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FIXED POINT INDEX AND SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In reflexive Banach spaces by constructing homotopies, we present a formula for the computation of the fixed point index for completely continuous operators by the solutions of the corresponding ordinary differential equations. The formula is then used to obtain fixed points for completely continuous operators defined on cones.

Key Words : Fixed Point Index; Completely Continuous Operators; Stirct *k*-set Contractions; Ordinary Differential Equations in Banach Spaces

1. PRELIMINARIES

Let X be a Banach space, $P \subset X$ a cone and $\Omega \subset X$ an open bounded set. For the strict k-set contraction $A : \overline{\Omega} \cap P \to P$ with $Ax \neq x$ on $x \in \partial\Omega \cap P$, the fixed point index $i(A, \Omega \cap P, P)$ is well-defined (see[3,4&9etc.]), which inherits the properties the topological degree deg (A, Ω, P) for strict k-set contractions defined on the whole $\overline{\Omega}$. Since the fixed point index plays an important role in the investigations of positive solutions to integral and differential equations etc., its computation is an important issue of research. Many results are available in this respect. In this paper, we present a new method for the computation of the fixed point index, which is quite different from those reported in the literature in the sense that it establishes a relationship between the fixed point index and the solutions to the corresponding ordinary differential equations (ODs for short) in Banach spaces.

Throughout the paper, we always assume that X is a reflexive Banach space and $A : P \to P$ is a completely continuous operator in the sense that $x_n \rightharpoonup x$ (weak convergence) implies that $Ax_n \to Ax$ where $P \subset X$ is a cone. Here and in the following the limits are taken as $n \to \infty$.

Lemma 1.1 — Under the above assumptions on A, A is uniformly continuous on every bounded subset of P.

PROOF : Suppose that $D \subset P$ is an open bounded set. In order to prove the uniform continuity of A on D, we only need to prove that the same property holds for $D_1 = \overline{\text{conv}\, D}$. Assume on the contrary that A is not uniformly continuous on D_1. Then there exists $\varepsilon_0 > 0$ and two sequences $\{x_n^1\}$, $\{x_n^2\}$ in D_1 such that

$$\left\| x_n^1 - x_n^2 \right\| \to 0, \qquad \text{... (1.1)}$$

$$\left\| Ax_n^1 - Ax_n^2 \right\| \geq \varepsilon_0, \qquad \text{... (1.2)}$$

By the reflexivity of X, the boundedness and the weak closedness of D_1, we can choose subsequences $\{x_{n_k}^1\}$ and $\{x_{n_k}^2\}$ such that $x_{n_k}^1 \rightharpoonup x_1$ and $x_{n_k}^2 \rightharpoonup x_2$, where $x_1, x_2 \in D_1$. Hence $x_{n_k}^1 - x_{n_k}^2 \rightharpoonup x_1 - x_2$, which implies that $x_1 = x_2$ by (1.1). On the other hand, it follows from the complete continuity of = A on D_1 that $Ax_{n_k}^1 \to Ax_1, Ax_{n_k}^2 \to Ax_2$, which contradicts (1.2).

Consider the ordinary differential equation in X

$$x'(t) = Ax - x, \quad x(0) = x_0, \qquad \text{... (1.3)}$$

where $x_0 \in P$. Its unique forward saturated solution exists by the standard theory of ODs in Banach spaces and is denoted by $x(t; x_0)$ with $t \in [0, T(x_0))$. Set $S(t)\, x_0 = x(t; x_0)$.

Lemma 1.2 — For every bounded $D \subset P$, there exists $\delta = \delta(D) > 0$ such that $S(t)$ is well-defined on D for every $0 < t \leq \delta$ and it can be decomposed as $S(t) = e^{-t} + I + \tilde{S}(t)$, where I is the identity operator and $\tilde{S}(t)(D)$ is relatively compact in X for every $0 < t \leq \delta$.

PROOF : For the fixed bounded set $D \subset P$, standard results in the ODs theory in Banach spaces ensure that $\delta_1 = \delta_1(D) > 0$ exists such that $S(t)$ is well-defined on D for every $0 < t \leq \delta$ and $S(t)\, D \subset P$ (e.g., see Theorem 4.4 in[2]).

It is well-known that (1.3) is equivalent to

$$x(t; x_0) = e^{-t} x_0 + \int_0^t e^{-(t-s)} Ax(s; x_0)\, ds.$$

Set

$$\tilde{S}(t)\, x_0 = \int_0^t e^{-(t-s)} Ax(s; x_0)\, ds.$$

Let $R > r > 0$ be two fixed numbers such that $D \subset B_r \cap P$, where $B_r = \{x \in X \mid \|x\| < r\}$. Suppose that $x_0 \in D$ and there exists $t \in [0, T(x_0))$ such that $\|S(t)\, x_0\| \geq R$. By setting $t^* = \min \{t \geq 0 \mid \|S(t)\, x_0\| \geq R\}$, we have $S(t)\, x_0 \in \overline{B}_R$ for all $0 \leq t \leq t^*$. Hence

$$R - r \leq \|S(t^*)\, x_0 - x_0\| = \left\| \int_0^{t^*} x'(s; x_0)\, ds \right\|$$

$$= \left\| \int_0^{t^*} (Ax(s; x_0) - x(s; x_0)\, ds \right\| .$$

Since $\|x(s; x_0\| \leq R$, $\forall s \in [0, t^*]$, we can find a positive constant $C(R)$ such that $\|Ax(s; x_0) - x(s; x_{0)}\| \leq C(R)$ for every $s \in [0, t^*]$. It follows from the above inequality that $R - r \leq t^* \cdot C(R)$, that is, $t^* \geq (R - r)/C(R)$. Now, if we choose $0 < \delta < \min \{\delta_1, (R - r)/C(R)\}$, then it will satisfy our needs. In fact, by the above argument, we only need to prove the relative compactness of the set $\tilde{S}(t)(D)$ for every $t \in (0, \delta]$.

Since, by the definition of $\delta, S(s)\, D \subset \overline{B}_R \cap P$ for $s \in [0, t] \subset [0, \delta]$, hence we have $AS(s)(D) \subset A(\overline{B}_R \cap P)$. Set $K = \bigcup_{s \in [0, t]} e^{-(t-s)} A(\overline{B}_R \cap P)$. It is easy to prove that K is relatively compact by the relative compactness of $A(\overline{B}_R \cap P)$. Therefore, $\overline{\text{conv}}\, K$ is compact. It follows that $\tilde{S}(t)(D)$ is relatively compact by the fact that $\int_0^t e^{-(t-s)} Ax(s; x_0)\, ds \in \overline{\text{conv}}\, K$ for every $0 < t \leq \delta$.

Lemma 1.3 — Assume that $\Omega \subset X$ is open bounded and $Ax \neq x$ for $x \in \partial\Omega \cap P$. Then there exists $\delta > 0$ such that $S(t)\, x_0 \neq x_0$ for $0 < t \leq \delta$ and $x_0 \in \partial\Omega \cap P$.

PROOF : The proof can be similarly given as Lemma 2.5 in[5].

Suppose that $\Omega \subset X$ is an open bounded subset and let $\delta = \delta(\overline{\Omega} \cap P) > 0$ be the smaller constant defined by Lemmas 1.2 and 1.3 with $D = \overline{\Omega} \cap P$ and $\partial\Omega \cap P$ respectively. Then $S(t)$ are strict e^{-t}-set contractions from $\overline{\Omega} \cap P$ to P for $0 < t \leq \delta$ by Lemma 1.2 and the fixed point index $i(S(t), \Omega \cap P, P)$ is well-defined from the typical result in the topological degree theory as we mentioned before.

2. Main Result and its Applications

Now we can present the main result of the paper.

***Theorem* 2.1** — *Assume that $\Omega \subset X$ is open bounded, $A : P \to P$ is completely continuous, locally Lipschitzian and $Ax \neq x$ for every $x \in \partial\Omega \cap P$.*

$$i(A, \Omega \cap P, P) = \lim_{t \to +} i(S(t), \Omega \cap P, P).$$

Proof : Let $\delta = \delta(\overline{\Omega} \cap P) > 0$ be the smaller constant defined by Lemmas 1.2 and 1.3 with $D = \overline{\Omega} \cap P$ and $\partial\Omega \cap P$ respectively. It suffices to prove that $i(A, \Omega \cap P, P) = i(S(t), \Omega \cap P, P)$ for $0 < t \leq \delta$.

$$\text{Set } H(t, x_0) = \begin{cases} x_0 - \dfrac{x_0 - S(t)\,x_0}{t}, & \text{for } 0 < t \leq \delta \\ Ax_0, & \text{for } t = 0. \end{cases}$$

Fi[illegible] by Lemma 1.2, we have

$$H(t, x_0) = \frac{t - 1 + e^{-t}}{t}\, x_0 + \frac{\tilde{S}(t)\, x_0}{t}$$

[illegible] 0 $t \leq$ [illegible] Henc[illegible] if [illegible] e se[illegible] $(\,) = (t - 1 + e^{-t})/t$, then $k(t)$ is an increasing function on $(0, \delta]$ and h[illegible] 0 $k(\,)$ $k(\delta) < 1$. Th[illegible] refor[illegible] $H(t\ \cdot) : \overline{\Omega} \cap P \to P$ are strict $k(\delta)$-set contractions for [illegible] $\in ($[illegible] $x_0) \neq$ [illegible] r every $t \in [0, \delta]$ a[illegible] $x_0 \in \partial\Omega \cap P$. In order to apply the homotopy invarian[illegible] x [illegible] r strict k [illegible] e[illegible] ontractions. We need to verify the corresponding [illegible] ions. [illegible] n [illegible] to [illegible] ve[illegible] al steps.

Step 1 : We prove that the continuity of $H(t, x_0)$ at every point $t_0 \in [0, \delta]$ is uniform with respect to $x_0 \in \overline{\Omega}$. There are two cases.

Case (*a*) : $t_0 \neq 0$.

Without loss of generality, we can assume that $t \geq t_0/2$. Hence, we have

$$\| H(t, x_0) - H(t_0, x_0) \|$$

$$= \frac{\| tx_0 - tS(t_0)x_0 - t_0 x_0 + t_0 S(t) x_0 \|}{tt_0}$$

$$\leq \frac{2}{t_0^2} \left(\left| t - t_0 \right| \| x_0 \| + t_0 \| S(t) x_0 - S(t_0) x_0 \| + | t - t_0 | \| S(t_0) x_0 \| \right).$$

Moreover,

$$\| S(t) x_0 - S(t_0) x_0 \| = \left\| \int_{t_0}^{t} x'(s; x_0)\, ds \right\|$$

$$= \left\| \int_{t_0}^{t} (Ax(s; x_0) - x(s; x_0))\, ds \right\| \leq C(\overline{\Omega} \cap P) | t - t_0 |,$$

where $C(\overline{\Omega} \cap P)$ is a constant dependent only on the set $\overline{\Omega} \cap P$. Hence the continuity of $H(t, x_0)$ at every point $t_0 \in [0, \delta]$ is uniform with respect to $x_0 \in \overline{\Omega}$.

Case (*b*) : $t_0 = 0$.

$$\| H(t, x_0) - H(0, x_0) \| = \left\| x_0 - \frac{x_0 - S(t) x_0}{t} - Ax_0 \right\|$$

$$= \frac{\left\| \int_0^t (x'(s; x_0) + x_0 - Ax_0)\, ds \right\|}{t}$$

$$= \frac{\left\| \int_0^t (Ax(s; x_0) - Ax_0 + x_0 - x(s; x_0))\, ds \right\|}{t}. \qquad \text{... (2.1)}$$

For every $\varepsilon > 0$, denote the ε-neighbourhood of $\overline{\Omega}$ by $\overline{\Omega}_\varepsilon$. By the uniform continuity of A on $\overline{\Omega}_\varepsilon \cap P$ (by Lemma 1.1), there exists $\delta_\varepsilon(\overline{\Omega})$ such that $\| Ax_1 - Ax_2 \| < \varepsilon$ for $x_1, x_2 \in$

$\overline{\Omega}_\varepsilon \cap P$ with $|x_1 - x_2| < \delta_\varepsilon(\overline{\Omega})$. As in the proof of Lemma 1.2, a constant $\delta_1(\varepsilon)$ can be chosen such that $x(t; x_0) \in \overline{\Omega}_\varepsilon \cap P$ for every $x_0 \in \overline{\Omega}$ and $t \in [0, \delta_1]$. Hence, we have $\| x(s; x_0 - x_0 \| \le L(\varepsilon)\, s \le L(\varepsilon)\, t$ for $s \le t \le \delta_1(\varepsilon)$ and some constant $L(\varepsilon)$. Denote by $\delta_2(\varepsilon)$ the constant determined by Lemma 1.3 with $D = \overline{\Omega} \cap P$. Then, by (2.1), $\| H(t, x_0) - H(0, x_0) \| < 2\varepsilon$ if

$$t < \min\left\{ \frac{\delta_\varepsilon(\overline{\Omega})}{L(\varepsilon)},\ \frac{\varepsilon}{L(\varepsilon)},\ \delta_1(\varepsilon),\ \delta_2(\varepsilon) \right\}$$

Step 2 : Prove that $H(t, x_0)$ is continuous on $[0, \delta] \times \overline{\Omega}$. Hence we first suppose that $(t_n, \overset{0}{x}_n) \to (t_0, x_0)$.

Case (*a*) : $t_0 \ne 0$. The limit $H(t_n, \overset{0}{x}_n) \to H(t_0, x_0)$ follows from the continuous dependence theorem on initial data in the ODEs theory in Banach spaces. Hence we only consider the following case.

Case (*b*) : $t_0 = 0$. By the argument in Step 1, we have

$$H(t_n < x_0) - H(t_0, x_0) \to 0.$$

On the other hand, we have

$$H\left(t_n, \overset{0}{x}_n\right) - H(t_n, x_0)$$

$$= \overset{0}{x}_n - x_0 + \frac{(x_0 - S(t_n)\, x_0) - \left(\overset{0}{x}_n - S(t_n)\, \overset{0}{x}_n\right)}{t_n}$$

$$= \overset{0}{x}_n - x_0 + \frac{\int_0^{t_n} \left(x'\left(s; \overset{0}{x}_n\right) - x'\left(s; \overset{0}{x}_n\right)\right) ds}{t_n}$$

$$= \overset{0}{x}_n - x_0 + \frac{\int_0^{t_n} \left(Ax\left(s; \overset{0}{x}_n\right) - Ax(s; x_0) + x(s; x_0) - x\left(s; \overset{0}{x}_n\right)\right) ds}{t_n}.$$

By Lemma 1.1 and the continuous dependence theorem on initial data in the theory of ODEs in Banach spaces, we get

$$H\left(t_n, \overset{0}{x}_n\right) - H\left(t_n, x_0\right) \to 0.$$

Hence

$$H\left(t_n, x_n^0\right) - H\left(t_0, x_0\right)$$
$$= H\left(t_n, x_n^0\right) - H\left(t_n, x_0\right) + H\left(t_n, x_0\right) - H\left(t_0, x_0\right) \to 0.$$

Construct a homotopy $\overline{H}(\cdot\,,\cdot) : \overline{\Omega} \cap P \to P$ from $H(\delta, \cdot)$ to $S(\delta)$ by

$$\overline{H}(t, x_0) = t\,S(\delta)\,x_0 + (1-t)\left(x_0 - \frac{x_0 - S(\delta)\,x_0}{\delta}\right).$$

It is easily verified that $\overline{H}(t, x_0) \neq x_0$ for every $t \in [0, 1]$ and $x_0 \in \partial\Omega \cap P$. By Lemma 1.2, we have

$$\overline{H}(t, x_0) = \left(te^{-\delta} + (1-t)\,\frac{\delta - 1 + e^{-\delta}}{\delta}\right)x_0 + t\,\tilde{S}(\delta)\,x_0 + (1-t)\,\frac{\tilde{S}(\delta)}{\delta}\,x_0.$$

Hence, $\overline{H}(t, \cdot)$ is a $k(t)$-set contraction on $\overline{\Omega} \cap P$ for every fixed $t \in [0, 1]$, where

$$k(t) = te^{-\delta} + (1-t)\,\frac{\delta - 1 + e^{-\delta}}{\delta}.$$

Obviously,

$$k(t) \leq \max\left\{e^{-\delta},\ \frac{\delta - 1 + e^{-\delta}}{\delta}\right\} < 1$$

for every $t \in [0, 1]$. Hence, it is obvious that $\overline{H}(t, \cdot)$ is the desired homotopy from $\overline{H}(\delta, \cdot)$ to $S(\delta)$.

Combining the above arguments and the homotopy invariance for strict k-set contractions we get

$$i(A, \Omega \cap P, P) = i(H(\delta, \cdot), \Omega \cap P, P) = i(S(t), \Omega \cap P, P)$$

for $0 < t \leq \delta$. This completes the proof.

Let $\Omega \subset X$ be an open bounded set. The open set $\Omega \cap P$ in P is said to have the line property with respect to θ, if every ray in P from θ intersects $\partial\Omega \cap P$ at only one point.

***Theorem* 2.2** — *Let $A : P \to P$ be a locally Lipschitzian completely continuous operator, $P \subset X$ a cone. Let $\Omega \subset X$, $\theta \in \Omega$ be an open bounded set and $\Omega \cap P$ have the line property with respect to θ. Assume that all solutions to (1.3) with initial data $x_0 \in \partial\Omega$ enter $\overline{\Omega} \cap P$ for $t \geq 0$. Then A has at last one fixed point in $\overline{\Omega} \cap P$.*

PROOF : Suppose that $Ax \neq x$ on $\partial \Omega \cap P$. Under the above assumptions, we have

$$S(t)\, x \neq \lambda x \quad \text{on } \partial \Omega \quad \text{for all } \lambda > 1.$$

Hence $i(S(t), \Omega \cap P, P) = 1$ for $t > 0$ sufficiently small. Therefore by Theorem 2.1, we have

$$i(A, \Omega \cap P, P) = \lim_{t \to 0+} i(S(t), \Omega \cap P, P) = 1.$$

This completes the proof.

Remark 2.1 : Under the conditions of the above theorem, if we further assume that $Ax \neq x$ on $\partial \Omega \cap P$, then $i(A, \Omega \cap P, P) = 1$. This result is different from those in the literature for the computation of the topological degree.

Theorem 2.3 — *Assume that $B_R(\theta), B_r(\theta)$ are two balls in X with radii $R > r > 0$. Let $A : P \to P$ be a locally Lipschitzian completely continuous operator, $P \subset X$ a cone in X. Assume that all solutions to (1.3) with initial data $x_0 \in \partial B_R(\theta) \cap P$ and $x_0 \in \partial B_r(\theta) \cap P$ enter (or leave) the conical shell $(\overline{B}_R \setminus B_r) \cap P$ for $t \geq 0$. Then A has at least one fixed point in $(\overline{B}_R \setminus B_r) \cap P$.*

PROOF : We only consider the case where all solutions to (1.3) enter the above conical shell for $t \geq 0$. We can suppose that A has no fixed points on $\partial B_R(\theta) \cap P$ and $\partial B_r(\theta) \cap P$. As the argument in the proof of Theorem 2.2, we can obtain that $i(S(t), B_R \cap P, P) = 1$ for $t > 0$ sufficiently small. Since $\| S(t) x_0 \| \geq \| x_0 \|$ for all $x_0 \in \partial B_r \cap P$ and $t > 0$ sufficiently small, by a result in[10], we get $i(S(t), B_r \cap P, P) = 0$. Hence

$$i(A, (\overline{B}_R \setminus B_r) \cap P, P) = \lim_{t \to 0+} i(S(t), (\overline{B}_R \setminus B_r) \cap P, P) = 1 \neq 0.$$

The conclusion follows immediately.

Remark : There are two kinds of "compression or expansion of conical shells" type theorems in the literature which involve the partial order induced by P and the norm in X respectively (e.g., see[4,10]) and Corollary 20.1 in[2]. They have wide applications in the literature. The above theorem can be regarded as a new kind of "compression or expansion of conical shells" type theorem which involves the corresponding differential equations. It also reveals some interesting properties on the solutions of ODEs with the forms of (1.3) in Banach spaces.

Finally, we investigate completely continuous operators in a Hilbert space H with gradient forms. To be more precise, a copletely continuous operator $A : P \to P$ is said to have a gradient form if there is a C^1-functional $f: H \to \mathbb{R}$ such that $f' = I - A$, where $P \subset H$ is a cone.

Corollary 2.1 — Let $A: P \to P$ be a locally Lipschitzian completely continuous operator in Hilbert space H which has the gradient form. Assume that there exist two constants $\beta > \alpha$ such that

(*i*) $\Omega = f^{-1}(-\infty, \beta)$ is an open bounded set and $\theta \in \Omega_1 = f^{-1}(-\infty, \alpha)$. Furthermore, there is an open bounded set $V \subset X$, $\theta \in V$ such that $V \cap P$ has the line property with respect to θ and $\partial V \subset \overline{\Omega} \setminus \Omega_1$.

(*ii*) $f'(x) \neq \theta, \quad \forall x \in f^{-1}([\alpha, \beta]) \cap P$.

Then $i(A, \Omega \cap P, P) = 1$.

PROOF : Set $\rho = \inf \{\| f'(x) \| \mid x \in \phi^{-1}([\alpha, \beta]) \cap P\}$. It is easy to prove that $\rho > 0$ by (*ii*). By a standard argument (e.g., see[11]), choose a locally Lipschitzian completely continuous operator $A^*: P \to P$ such that $\| A^* x - Ax \| < \rho/2, \quad \forall x \in P$. Hence by the homotopy invariance for the fixed point index for completely continuous operators $i(A, \Omega \cap P, P) = i(A^*, \Omega \cap P, P)$. It is easy to verify that $\overline{\Omega} \cap P$ is forward flow invariant for the eq. (1.3) with A replaced by A^* (see[1]). Its saturated solution is still denoted by $x(t; x_0)$. By the boundedness and the flow invariance of $\overline{\Omega} \cap P$, any solution $x(t, x_0)$ with $x_0 \in \overline{\Omega} \cap P$ exists on $[0, \infty)$ (see Theorem 4.4 in[2]). Now, by

$$\frac{df(x(t; x_0))}{dt} = \langle f'(x(t; x_0)), x'(t; x_0) \rangle$$

$$= \langle (x(t; x_0)) - Ax(t; x_0), A^* x(t; x_0) - x(t; x_0) \rangle$$

$$\leq - \| x(t; x_0) - Ax(t; x_0) \|^2 + \| x(t; x_0) - Ax(t; x_0) \|$$

$$\| A^* x(t; x_0) - Ax(t; x_0) \| < 0,$$

for $x_0 \in (\overline{\Omega} \setminus \Omega_1) \cap P$ and $t > 0$ small, we get $S(t)x_0 \neq x_0$ for all $t > 0$ and $x_0 \ (\overline{\Omega} \setminus \Omega_1) \cap P$. Furthermore, by the argument in Lemmas 1.2 and 1.3, $i(S(t), \Omega \cap P, P)$ is well-defined an $i(A, \Omega \cap P, P) = i(S(t), \Omega \cap P, P)$ for all $t > 0$. It is also easy to verify from the above inequality that $S(s)x_0 \in \overline{\Omega}_1 \cap P$ for every $x_0 \in \partial \Omega$ with $s = 2(\beta - \alpha)\rho^{-2}$ (see[1]). Since

$S(s)$ has no fixed point in $f^{-1}[\alpha, \beta] \cap P$, we have $i(S(s), \Omega \cap P, P) = i(S(s), V \cap P, P) = 1$, where the latter equality is obtained by the line property of the set $V \cap P$. This completes the proof.

Corollary 2.1 — Let $f: \mathbb{R}^n \to R$ be of class C^1 with the form $f(x) = \frac{1}{2}|x|^2 - V(x)$ and such that

(*i*) $f(x) \to \infty$ as $|x| \to \infty$,

(*ii*) there exists $r_1 > 0$ such that $grad f(x) \neq 0$ for every x with $|x| \geq r_1$,

(*iii*) $grad\, V(x)\, P \subset P$, where P is the canonical positive cone in $\mathbb{R}^n$.

Then for every $r \geq r_1$

$$i(grad\, V(x), B_r(\theta) \cap P, P) = 1.$$

Remark 2.3 : The above corollary represents an extension of the corresponding resulting of Krasnosel'skii[7]. Related results with a different method can be found in[8].

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REFERENCES

1. H. Amann, *Porc. Amer. Math. Soc.*, **84** (1982), 591-95.
2. K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lect. Notes in Mathematics, **596**, Springer-Verlag, New York, 1977.
3. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
4. D. J. Guo, *Nonlinear Functional Analysis*, Shandong Sci. Tech. Publishing House, 1985.
5. Z. Q. Han, *Quaest. Math.*, **22** (1999), 171-81.
6. Z. Q. Han, *Acta. Math. Sci.*, **17** (1997), 455-62.
7. M. A. Krasnosel'skii, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, New York, 1984.
8. G. F. Lang, *Northeast. Math. J.*, **10** (1994), 190-98.
9. R. D. Nussbaum, *Ann. Mat. Pura Appl.*, **89** (1971), 217-58.
10. J. X. Sun, *J. Math. Anal. Appl.*, **126** (1987), 566-73.
11. J. X. Sun and S. C. Hu, *Discrete Contin. Dyn. Syst.*, **9** (2003), 483-96