the result. □

Corollary 2.8 — Let I be a complete intersection ideal of the polynomial ring $A[T]$ that contains a monic polynomial. Suppose $\mu(I/I^2) = r\geq$ *dim* $(A[T]/I) + 2$ and dim $(A[T]) = r$. Suppose T is not a divisor of zero in $A[T]/I$ and $I(0) = \langle a_1, ..., a_r\rangle$ over A. Then there exist $f_1, ..., f_r\in I$ such that (*i*) $(I)_p = \langle f_1, \ldots, f_r \rangle_p$ for all prime $p \in V(I)$

(*ii*) $f_i(0) = a_i$ for $i \in \{1, \ldots, r\}$.

PROOF : Since I contains a monic polynomial and $\mu(I/I^2) = r \geq dim\,(A\,[T]/I) + 2$ by theorem 1.2, $\mu(I) = \mu(I/I^2) = r$. Since I is a complete intersection ideal, $ht(I) = \mu(I) = r$. Therefore $dim\,(A\,[T]) = ht\,(I)$ and I is a zero dimensional ideal. Then, by theorem 2.6, we get the required. □

ANOTHER PROOF OF THE THEOREM 2.6

If T is not a divisor of zero in $A\,[T]/I$, then T is not in any of the associated prime ideal of $A\,[T]/I$. If p is a minimal prime ideal over I, then $p \in Ass_{A\,[T]}\,(A\,[T]/I)$, where $Ass_{A\,[T]}\,(A\,[T]/I)$ denotes the set of associative prime ideals. Since I is a zero dimensional ideal, any maximal ideal containing I is a associative prime. Let $\mathcal{M}$ be the annihilator of b, denoted as $Ann\,(b)$, where $b \in A\,[T]$ and $b \notin I$. We know that the set of divisors of zero equals $\bigcup p$, where $p \in Ass\,(\mathcal{M})$. This implies that T is not in any maximal ideal containing I. Thus T is a unit modulo I. Hence $(I, T) = A\,[T]$ and $I\,(0) = A$. Therefore theorem follows from corollary 2.3.

3. GENERAL ANSWER OF THE PROBLEM

In general, the answer to both the questions of the problem is negative. For the first question we give an example, where the extension of Theorem 2.6 for one dimensional ideal fails.

Example 1 — Let $A = \mathcal{R}[X, Y, Z]$, where $\mathcal{R}$ denotes the field of real numbers and $I = (X, Y, T + Z^2 + 1)$ be an ideal of $A[T]$ generated by $\{X, Y, T + (Z^2 + 1)\}$. Then $I(0) = (X, Y, Z^2 + 1)$. Note that $A\,[T]/I = \mathcal{R}[X, Y, Z, T]/(X, Y, T + Z^2 + 1) \cong \mathcal{R}(Z)$. Further $A/I\,(0) = \mathcal{R}[X, Y\,Z]/(X, Y, Z^2 + 1) \cong \mathcal{R}(Z)/(Z^2 + 1) \cong C$, the field of complex numbers. Clearly $dim\;(\mathcal{R}) = 1$ and $dim\;(C) = 0$, $\mu(I) = ht\,(I) = 3$ and $\mu(I\,(0)) = ht\,(I\,(0)) = 3$. This implies that I and $I\,(0)$ are complete intersection ideals of $A[T]$ and A respectively. Hence I/I^2 and $I\,(0)/I\,(0)^2$ are free modules of rank 3 over $A\,[T]/I$ and $A/I(0)$ respectively. It follows that $(\overline{X}, \overline{Y}, \overline{Z^2 + 1})$ is a free basis of $I(0)/I(0)^2$ over $A/I(0)$. Given any invertible matrix $M \in Gl_3(A/I\,(0))$, we have a free basis $\{\overline{f_1\,(X, Y, Z)}, \overline{f_2\,(X, Y, Z)}, \overline{f_3\,(X, Y, Z)}\}$ of $I\,(0)/I\,(0)^2$ over $A/I\,(0)$ given by

$$\begin{pmatrix} \overline{f_1\,(X, Y, Z)} \\ \overline{f_2\,(X, Y, Z)} \\ \overline{f_3\,(X, Y, Z)} \end{pmatrix} = M \begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z^2 + 1} \end{pmatrix},$$

where $\{f_1\,(X, Y, Z), f_2\,(X, Y, Z), f_3\,(X, Y, Z)\}$ generates $I(0)$ (Note that $I(0)$ contains a monic polynomial), and $M = [u_{ij}\,(X, Y, Z)]$, $1 \leq i, j \leq 3$. Suppose that $\{\overline{f_1\,(X, Y, Z)}, \overline{f_2\,(X, Y, Z)}, \overline{f_3\,(X, Y, Z)}\}$ has a lifting $\{g_1\,(X, Y, Z, T), g_2\,(X, Y, Z, T), g_3\,(X, Y, Z, T)\}$ to a generating set of I. It follows that

$\{\overline{g_1(X, Y, Z, T)}, \overline{g_2(X, Y, Z, T)}, \overline{g_3(X, Y, Z, T)}\}$ is a free basis of I/I^2 over $A[T]/I$. Then there exists an invertible matrix $N \in GL_3(A[T]/I)$ such that

$$\begin{pmatrix} \overline{g_1(X, Y, Z, T)} \\ \overline{g_2(X, Y, Z, T)} \\ \overline{g_3(X, Y, Z, T)} \end{pmatrix} = N \begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{T + Z^2 + 1} \end{pmatrix},$$

Suppose $N = \overline{[h_{ij}[X, Y, Z, T]}]$, where $h_{ij}[X, Y, Z, T] \in A[T]$, $1 \leq i, j \leq 3$. Since $A[T]/I \simeq \mathcal{R}[Z]$ and the units of $\mathcal{R}[Z]$ are non-zero members of $\mathcal{R}$, Det $[h_{ij}(0, 0, Z, -Z^2 - 1)]$ is free from Z and is a non-zero real number. Putting $T = 0$, we get

$$\begin{pmatrix} \overline{g_1(X, Y, Z)} \\ \overline{g_2(X, Y, Z)} \\ \overline{g_3(X, Y, Z)} \end{pmatrix} = N \begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z^2 + 1} \end{pmatrix}.$$

This implies that $Det[u_{ij}(0, 0, i)] = Det[h_{ij}(0, 0, 0, -1)]$, which is a real number. Thus for example if we take

$$M = \begin{pmatrix} \overline{Z} & \overline{0} & \overline{0} \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix},$$

$$Det \begin{pmatrix} \overline{i} & \overline{0} & \overline{0} \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} = i.$$

Hence $\{\overline{f_1(X, Y, Z)}, \overline{f_2(X, Y, Z)}, \overline{f_3(X, Y, Z)}\}$, where

$$\begin{pmatrix} \overline{f_1(X, Y, Z)} \\ \overline{f_2(X, Y, Z)} \\ \overline{f_3(X, Y, Z)} \end{pmatrix} = \begin{pmatrix} \overline{Z} & \overline{0} & \overline{0} \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z^2 + 1} \end{pmatrix},$$

does not have a lifting to a generating set of I, i.e. the generator $\{XZ, Y, Z^2 + 1\}$ of $I(0)$ cannot be lifted to a generating set of I. The generators of $I(0)$ can be lifted to a set of generators of I if and only if *det* (M) can be lifted to a unit of $A[T]/I$.

For the second question of the problem we show by an example that a generating set with one of them monic may not exist.

Example 2 — Let A be a regular affine domain over the complex numbers C such that there exists a maximal ideal m of A, which is not a complete intersection i.e. $F^n K_0 A \neq 0$, where $F^n K_0 A$ denotes the subgroup of the Grothendieck group $K_0(A)$ generated by all the residue fields of all smooth maximal ideals of height n. For example $A = C[X_1, \ldots. X_{n+1}] / (X_1^{n+2} + \ldots + X_{n+1}^{n+2} - 1)$. Consider a maximal ideal m in A such that $[A/m] \neq 0$ in $K_0 A$. Suppose $R = A\ [T]$ and $M = (m, T)$. Let $\{f_1, \ldots, f_{n+1}\}$ be the minimal set of generators of M with f_1 monic. Then $\overline{R} = R/(f_1)$ is an integral extension of A, $\overline{M} = (\overline{f}_2, \ldots, \overline{f}_{n+1})$ and $\overline{R}/\overline{M} \cong R/M$ as an A-Module. Also $\overline{R}/\overline{M} \cong A/m$ as an A-module. Since A is regular, any finitely generated module over A has finite homogical dimension, $G_0(A) = K_0 A$. Since f_1 is monic, any finitely generated module over $\overline{R}$ remains finitely generated over A. Thus we have a natural map $\phi : G_0(\overline{R}) \to G_0(A)$, which sends $[\overline{R}/\overline{M}]$ to $[A/m]$. Let $\overline{N} = (\overline{f}_2, \ldots, \overline{f}_n)$. Then we get a short exact sequence of finitely generated modules $\overline{R}$-modules, and hence finitely generated A-modules $0 \to \overline{R}/\overline{N} \to \overline{R}/\overline{N} \to \overline{R}/\overline{M} \to 0$, where the first map is multiplication by $\overline{f_{n+1}}$. Since $\overline{R}/\overline{N} \cong R/N$, $(R/N)_{\overline{f_{n+1}}}$ is regular local ring, which is a integral domain. We claim that $\overline{f_{n+1}}$ is a nonzero divisor in R/N. Suppose there exists nonzero $\overline{x} \in R/N$ such that $\overline{x} \cdot \overline{f_{n+1}} = 0$. Since $(R/N)_{\overline{f_{n+1}}}$ is integral domain, $\overline{x}/1 = 0$ in $(R/N)_{\overline{f_{n+1}}}$, then there exist $\overline{s} \in (R/N) - (\overline{f_{n+1}})$ such that $\overline{s} \cdot \overline{x} = 0$. Consider $Ann\ (\overline{x})$, which is an ideal in R/N. But $(\overline{f_{n+1}}) \subset Ann\,(\overline{x})$ and $s \in Ann\,(\overline{x})$. This implies that $Ann\ (\overline{x}) = R/N$ and hence $\overline{x} = 0$ in R/N. This contradicts the assumption and hence proves the claim. Therefore the multiplication map $\overline{f_{n+1}}$ is injective. Hence we get $[\overline{R}]/\overline{M}] - [\overline{R}/\overline{N}] - [\overline{R}/\overline{N}] = 0$ in $G_0 A$. This implies that $[A/m] = 0$ in $G_0 A = K_0 A$. This contradicts the choice of m.

4. Sections of Projective Module Over Laurent Polynomial Rings

In this section we discuss a proof of the Laurent polynomial analogue of Nori's Homotopy conjecture (Theorem 1.1 given in the introduction), which is based on induction. We start with a lemma, which we will be using in the proof of subsequent results. In this section A denotes commutative Noetherian ring with identity. The range of a map G will be denoted $Im\ (G)$.

Lemma 4.1 — Suppose P is a projective A-module of rank r and I is an ideal of A. Then

$$Hom_A\,(P, I) \cong I\,Hom_A\,(P, A).$$

PROOF : Note that $Hom_A\,(P, A)$ is an A-module. Any homomorphism from P to I can be thought of as homomorphism from P to A. This gives us a homomorphism $G : Hom_A\,(P, I) \to Hom_A\,(P, A)$. The homomorphism G is an imbedding. First we show that $\mathrm{Im}\,(G) \subseteq I\,Hom_A\,(P, A)$. Note that $Hom_A\,(A, I) \cong I$, $Hom_A\,(A, A) \cong A$. Since P is a projective module, there exists A-module Q such that $P \oplus Q = A^r$ for some r. Therefore $Hom_A\,(P, I)$ is embedded in $Hom\ (P \oplus Q, I) = Hom_A\,(A^r, I) = Hom_A\,(A, I)^r = I^r\,Hom_A\,(A, A) = I^r A = I^r$. This implies that $G\,(Hom\,(P, I)) \subset I\,Hom_A\,(P, A)$.

Next, we show that $Im\ (G) = I\ Hom\ (P, A)$. It is enough to show that

$$(I, Hom_A\,(P, A)/\mathrm{Im}\,(G))_p = 0 \text{ for all } p \in Spec\,(A).$$

$$Hom_A\,(P, I)_p \cong Hom_{A_p}\,(P_p, I_p)$$

$$\cong Hom_{A_p}\,(A_p^r, I_p)$$

$$\cong \oplus\, Hom_{A_p}\,(A_p, I_p) \ (r \text{ times})$$

$$\cong \oplus\, I_p \ (r \text{ times})$$

and

$$I\,Hom_A\,(P, A)_p \cong I_p\,Hom_{A_p}\,(P_p, A_p)$$

$$\cong I_p\,Hom_{A_p}\,(A_p^r, A_p)$$

$$\cong \oplus\, I_p\,Hom_{A_p}\,(A_p, A_p) \ (r \text{ times})$$

$$\cong \oplus\, I_p \ (r \text{ times})$$

Therefore

$$Hom_A\,(P, I)_p \cong I_p\,Hom_{A_p}\,(P, A)_p$$

$$\Rightarrow Hom_A\,(P, I)_p \cong I_p\,Hom_{A_p}\,(P_p, A_p)$$

$$\Rightarrow Coker\,(G_p) = 0, \text{ for all } p \in Spec\,(A)$$

$$\Rightarrow Coker\,(G) = 0.$$

$\Rightarrow$ G is surjective. Therefore G is an isomorphism and hence

$$Hom_A\,(P, I) \cong I\,Hom_A\,(P, A).$$

Proposition 4.2 — Let I be an ideal of the Laurent polynomial ring $A\,[T, T^{-1}]$ and P be A-projective module. Write $I\ (1) = \{f(1) \mid f \in I\}$. Let $S : P \to I\,(1)$ and $\phi : P\,[T, T^{-1}] \to I/I^2$ be two surjective homomorphisms such that $\phi\,(1) \equiv S$ modulo $(T-1, I\,(1)^2)$. Then there exists a lift $\psi : P\,[T, T^{-1}] \to I$ of ϕ such that $\psi\,(1) = S$.

PROOF : Consider the following diagram

$$\begin{tikzcd}
 & P[T, T^{-1}] \arrow[dl, "\eta"'] \arrow[d, "\phi"] & \\
I \arrow[r, "\nu"'] & I/I^2 \arrow[r] & 0,
\end{tikzcd}$$

where ν is the quotient map. Since $P[T, T^{-1}]$ is also a projective module over $A[T, T^{-1}]$, there exists a lift η of ϕ such that $\nu \circ \eta = \phi$ and η can be written as

$\eta_0 T^{-k} + \ldots + \eta_k \oplus \eta_{k+1} T + \ldots + \eta_{k+m} T^m$, where $\eta_i \in Hom_A(P, A)$, for $0 \le i \le k+m$. Since $T \to (T-1)$ determines an automorphism of $P[T]$, we can write

$\eta = \eta_0' + \eta_1' (T-1) + \ldots + \eta_l' (T-1)^l / T^k$, where $l = k + m$ and $\eta(1) = \eta_0'$. We also have $\eta(1) \equiv S \bmod I(1)^2$. This implies that $\eta(1) - S$ maps into $I(1)^2$. Therefore $S - \eta(1) = f_1(1) g_1(1) \lambda_1 + \ldots + f_n(1) g_n(1) \lambda_n$, where $f_i, g_i \in I$ and $\lambda_i \in Hom_A(P, A)$, for $i \in 1, \ldots, r$. Let $\lambda = f_1(T, T^{-1}) g_1(T, T^{-1}) \lambda_1 + \ldots + f_n(T, T^{-1}) g_n(T, T^{-1}) \lambda_n$.

Define $\psi = \eta + \lambda$.Then it clear that ψ satisfies the required properties.

***Theorem* 4.3** — *Let I be an ideal of the Lauren polynomial ring $A[T, T^{-1}]$ containing a doubly monic polynomial and P be a projective A-module of rank $r \ge \dim(A[T, T^{-1}]/I) + 2$. Write $I(1) = \{f(1) \mid f \in I\}$. Let $S : P \to I(1)$ and $\phi ; P[T, T^{-1}] \to I/I^2$ be two surjective homomorphisms such that $\phi(1) \equiv S \bmod (T-1, I(1)^2)$. Then there exisst a surjective homomorphism $\psi : P[T, T^{-1}] \to I$ such that $\psi(1) = S$ and ψ lifts ϕ.*

PROOF : We first establish some lemmas which are required to prove the theorem.

Lemma 4.4 — Under the hypothesis of theorem 4.3, let $J = I \cap A[T]$. Then $A[T]/J \cong A[T, T^{-1}]/I$.

PROOF : Since $A[T] \subset A[T, T^{-1}]$, we can define a map $\phi : A[T] \to A[T, T^{-1}]/I$ by $\phi(f(T)) = f(T) + I$. Then ϕ is clearly an injective A-module homomorphism. It is enough to prove that ϕ is a surjective. Let $h(T, T^{-1}) + I$ be an arbitrary element of $A[T, T^{-1}] + I$, where $h(T, T^{-1}) = a_{-k} T^{-k} + a_{-k+1} T^{-(k-1)} + \ldots + a_0 + a_1 T + \ldots + a_{l-1} T^{l-1} + a_l T^l$.

We show the existence of an element $g(T, T^{-1}) \in I$ such that $h(T, T^{-1}) - g(T, T^{-1}) \in A[T]$. We prove it by induction on k (the order of the pole of the Laurent polynomial $h(T, T^{-1})$. If $k = 0$, then $h(T, T^{-1}) \in A[T]$ and then we have nothing to show. Suppose it is true when the order of pole of $h(T, T^{-1})$ is $k - 1$. Then we prove it when this order equals k. Let $u(T, T^{-1}) = T^{-r} + b_{-r+1} T^{-(r-1)} + \ldots + b_0 + b_1 T + \ldots + b_{s-1} T^{s-1} + T^s$ be a doubly monic polynomial in I. Then consider $h(T, T^{-1}) - u(T, T^{-1}) a_{-k} T^{r-k}$. Observe that the first term $a_{-k} T^{-k}$ of $h(T, T^{-1})$ will vanish and the order of the pole in $h(T, T^{-1}) - u(T, T^{-1}) a_{-k} T^{r-k}$ is $k - 1$ for which it is true by induction hypothesis. Hence it is true for all k. Also note that

$u(T, T^{-1})\, a_k T^{r-k} \in I$. Therefore $h(T, T^{-1}) + I$ can be written as $f(T) + u(T, T^{-1})\, \alpha(T, T^{-1})$, where $f(T) \in A[T]$ and $u(T, T^{-1})\, \alpha(T, T^{-1}) \in I$. Take $g(T, T^{-1}) = u(T, T^{-1})\ \alpha(T, T^{-1})$. Then $\phi(f(T)) = f(T) + g(T, T^{-1}) + I = f(T) + I$. Thus ϕ is a surjective homomorphism and ker $\phi = J$. Then, by a standard algebraic result, $A[T]/J \cong A[T, T^{-1}]/I$. This completes the proof of the lemma.

Lemma 4.5 — Under the hypothesis of the theorem let $J = I \cap A[T]$. Then $J/J^2 \cong I/I^2$.

PROOF : Note that $J \subset I$ and I^2 also contains a doubly monic polynomial. Define a map $\eta : J \to I/I^2$ by $\eta(f(T)) = f(T) + I^2$. The remainder of the proof is similar to the proof of the lemma 4.4.

Lemma 4.6 — Let P be a projective module over A and $J = I \cap A[T]$. Then $P[T]/JP[T] \cong P[T, T^{-1}]\ IP[T, T^{-1}]$.

PROOF : From Lemma 4.4, we have $A[T]/J \cong A[T, T^{-1}]/I$. Since P is a projective module over A, tensoring with P we get $P \otimes A[T]/J \cong p \otimes A[T, T^{-1}]/I$. This implies that $P[T]/JP[T] \cong P[T, T^{-1}]\ /\ IP[T, T^{-1}]$. This completes the proof of the lemma.

PROOF OF THEOREM 4.3 : Let $J(1) = \{f(1) \mid f \in J\}$. By hypothesis there is a surjective homomorphism $\phi : P[T, T^{-1}] \to I/I^2$. Then we get a surjective homomorphism $P[T, T^{-1}]\ /\ IP[T, T^{-1}] \to I/I^2$. By Lemma 4.5, we have $J/J^2 \cong I/I^2$ and by lemma 4.6, we have $P[T]/JP[T] \cong P[T, T^{-1}]/IP[T, T^{-1}]$. Therefore there is a surjection from $P[T]/JP[T]$ onto J/J^2. Then $F : P[T] \to J/J^2$ is a surjective homomorphism and $F(1) \equiv S \bmod J(1)^2$, for $JA[T, T^{-1}] = I$ and $J(1) = I(1)$. Then, by theorem 1.1, we have a surjective homomorphism $G : P[T] \to J$ such that G lifts F and $G(1) = S$. Now by localizing at T, we get a surjective homomorphism $\psi : P[T]_T \to J_T$ defined by $\psi(f/T^r) = G(f)/T^r$, which satisfies the required properties because $P[T]_T = P[T, T^{-1}]$ and $J_T = I$. This completes the proof of the theorem.

Note — We have used the theorem 1.1 for J (1) The proof is similar to the proof of J (0).

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