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A NEW CONSTRUCTION OF EWELL'S OCTUPLE PRODUCT IDENTITY

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In this article, we establish an octuple product identity motivated by the work of Carlitz and Subbarao in which they use Jacobi's triple product identity only to prove the quintuple product identity and Winquist's identity. Our work turns out to be a new construction of Ewell's octuple product identity. On the other hand, we offer an alternative proof for the octuple product identity by appealing to functional equations satisfied by related infinite products.

Key Words: **Jacobi's Triple Product Identity; Pentagonal Number Theorem; Euler's Identity; Quintuple Product Identity; Octuple Product Identity; Winquist's Identity**

1. INTRODUCTION

Series-product identities play important roles in several research areas, such as the derivation of identities, number theory, the theory of special functions, combinatories, etc. The most famous and important example is Jacobi's triple product identity which states that

$$\prod_{n=1}^{\infty} (1-x^n)(1-ax^n)(1-a^{-1}x^{n-1}) = \sum_{n=-\infty}^{\infty} (-1)^n a^n x^{n(n+1)/2}.$$

Here and in the sequel, we always assume that $|x| < 1$ and $a \neq 0$. Also, for convenience, we shall always refer to the identity above as J.T.P.I.

Two other well-known identities of this kind are the quintuple product identity and Winquist's identity which can be stated, respectively, as

$$\prod_{n=1}^{\infty} (1-x^n)(1-ax^n)(1-a^{-1}x^{n-1})(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$

$$= \sum_{n=-\infty}^{\infty} (a^{3n} - a^{-3n-1})\, x^{n(3n+1)/2}$$

and

$$\prod_{n=1}^{\infty} (1-x^n)^2 (1-ax^{n-1}) (1-bx^{n-1}) (1-ab^{-1} x^{n-1}) (1-abx^{n-1})$$

$$\times (1-a^{-1} x^n) (1-b^{-1} x^n) (1-a^{-1} bx^n) (1-a^{-1} b^{-1} x^n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} (a^{-3m} b^{-3n} - a^{-3n} b^{3n+1} - b^{-3m-1} a^{-3n+1}$$

$$+ b^{-3m-1} a^{3n+2})\, x^{(3m^3+3m+3n^2+n)/2},$$

where a and b are both nonzero.

There have appeared several proofs in the literature for the identities above. We mention some of them for reference. Jacobi's triple product identity was established in his famous *Fundamenta Nova*[11,12] but, in fact, first proved by Gauss[9] [p. 464]. Also, see[1&5] for simple proofs of J.T.P.I. In[2] [p. 83], Berndt gave a thorough survey on the history of the quintuple product identity. Winquist's identity was established in[16] in order to prove a congruence property modulo 11 of Ramanujan[15] for the partition function. Since then, various proofs for Winquist's identity have been found in[4,10&13] etc.

In[3&4], Carlitz and Subbarao employed J.T.P.I. to obtain the quintuple product identity and Winquist's identity through a uniform and constructive approach. In Section 2, we are motivated by the method of Carlitz and Subbarao to construct an octuple product identity (Theorem 2.1) from which we deduce some special cases including a famous identity of Euler (Corollary 2.3). After the construction of Theorem 2.1, we learned that Ewell[6] had established an equivalent octuple product identity by combining both J.T.P.I. and the quintuple product identity. In Section 3, we will compare our octuple product identity with the one obtained by Ewell[6]. In Section 4, we will adopt similar arguments to give a simple proof for an identity which was partially derived by Gauss[14] and completed by Ewell[7]. In Section 5, we offer an alternative proof for the octuple product identity by appealing to functional equations satisfied by related infinite products.

2. An Octuple Product Identity

***Theorem* 2.1** — *For $a \neq 0$ and $|x| < 1$, we have*

$$\prod_{n=1}^{\infty} (1-x^n)^2 (1-ax^n) (1-a^{-1} x^n) (1-ax^{n-1}\ 1-a^{-1} x^{n-1})$$

$$\times (1-a^2 x^{2n-1}) (1-a^{-2} x^{2n-1})$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^m \left(2(-1)^n a^{4n} x^{6m^2+2n^2-2m} - a^{2n+1} (-x)^{(3m^2+n^2-m+n)/2} \right).$$

To simplify notation, we shall often write $\sum_n$ and $\prod_n$ for $\sum_{n=-\infty}^{\infty}$ and $\prod_{n=1}^{\infty}$, respectively, when no confusion occurs.

PROOF : For brevity, let $R(a,x)$ denote the left hand side of Theorem 2.1 and let $A(a,x) = R(a,x) \prod_n (1-x^{2n})$. By J.T.P.I., we have

$$A(a,x) = \sum_m (-1)^m a^m x^{(m^2+m)/2} \sum_n (-1)^n a^n x^{(n^2-n)/2} \sum_l (-1)^l a^{2l} x^{l^2}$$

$$= \sum_{m,n,l} (-1)^{m+n+l} a^{m+n+2l} x^{(m^2+n^2+2l^2+m-n)/2}.$$

Replacing m and n by $u-l$ and $v-l$, respectively, in the last sum, we have

$$A(a,x) = \sum_{u,v} (-1)^{u+v} a^{u+v} x^{(u^2+v^2+u-v)/2} \sum_l (-1)^l x^{2l^2+(u+v)l}.$$

Observe that the inner sum of the last identity vanishes when $u+v \equiv 2 \pmod 4$.

This can be easily seen by replacing l with $l+\frac{1}{2}(u+v)$ in the inner sum. Hence,

$$A(a,x) = A_0 + A_1 + A_{-1},$$

where

$$A_i = \sum_{u+v\equiv i \pmod 4} (-1)^{u+v} a^{u+v} x^{(u^2+v^2+u-v)/2} \times \sum_l (-1)^l x^{2l^2+(u+v)l},$$

for i = 0, 1, −1. To evaluate A_0, we substitute $4u + b$ and $4v - b$ for u and v, respectively, where $u, v \in \mathbb{Z}$ and b = −1, 0, 1, 2. Then,

$$A_0 = \sum_{b=-1}^{2} \left(\sum_{u,v} a^{4u+4v} x^{8(u^2+v^2)+(4b-2)(u-v)+b(b+1)} \times \sum_t (-1)^l x^{2(l-(u+v))^2-2(u+v)^2} \right)$$

$$= \sum_{b=-1}^{2} \left(\sum_{l} (-1)^l x^{2l^2} \sum_{u,v} (-1)^{(u+v)} a^{4u+4v} \times x^{2(u^2+v^2)+(2(u-v)+b)(2(u-v)+b+1)} \right),$$

after replacing l by $l+u+v$. Next, denote the sum $\sum_{l} (-1)^l x^{2l^2}$ by $P_0(x)$ and substitute $(s+t)/2$ and $(t-s)/2$ for u and v, respectively, in the last sum. Then,

$$A_0 = P_0(x) \sum_{b=-1}^{2} \sum_{s \equiv t \,(\mathrm{mod}\, 2)} (-1)^t a^{4t} x^{2t^2+(2s+b)(2s+b+1)}$$

$$= P_0(x) \sum_{b=-1}^{2} \sum_{s,t} \left(a^{8t} x^{8t^2+(4s+b)(4s+b+1)} - a^{8t+4} x^{2(2t+1)^2+(4s+b-1)(4s+b-2)} \right).$$

Replacing b and s by $1-b$ and $-s$, respectively, in the second part of the last sum, we arrive at

$$A_0 = P_1(x) \sum_{t} \left(a^{8t} x^{8t^2} - a^{8t+4} x^{2(2t+1)^2} \right)$$

$$= P_1(x) \sum_{n} (-1)^n a^{4n} x^{2n^2}, \qquad \text{... (2.1)}$$

where

$$P_1(x) = P_0(x) \sum_{b=-1}^{2} \sum_{s} x^{(4s+b)(4s+b+1)} \qquad \text{... (2.2)}$$

By similar arguments and the observation that

$$\sum_{b=-1}^{2} \sum_{s} x^{(4s+b)^2} = \sum_{b=-1}^{2} \sum_{s} x^{(4s+b+1)^2},$$

we obtain

$$A_1 = -Q_1(x) \sum_{t} \left(a^{8t+1} x^{8t^2+2t} - a^{8t+5} x^{8t^2+10t+3} \right) \qquad \text{... (2.3)}$$

and

$$A_{-1} = -Q_1(x) \sum_t \left(a^{8t-1} x^{8t^2-2t} - a^{8t+3} x^{8t^2+6t+1} \right), \qquad \ldots (2.4)$$

where

$$Q_1(x) = \sum_l (-1)^l x^{2l^2+l} \sum_{b=-1}^{2} \sum_s x^{(4s+b)^2}. \qquad \ldots (2.5)$$

Next, adding up (2.3) and (2.4), we obtain

$$A_1 + A_{-1} = -Q_1(x) \sum_n a^{2n+1} (-x)^{n(n+1)/2}. \qquad \ldots (2.6)$$

Combining (2.1) and (2.6), we obtain

$$A(a,x) = P_1(x) \sum_n (-1)^n a^{4n} x^{2n^2} - Q_1(x) \sum_n a^{2n+1} (-x)^{n(n+1)/2}. \qquad \ldots (2.7)$$

Now, it remains to evaluate $P_1(x)$ and $Q_1(x)$. By taking $a = 1$ and -1 in (2.7), we have

$$0 = P_1(x) \sum_n (-1)^n x^{2n^2} - Q_1(x) \sum_n (-x)^{n(n+1/2} \qquad \ldots (2.8)$$

and

$$4 \prod_n (1-x^{2n})^3 = P_1(x) \sum_n (-1)^n x^{2n^2} + Q_1(x) \sum_n (-x)^{n(n+1)/2}, \qquad \ldots (2.9)$$

where the left sides of (2.8) and (2.9) follow from the definition of $A(a,x)$ at the beginning of the proof. By adding (2.8) to (2.9) and subtracting (2.8) from (2.9), respectively, we have

$$P_1(x) = 2 \prod_n (1-x^{2n})(1-x^{4n}) \qquad \ldots (2.10)$$

and

$$Q_1(x) = \prod_n (1-x^{2n})^2 (1+x^{2n-1}), \qquad \ldots (2.11)$$

with the help of J.T.P.I. By (2.10), (2.11) and dividing both sides of (2.7) by $\prod_n (1-x^{2n})$, we arrive at

$$R(a,x) = 2P(x) \sum_n (-1)^n a^{4n} x^{2n^2} - Q(x) \sum_n a^{2n+1} (-x)^{n(n+1)/2}, \qquad \ldots (2.12)$$

where

$$P(x) = \prod_{n} (1 - x^{4n})$$

and

$$Q(x) = \prod_{n} (1 + x^{2n-1})(1 - x^{2n}).$$

On the other hand, Euler's pentagonal number theorem (which also follows from J.T.P.I.) gives that

$$P(x) = \sum_{m} (-1)^m x^{6m^2 - 2m} \qquad \text{... (2.13)}$$

and

$$Q(x) = \prod_{n} (1 - (-x)^n) = \sum_{m} (-1)^{(3m^2 + m)/2} x^{(3m^2 - m)/2}. \qquad \text{... (2.14)}$$

Finally, plugging (2.13) and (2.14) into (2.12) gives the desired result. □

Corollary 2.2 — For $|x| < 1$, we have

$$\prod_{n=1}^{\infty} \frac{(1 - x^{2n})(1 + x^{2n})}{(1 - x^{2n-1})(1 + x^{2n-1})} = \sum_{n=-\infty}^{\infty} x^{16n^2 + 4n}(1 + x^{8n+2})$$

and

$$\prod_{n=1}^{\infty} \frac{(1 - x^{2n})(1 + x^{2n-1})}{(1 + x^{2n})(1 - x^{2n-1})} = \sum_{n=-\infty}^{\infty} x^{16n^2}(1 + 2x^{8n+1} + x^{16n+4}).$$

PROOF : From (2.2), we have

$$\frac{P_1(x)}{P_0(x)} = \sum_{s} \left(x^{4s(4s-1)} + x^{4s(4s+1)} + x^{(4s+1)(4s+2)} + x^{(4s+2)(4s+3)} \right)$$

$$= 2 \sum_{n} \left(x^{4n(4n+1)} + x^{(4n+1)(4n+2)} \right).$$

Then, by (2.10) and writing $P_0(x)$ into products using J.T.P.I., we arrive at the first equality of Corollary 2.2. The second equality of Corollary 2.2 follows from (2.5) and (2.11) by similar arguments. □

Remark 1 : Note that, by using J.T.P.I., we obtain

$$\prod_{n=1}^{\infty} (1 + x^{4n-1})(1 + x^{4n-3})(1 - x^{4n}) = \frac{1}{2} \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}$$

and

$$\prod_{n=1}^{\infty} (1+x^{2n-1})^2 (1-x^{2n}) = \sum_{n=-\infty}^{\infty} x^{n2}$$

which are equivalent to the first and second identities in Corollary 2.2, respectively after splitting each of the last two sums into four party according to the residues of a modulus.

Corollary 2.3 — For $a \neq 0$ and $|x| < 1$, we have

$$(2-(a+a^{-1})) \prod_n (1-x^{2n-1})^2 (1-ax^n)^2 (1-a^{-1}x^n)^2$$

$$= 2 \prod_n (1+x^{2n})^2 (1+a^2x^{2n-1})(1+a^{-2}x^{2n-1})$$

$$-(a+a^{-1}) \prod_n (1+x^{2n-1})^2 (1+a^2x^{2n})(1+a^{-2}x^{2n}).$$

In particular, the case $a = -1$ gives Euler's identity

$$\prod_n (1+x^n) = \prod_n \frac{1}{1-x^{2n-1}}$$

and the case $a = e^{2\pi i/3}$ gives

$$3 \prod_n (1+x^n) \left(\frac{1-x^{3n}}{1-x^{2n}}\right)^z = 2 \prod_n (1+x^{2n})^3 (1+x^{6n-3})$$

$$+ \prod_n (1+x^{2n-1})^3 (1+x^{6n}).$$

PROOF : This follows directly from applying Theorem 2.1, in its equivalent form (2.12) by transforming the two sums on the right side of (2.12) into infinite products using J.T.P.I. □

3. EWELL'S OCTUPLE PRODUCT IDENTITY

In 1982, Ewell[6] established the following octuple product identity by combining both J.T.P.I. and the quintuple product identity.

Theorem 3.1 — *(Ewell). For $a \neq 0$ and $|x| < 1$, we have*

$$\prod_n (1-x^n)^2 (1-ax^n)(1-a^{-1}x^n)(1-ax^{n-1})(1-a^{-1}x^{n-1})$$

$$\times (1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$

$$= 2P(x) \sum_n (-1)^n a^{4n} x^{2n^2} - Q_2(x) \sum_n (-1)^n a^{4n} x^{2n^2} (ax^n + a^{-1} x^{-n}) \qquad \ldots (3.1)$$

where $P(x) = \prod_n (1 - x^{4n})$

and

$$Q_2(x) = \prod_n (1 - x^{12n})(1 - x^{12n-5})(1 - x^{12n-7})$$

$$+ x \prod_n (1 - x^{12n})(1 - x^{12n-1})(1 - x^{12n-11}).$$

Observe the resemblance between (3.1) and (2.12). The left sides of both identities are the same. Also, observe that the two sums $\sum_n a^{2n+1} (-x)^{n(n+1)/2}$ and $\sum_n (-1)^n a^{4n} x^{2n^2}$ $(ax^n + a^{-1} x^{-n})$ appearing on the right sides are identical which can be seen by splitting the former sum into two parts according to the parity of n. And so, we arrive at that $Q(x) = Q_2(x)$, namely

$$\prod_n (1 - (-x)^n) = \prod_n (1 - x^{12n})(1 - x^{12n-5})(1 - x^{12n-7})$$

$$+ x \prod_n (1 - x^{12n})(1 - x^{12n-1})(1 - x^{12n-11}).$$

The last equality can be verified by separating the odd and even index terms of the infinite sum in (2.14) for $Q(x)$ and then applying J.T.P.I. to the two separated sums. The identity (2.12) improves Ewell's result in the sense that $Q(x)$ owns a simpler form than $Q_2(x)$ does. On the other hand, we should emphasize that Ewell obtained Theorem 3.1 by combining both J.T.P.I. and the quintuple product identity, but we derive (2.12) with only the help of J.T.P.I.

Next, if we multiply both sides of (2.12) by a, differentiate the resulting identity with respect to a twice, and let $a = 1$, we arrive at

$$\prod_n (1 - x^n)^6 (1 - x^{2n-1})^2 = -2P(x) \sum_{n=1}^{\infty} (-1)^n (4n)^2 x^{2n^2}$$

$$+ Q(x) \sum_{n=0}^{\infty} (-1)^n (2n+1)^2 (-x)^{n(n+1)/2},$$

which is equivalent to the corollary in[6]. Multiplying both sides of the last identity by

$\prod_n (1-x^{2n})^2$, we obtain

$$\prod_n (1-x^n)^8 = -2\left(\prod_n (1+x^{2n})(1-x^{2n})^3\right)\sum_{n=1}^{\infty} (-1)^n (4n)^2 x^{2n^2}$$

$$+\left(\prod_n (1+x^{2n-1})(1-x^{2n})^3\right)\sum_{n=0}^{\infty} (-1)^n (2n+1)^2 (-x)^{n(n+1)/2}.$$

4. An Identity of Gauss

The following result, according to MacMohan [14, p. 79], was first derived by Gauss without identifying the $A_0(x)$ and $A_1(x)$. The derivation was completed by Ewell[7] in 1982. Ewell also deduced a special case of Gauss's identity by which he gave a new proof of Ramanujan's congruence property modulo 7 for partition function. Here, we will use similar (but much easier) arguments as in the proof of Theorem 2.1 to reconstruct Gauss's result.

***Theorem* 4.1 (Gauss-Ewell)** — *For $a \neq 0$ and $|x| < 1$, we have*

$$\prod_{n=1}^{\infty} (1+ax^{2n-1})^2 (1+a^{-1}x^{2n-1})^2 \qquad \text{... (4.1)}$$

$$= A_0(x) \sum_{m=-\infty}^{\infty} a^{2m} x^{2m^2} + A_1(x) \sum_{m=0}^{\infty} x^{2m(m+1)} (a^{2m+1} + a^{-2m-1}),$$

where

$$A_0(x) = \left(\prod_n (1-x^{2n})^{-2}\right) \sum_n x^{2n^2}$$

and

$$A_1(x) = \left(\prod_n (1-x^{2n})^{-2}\right) x \sum_n x^{2n(n+1)}.$$

Proof : For brevity, we denote the left side of (4.1) by $G(a,x)$ and let $H(a,x) = G(a,x) \prod_n (1-x^{2n})^2$. Then, by J.T.P.I., we have

$$H(a,x) = \sum_{n,l} a^{n+l} x^{n^2+l^2} = \sum_m a^m x^{m^2} \sum_n x^{2n^2+2mn}$$

(replace l by $m-n$)

$$= \sum_m a^{2m} x^{4m^2} \sum_n x^{2n^2+4mn} + \sum_m a^{2m+1} x^{(2m+1)^2} \sum_n x^{2n^2+2(2m+1)n}$$

$$= \sum_m a^{2m} x^{2m^2} \sum_n x^{2(n+m)^2} + x \sum_m a^{2m+1} x^{2m^2+2m} \sum_n x^{2(n+m)^2+2(n+m)}$$

Observe that the last two inner sums are independent of m after the replacement of n by $n-m$. Hence, we obtain

$$H(a,x) = \sum_n x^{2n^2} \sum_m a^{2m} x^{2m^2} + x \sum_n x^{2n(n+1)} \sum_m a^{2m+1} x^{2m(m+1)}$$

Now, (4.1) follows from dividing both sides of the last equality by $\prod_n (1-x^{2n})^2$ and the simple fact that

$$\sum_{m=-\infty}^{\infty} a^{2m+1} x^{2m(m+1)} = \sum_{m=0}^{\infty} (a^{2m+1} + a^{-2m-1})\, x^{2m(m+1)}$$

Remark 2 : Note that we can restate (4.1) equivalently as

$$\prod_n (1-x^{2n})^2 (1+ax^{2n-1^2})(1+a^{-1}x^{2n-1})^2$$

$$= \sum_m x^{2m^2} a^{2m} \sum_n x^{2n^2} + \sum_m x^{2m(m+1)} x^{2m+1} \sum_n x^{2n(n+1)} \quad \ldots (4.2)$$

In 1995, Ewell[8] established a far-reaching generalization of (4.2):

$$\prod_n (1-x^{2n})^2 (1+abx^{2n-1})(1+a^{-1}b^{-1}x^{2n-1})(1+ab^{-1}x^{2n-1})(1+a^{-1}bx^{2n-1})$$

$$= \sum_m x^{2m^2} a^{2m} \sum_n x^{2n^2} b^{2n} + \sum_m x^{2m(m+1)} a^{2m+1} \sum_n x^{2n(n+1)} b^{2n+1},$$

from which he deduced two interesting formulas, one giving the number of representations of a given positive integer by sums of four triangular numbers and the other giving the number of representations by sums of eight triangular numbers.

5. A Second Proof of Theorem 2.1

In this section, we offer a second proof of Theorem 2.1 by using functional equations satisfied by related infinite products. Let $R(a,x)$ be as defined in the proof of Theorem 2.1. Write

$$R(a,x) = \sum_{n=-\infty}^{\infty} A_n(x)\, a^n \qquad \text{... (5.1)}$$

Observe that $R(ax, x) = -a^{-4} x^{-2} R(a, x)$. Then, we have

$$A_n(x) = -x^{n-2} A_{n-4}(x), \qquad \text{... (5.2)}$$

which implies that

$$A_{4n+s}(x) = (-1)^n x^{2n^2+sn} A_s(x),$$

for $s = 0, 1, 2, 3$, and so

$$R(a,x) = \sum_{s=0}^{3} \sum_{n} A_{4n+s}(x)\, a^{4n+s} = \sum_{s=0}^{3} \sum_{n} (-1)^n x^{2n^2+sn} A_s(x)\, a^{4n+s}. \qquad \text{... (5.3)}$$

Also, since $R(a^{-1}, x) = R(a, x)$, we have $A_n(x) = A_{-n}(x)$. Combined with (5.2), we have

$$A_n(x) = -x^{n-2} A_{-n+4}(x).$$

In particular,

$$A_2(x) = 0 \quad \text{and} \quad A_3(x) = -xA_1(x). \qquad \text{... (5.4)}$$

On the other hand,

$$R(a,x) \prod_n (1-x^{2n}) = \prod_n (1-x^n)(1-ax^n)(1-a^{-1}x^{n-1})(1-x^n)(1-a^{-1}x^n)$$

$$\times (1-ax^{n-1})(1-x^{2n})(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$

$$= \sum_{i,j,k} (-1)^{i+j+k} x^{1/2(i^2+i+j^2-j+2k^k)} a^{i+j+2k},$$

where each of i, j and k runs through all integers. For each given pair of i and k, we replace j by $s - i - 2k$ to obtain

$$R(a,x) \prod_n (1-x^{2n})$$

$$= \sum_s (-1)^s x^{\frac{1}{2}(s^2-s)} \left(\sum_{i,k} (-1)^k x^{(i+k)^2 + (-s+1)i + 2k^2 + (-2s+1)k} \right) a^s.$$

For each given pair of s and k, we replace i by $l - k$ to obtain

$$R(a,x)\prod_n (1-x^{2n})$$

$$= \sum_s (-1)^s x^{\frac{1}{2}(s^2-s)} \left(\sum_k (-1)^k x^{2k^2-sk} \sum_l x^{l^2+(-s+1)l} \right) a^s$$

$$= \sum_s (-1)^s x^{\frac{1}{2}(s^2-s)} \left(\prod_n (1-x^{4n})(1-x^{4n-2+s})(1-x^{4n-2-s}) \right. \quad \ldots (5.5)$$

$$\times (1-x^{2n})(1+x^{2n-2+s})(1+x^{2n-s})\, a^s,$$

where the last equality follows from J.T.P.I. Comparing the coefficients of a^s in (5.1) and (5.5), we have

$$A_s(x) = (-1)^s x^{\frac{1}{2}(s^2-s)} \prod_n (1-x^{4n})(1-x^{4n-2+s})$$

$$\times (1-x^{4n-2-s})(1+x^{2n-2+s})(1+x^{2n-s}).$$

In particular,

$$A_0(x) = \prod_n (1-x^{4n})(1-x^{4n-2})^2(1+x^{2n})(1+x^{2n-2}) = 2P(x) \quad \ldots (5.6)$$

and

$$A_1(x) = -\prod_n (1-x^{4n})(1-x^{4n-1})(1-x^{4n-3})(1+x^{2n-1})^2 = -Q(x). \quad \ldots (5.7)$$

Plugging (5.6), (5.7) and (5.4) into (5.3), we have the desired result.

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