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PERIODIC BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS*

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In the present paper, by means of comparison principle and monotone iterative techniques, we obtain a new theorem about the existence of solutions for impulsive differential-difference equations.

Key Words : Comparison Principle; Monotone-Iterative Technique; Impulsive Delay Equations; Periodic Boundary Value Problem

1. INTRODUCTION

In the mathematical simulation in various fields of science and technology impulsive differential equations are often used[1,2]. This leads to the necessity of justification of methods for their approximate solution, in particular, for periodic solution. The existence analysis of periodic boundary value problems ordinary or functional differential equations with impulsive effects has been the subject of many investigations[3-7, 13-15] in recent years, and various interesting results have been reported. However, the corresponding theory for impulsive delay differential equations has not yet been fully developed. There are a number of difficulties one must face in developing the corresponding theory of impulsive delay differential equations. Recently existence and uniqueness results for impulsive delay differential equations are presented in[17], and continuous dependence on initial value for impulsive delay differential equations is presented in[12], and periodic problems are considered in[8,9,16]. In[9] it is considered using topological degree method and in[8] the authors considered the problem by means of monotone iterative method. We also note that the impulsive differential inequalities are a very valid tool for proving the comparison result, see[11].

In this paper, via a new comparison principle and monotone iterative method, we shall study the problem

$$x'(t) = f(t, x(t), x(t-h)), \quad t \in J_0 = J \setminus \{t_1, ..., t_p\},$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, ..., p,$$

$$x(t) = x(0), \qquad \text{for } t \in [-h, 0],$$

$$x(0) = x(T), \qquad \text{... (1.1)}$$

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where $J = [0, T]$, $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T, h > 0$ is a constant, $I_k \in C(R, R), k = 1, \ldots, p$, $f : J \times R \times R \to R$ is continuous everywhere except at $\{t_k\} \times R \times R$; $f\left(t_k^+, x, y\right)$ and $f\left(t_k^-, x, y\right)$ exist, $f\left(t_k^-, x, y\right) = f(t_k, x, y)$ for all $(x, y) \in R \times R$, $\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right)$.

We first introduce the following set $PC([a, b], R) = \{u : [a, b] \to R$ is continuous at $t \neq t_k$ and the left continuous at $t = t_k$ and the right limit $u\left(t_k^+\right)$ exists for $k = 1, 2, \ldots, p\}$ and $PC^1([a, b], R) = \{u \in PC([a, b], R)$ is continuously differentiable for $t \in (a, b), t \neq t_k$ and has continuous left derivatives at the points $t_k \in (a, b)\}$. Let $\Omega = PC([-h, T], R) \cap PC^1([0, T], R)$. For $u \in \Omega$, define its norm by $\| u \|_\Omega = \max\{\| u \|_1, \| u \|_2\}$, where

$$\| u \|_1 = \sup\{| u(t) | : t \in [-h, T]\}, \quad \| u \|_2 = \sup\{| u'(t) | : t \in [0, T]\}.$$

In order to prove our comparison result, we need the following lemma.

Lemma 1[2] — Let $s \in [0, T), c_k \geq 0, \alpha_k, k = 1, \ldots, p$ are constants and let $p, q \in PC(J, R)$, $x \in PC^1(J, R)$. If

$$\begin{cases} x'(t) \leq p(t)\, x(t) + q(t), & t \in [s, T),\ t \neq t_k, \\ x\left(t_k^+\right) \leq c_k\, x(t_k) + \alpha_k, & t \in [s, T), \end{cases}$$

then for $t \in [s, T]$

$$\begin{aligned} x(t) \leq\ & x(s^+) \left(\prod_{s < t_k < t} c_k\right) \exp\left(\int_s^t p(u)\, du\right) \\ & + \int_s^t \left(\prod_{u < t_k < t} c_k\right) \exp\left(\int_u^t p(\tau)\, d\tau\right) q(u)\, du \\ & + \sum_{s < t_k < t} \left(\prod_{t_k < t_i < t} c_i\right) \exp\left(\int_{t_k}^t p(\tau)\, d\tau\right) \alpha_k. \end{aligned}$$

2. Comparison Result

We will establish a new comparison result, which plays an important role on monotone iterative technique.

Lemma 2 — Let function $m \in \Omega$ satisfy the inequalities

$$m'(t) \le -M\,m(t) - N\,m(t-h), \qquad t \in J_0,$$

$$\Delta m(t) \le -L_k\, m(t), \quad t = t_k,$$

$$m(t) = m(0), \qquad \text{for } t \in [-h, 0],$$

$$m(0) \le m(T), \qquad \text{... (2.1)}$$

where $M > 0, N > 0, h > 0, 0 \le L_k < 1,\ k = 1, 2, ..., p$ and

$$M + N \le \frac{\prod_{k=1}^{p} (1 - L_k)}{\int_0^T \prod_{s < t_k < T} (1 - L_k)\, ds} \qquad \text{... (2.2)}$$

Then $m(t) \le 0$ for $t \in [-h, 0]$.

PROOF : Suppose the conclusion is not valid. It follows that there exists $t^* \in [0, T]$ such that $m(t^*) > 0$. We will consider the following two possible cases.

Case 1 : For all $t \in [-h, T], m(t) \ge 0$. Then $m'(t) \le 0, (\not\equiv 0),\ t \in J_0$ and since $m\left(t_k^+\right) \le (1 - L_k)\ \ m(t_k) \le m(t_k), k = 1, 2, ..., p,$ so $m(t)$ is non-increasing, thus we have $m(0) \ge m(t^*) > 0$.

Consider the inequalities

$$\begin{cases} m'(t) \le 0\ (\not\equiv 0), & t \in J_0 \\ m\left(t_k^+\right) \le (1 - L_k)\, m(t_k). \end{cases}$$

By Lemma 1, we can obtain

$$m(t) < m(0) \prod_{0 < t_k < t} (1 - L_k).$$

Let $t = T$, we have

$$m(t) < m(0) \prod_{k=1}^{p} (1 - L_k) < m(0),$$

which is a contradiction with $m(0) \le m(T)$.

Case 2 : $m(t) < 0$ for some $t \in [0, T]$.

(*i*) Suppose $m(T) \geq 0$. Let $m(\bar{t}) = \inf_{s \in J} m(s) = -\lambda, \lambda > 0$, $\bar{t} \in (t_i, t_{i+1}]$. Set $\bar{t} \neq t_i^+$ (if, $\bar{t} = t_i^+$ we can prove in the same way). From $m(t) \geq -\lambda$ and $m(t-h) \geq -\lambda$, $t \in [0, T]$, we can get

$$m'(t) \leq -M\,m(t) - N\,m(t-h) \leq (M+N)\,\lambda,\ t \in J_0.$$

We consider the inequalities

$$\begin{cases} m'(t) \leq \lambda(M+N),\ t \neq t_k,\ t \in [\bar{t}, T], \\ m\left(t_k^+\right) \leq (1-L_k)\,m(t_k),\ k = i+1, \ldots, p. \end{cases}$$

From Lemma 1, we have

$$m(t) \leq m(\bar{t}) \prod_{\bar{t} < t_k < T} (1-L_k) + \int_{\bar{t}}^{t} \prod_{s < t_k < t} (1-L_k)\,\lambda(M+N)\,ds.$$

Let $t = T$, we obtain

$$m(T) \leq m(\bar{t}) \prod_{\bar{t} < t_k < T} (1-L_k) + \int_{\bar{t}}^{T} \prod_{s < t_k < T} (1-L_k)\,\lambda(M+N)\,ds$$

$$< -\lambda \prod_{k=1}^{p} (1-L_k) + \lambda(M+N) \int_{0}^{T} \prod_{s < t_k < T} (1-L_k)\,ds,$$

since $m(T) \geq 0$, so

$$\frac{\prod_{k=1}^{p} (1-L_k)}{\int_{0}^{T} \prod_{s < t_k < T} (1-L_k)\,ds} < M+N,$$

which contradicts (2.2).

(*ii*) Suppose $m(T) < 0$, $m(0) \leq m(T) < 0$. Let $\hat{t} \in [0, t^*]$ such that $m(\hat{t}) = \inf_{t \in [0, t^*]} m(t) = -\lambda_1 > 0$, in a similar way as in the proof of (*i*), for $t \in [0, t^*]$ we can obtain the following inequalities

$$\begin{cases} m'(t) \le \lambda_1 (M+N), & t \ne t_k \\ m\left(t_k^+\right) \le (1-L_k)\, m(t_k), & \end{cases}$$

then

$$m(t^*) \le m(\hat{t}) \prod_{\hat{t}<t_k<t^*} (1-L_k) + \int_{\hat{t}}^{t^*} \prod_{s<t_k<t^*} (1-L_k)\, \lambda_1 (M+N)\, ds,$$

since $m(t^*) > 0$, $m(\hat{t}) = -\lambda_1$, we have

$$0 < -\lambda_1 \prod_{0<t_k<t^*} (1-L_k) + \int_0^{t^*} \prod_{s<t_k<t^*} (1-L_k)\, \lambda_1 (M+N)\, ds,$$

thus

$$\frac{\prod\limits_{0<t_k<t^*}^{p} (1-L_k)}{\int_0^{t^*} \prod\limits_{s<t_k<t^*} (1-L_k)\, ds} < M+N,$$

$$\frac{\prod\limits_{k=1}^{p} (1-L_k)}{\int_0^{T} \prod\limits_{s<t_k<T} (1-L_k)\, ds} \le \frac{\prod\limits_{0<t_k<t^*}^{p} (1-L_k)}{\int_0^{t^*} \prod\limits_{s<t_k<t^*} (1-L_k)\, ds} < M+N.$$

This contradicts (2.2). Therefore, we must have $m(t) \le 0, t \in [-h, T]$. The proof of the theorem is complete.

By Lemma 2, we can deduce the Lemma 1 in[9].

Corollary 1 — Let function $m \in \Omega$ satisfy the inequalities (2.1) and

$$(M+N)\, p\tau < (1-L)^p, \qquad \text{... (2.3)}$$

where M and N are positive constants, $0 \le L_i < 1$, $(i = 1, ..., p)$,

$$\tau = \max\{t_k - t_{k-1},\ k = 1, 2, ..., p+1\},$$

$$L = \max\{L_i : i = 1, ..., p\}.$$

Then $m(t) \leq 0$ to $t \in [-h, T]$.

PROOF : We have

$$\int_0^T \prod_{s<t_k<T} (1-L_k)\,ds = \int_0^{t_1} \prod_{s<t_k<T} (1-L_k)\,ds$$

$$+ \int_{t_1^+}^{t_2} \lambda(M+N)\,ds \prod_{s<t_k<T} (1-L_k)\,ds$$

$$+ \ldots + \int_{t_{p-1}^+}^{t_p} \prod_{s<t_k<T} (1-L_k)\,ds \quad + \int_{t_p^+}^{T} \prod_{s<t_k<T} (1-L_k)\,ds$$

$$= \prod_{k=1}^{p} (1-L_k)(t_1-t_0) + \prod_{k=2}^{p} (1-L_k)(t_2-t_1)$$

$$+ \ldots + (1-L_p)(t_p-t_{p-1}) + (t_{p+1}-t_p)$$

$$\leq p\tau,$$

that is

$$\int_0^T \prod_{s<t_k<T} \cdot (1-L_k)\,ds \leq p\tau. \qquad \ldots (2.4)$$

By (2.3) and (2.4) yield,

$$(M+N) \int_0^T \prod_{s<t_k<T} (1-L_k)\,ds$$

$$\leq (M+N)\,p\tau < (1-L)^p \leq \prod_{k=1}^{p} (1-L_k),$$

so

$$M+N < \frac{\prod_{k=1}^{p} (1-L_k)}{\int_0^T \prod_{s<t_k<T} (1-L_k)\,ds},$$

this shows that the inequality (2.2) holds. By Lemma 2, we have $m(t) \leq 0$, $t \in [-h, T]$. The proof is complete.

3. Existence Criteria

In this section, we consider the linear problem of (1.1)

$$x'(t) + M x(t) - N x(t-h) = \sigma(t) \qquad t \in [0, T],\ t \neq t_k,$$

$$\Delta x(t) + L_k x(t) = L_k \eta(t) + I_k(\eta(t)), \quad t = t_k,$$

$$x(t) = x(0), \qquad \text{for } t \in [-h, 0],$$

$$x(0) \leq x(T), \qquad \dots (3.1)$$

where $\sigma(t) \in PC(J, R)$, $\eta(t) \in \Omega$.

We need the following well-known theorem[2].

Lemma 3 — $x \in \Omega$ is a solution of PBVP (3.1) if and only if $x \in PC([-h, T], R)$ is a solution of the integral equation

$$x = \begin{cases} \displaystyle\int_0^T G(t, s)[\sigma(s) - N x(s-h)]\, ds \\ \displaystyle + \sum_{k=1}^{p} G(t, t_k)[-L_k(x(t_k) - \eta(t_k)) + I_k(\eta(t_k))], \quad t \in [0, T], \\ x(0), \quad t \in [-h, 0], \end{cases}$$

where

$$G(t, s) = \frac{1}{1 - e^{-MT}} \begin{cases} e^{-M(t-s)}, & 0 \leq s < t \leq T, \\ e^{-M(T+t-s)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Using Banach's fixed point theorem and Lemma 3, we can obtain the following result.

***Theorem* 1** — *Let $M > 0$, $N > 0$, $0 \leq L_k < 1$, $(k = 1, \dots, p)$, $I_k \in C(R, R)$, $\sigma \in PC(J, R)$ and $\eta \in \Omega$, if*

$$\frac{N}{M} + \frac{1}{1 - e^{-MT}} \sum_{k=1}^{p} L_k < 1, \qquad \dots (3.2)$$

then PBVP (3.1) has a unique solution $x \in PC([-h, T], R)$.

Proof : Let $\Omega_0 = \{x \in PC([-h, T], R), x(t) = x(0),\ \forall t \in [-h, 0]\}$. It is clear that Ω_0 is a Banach space with the norm

$$\|x\| = \sup\{|x(t)| : t \in [-h, T]\}.$$

For any $x \in \Omega_0$, we define the operator F by

$$[Fx](t) = \begin{cases} \int_0^T G(t,s)\,[\sigma(s) - N x(s-h)]\,ds \\ \quad + \Sigma_{0<t_k<T} G(t,t_k)\,[-L_k x(t_k) + L_k(\eta(t_k)) + I_k(\eta(t_k))], \; t \in [0,T], \\ FX(0), \; t \in [-h, 0], \end{cases}$$

then $Fx \in \Omega_0$, that is $F\Omega_0 \subset \Omega_0$.

For any $x, y \in \Omega_0$, $t \in [0, T]$, we have

$$\begin{aligned} |[Fx](t) - [Fy](t)| &\le N \int_0^T |G(t,s)|\,|x(s-h) - y(s-h)|\,ds \\ &\quad + \sum_{k=1}^{p} G(t,t_k)\,L_k\,|x(t_k) - y(t_k)| \\ &\le \frac{N}{M}\,\|x-y\| + \frac{1}{1-e^{-MT}} \sum_{k=1}^{p} L_k\,\|x-y\| \\ &= \left(\frac{N}{M} + \frac{1}{1-e^{-MT}} \sum_{k=1}^{p} L_k \right) \|x-y\|. \end{aligned}$$

thus

$$\begin{aligned} \|Fx - Fy\| &= \sup_{t \in [-h,T]} |[Fx](t) - [Fy](t)| \\ &= \sup_{t \in [0,T]} |[Fx](t) - [Fy](t)| \\ &\le \left(\frac{N}{M} + \frac{1}{1-e^{-MT}} \sum_{k=1}^{p} L_k \right) \|x-y\|. \end{aligned}$$

By (3.2), F is a contraction on Ω_0. So there exists unique $x \in \Omega_0$ such that $x = Fx$, that is, x is a unique solution of PBVP (3.1).

4. Monotone Iterative Technique

In this section, we shall establish existence criteria for solutions of the PBVP (1.1), by the method of lower and upper solutions and the monotone iterative technique. The comparison result developed in Section 2 will be used frequently.

First we introduce the concept of lower and upper solutions.

Definition 1 — Let $w \in \Omega \cap \Omega_0$, then w is called an upper solution of PBVP (1.1) if

$$
\begin{aligned}
& w'(t) \geq f(t, w(t), w(t-h)), && t \in J_0, \\
& \Delta w(t) \geq I_k(w(t)), && t = t_k, \\
& w(t) = w(0), && t \in [-h, 0], \\
& w(0) \geq w(T). && \qquad \text{... (4.1)}
\end{aligned}
$$

A lower solution $\upsilon(t)$ can be defined similarly by reversing the inequalities in (4.1). Defines $[\upsilon, w] = \{z \in \Omega \cap \Omega_0 : \upsilon \leq z \leq w\}$.

Theorem 2 — *Let the following conditions hold.*

(A_1) *υ, w are lower and upper solutions of (1.1) respectively, and $\upsilon(t) \leq w(t)$ for $t \in [-h, T]$.*

(A_2) *The function $f \in PC([0, T] \times R \times R, R)$ safisfies*

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \geq -M(x - \bar{x}) - N(y - \bar{y})$$

where $\upsilon(t) \leq \bar{x}(t) \leq x(t) \leq w(t)$, $\upsilon_t(\theta) \leq \bar{y}_t(\theta) \leq x_t(\theta) \leq w_t(\theta)$, $t \in [0, T]$, $\theta \in [-h, 0]$, $M > 0$, $N > 0$.

(A_3) *The functions $I_k \in C(R, R)$, moreover*

$$I_k(x) - I_k(y) \geq -L_k(x - y),$$

where

$$\upsilon(t_k) \leq y(t_k) \leq x(t_k) \leq w(t_k), \ 0 \leq L_k < 1, \ k = 1, 2, ..., p.$$

(A_4) *The inequalities (2.2) and (3.2) are valid.*

Then there exist monotone sequences $\{\upsilon_n\}$, $\{w_n\}$, $n = 0, 1, \ldots$ with $\upsilon_0 = \upsilon$, $w_0 = w$ such that $\upsilon_n(t) \nearrow \rho(t)$ and $w_n(t) \searrow r(t)$ uniformly on $[-h, T]$, which $\rho(t)$ and $r(t)$ are the minimal and maximal solutions of PBVP (1.1), respectively, that is, if $x \in \Omega$ is any solution of (1.1), then $\rho(t) \leq x(t) \leq r(t)$.

PROOF : For any $\eta(t) \in [\upsilon, w]$, we consider

$$
\begin{aligned}
x'(t) + M x(t) + N x(t-h) &= f(t, \eta(t), \eta(t-h)) \\
&+ M \eta(t) + N \eta(t-h), \ t \in J_0,
\end{aligned}
$$

$$\Delta x(t) + L_k x(t) = L_k \eta(t) + I_k(\eta(t)), \quad t = t_k, \ k = 1, \dots, p,$$

$$x(t) = x(0), \quad t \in [-h, 0],$$

$$x(0) = x(T). \qquad \dots (4.2)$$

Thus by Theorem 1, PBVP (4.2) has a unique solution $x \in \Omega \cap \Omega_0$. Hence we can define a mapping $A ; [\upsilon, w] \to \Omega \cap \Omega_0$ by $A\eta = x$. Next, we prove that the mapping A enjoys the following properties :

(*a*) $\upsilon \le A\upsilon, \quad w \ge Aw$;
(*b*) For any $\eta_1, \eta_2 \in [\upsilon, w]$ with $\eta_1 \le \eta_2$, then $A\eta_1 \le A\eta_2$.

The proof of property (*a*). Let $A\upsilon = \upsilon_1, m(t) = \upsilon(t) - \upsilon_1(t), t \in [-h, T]$. If follows that we have for $t \ne t_k$

$$\begin{aligned} m'(t) &= \upsilon'(t) - \upsilon_1'(t) \\ &\le f(t, \upsilon(t)\ \upsilon(t-h)) - \\ &[f(t, \upsilon(t), \upsilon(t-h)) + M(\upsilon(t) - \upsilon_1(t)) + N(\upsilon(t-h) - \upsilon_1(t-h))] \\ &= -Mm(t) - Nm(t-h), \ t \in J_0, \end{aligned}$$

and for $t = t_k$

$$\begin{aligned} \Delta m(t_k) &= \Delta \upsilon(t_k) - \Delta \upsilon_1(t_k) \\ &\le I_k(\upsilon(t_k)) - L_k[\upsilon(t_k) - \upsilon_1(t_k)] - I_k(\upsilon(t_k)) \\ &= -L_k m(t_k), \end{aligned}$$

also

$$m(t) = \upsilon(t) - \upsilon_1(t) = \upsilon(0) - \upsilon_1(0) = m(0), t \in [-h, 0],$$

$$m(0) = \upsilon(0) - \upsilon_1(0) \le \upsilon(T) - \upsilon_1(T) = m(T).$$

By Lemma 2, we have $m(t) \le 0$, that is, $\upsilon(t) \le \upsilon_1(t)$, $t \in [-h, T]$. Analogously, it is proved that $w_1(t) = Aw(t) \le w(t)$ on $[-h, T]$.

Next, we prove the property (*b*). For any $\eta_1, \eta_2 \in [\upsilon, w]$ with $\eta_1 \le \eta_2$, let $x_1 = A\eta_1$, $x_2 = A\eta_2$ and $m(t) = x_1 - x_2$, then we get for $t \in J_0$

$$m'(t) = x_1'(t) - x_2'(t)$$

$$= f(t, \eta_1(t), \eta_1(t-h)) + M[\eta_1(t) - x_1(t)]$$

$$+ N[\eta_1(t-h) - x_1(t-h)]$$

$$- [f(t, \eta_2(t), \eta_2(t-h)) + M(\eta_2(t) - x_2(t))$$

$$+ N(\eta_2(t_h) - x_2(t-h))]$$

$$\leq M[\eta_2(t) - \eta_1(t)] + N[\eta_2(t-h) - \eta_1(t-h)) - M[\eta_2(t) - \eta_1(t)]$$

$$- N[\eta_2(t-h) - \eta_1(t-h)] - M[x_1(t) - x_2(t)]$$

$$- N[x_1(t-h) - x_2(t-h)]$$

$$= -M m(t) - N m(t-h),$$

and for $t = t_k, k = 1, \ldots, p,$

$$\Delta m(t) = \Delta x_1(t) - \Delta x_2(t)$$

$$= I_k(\eta_1(t)) - L_k(x_1(t) - \eta_1 t)) - I_k(\eta_2(t)) + L_k(x_2(t) - \eta_2(t))$$

$$\leq L_k(\eta_2(t) - \eta_1(t)) - L_k(x_1(t) - \eta_1(t)) + L_k(x_2(t) - \eta_2(t))$$

$$= -L_k(x_1(t) - x_2(t)) = -L_k m(t),$$

also

$$m(t) = x_1(t) - x_2(t) = x_1(0) - x_2(0) = m(0), \quad t \in [-h, 0]$$

$$m(0) = x_1(0) - x_2(0) = x_1(T) - x_2(T) = m(T).$$

Hence

$$m(t) \leq 0, \qquad t \in [-h, T],$$

i.e.,

$$A\eta_1 \leq A\eta_2.$$

Define sequences $\{v_n(t)\}$, $\{w_n(t)\}$ by

$$v_{n+1} = A v_n, \qquad w_{n+1} = Aw_n, \quad n = 0, 1, 2, \ldots$$

where $v_0 = v$, $w_0 = w$.

It follows, from the properties (a) and (b), that

$$v_0 \le v_1 \le \ldots \le v_n \le w_n \le \ldots \le w_1 \le w_0$$

in $[-h, T]$ and by standard arguments we conclude that

$$\lim_{n\to\infty} v_n = \rho(t), \qquad \lim_{n\to\infty} w_n = r(t),$$

and

$$v_n(t) \le \rho(t) \le r(t) \le w_n(t).$$

It is easy to show that ρ and r are solutions of the PBVP (1.1) using the fact that v_n, w_n satisfy the relations

$$v'_{n+1}(t) + M v_{n+1}(t) + N v_{n+1}(t-h)$$

$$= f(t, v_n(t), v_n(t-h)) + M v_n(t) + N v_n(t-h), \ t \ne t_k,$$

$$\Delta v_{n+1}(t) + L_k v_{n+1}(t)$$

$$= L_k v_n(t) + I_k(v_n(t)), \ t = t_k, \quad k = 1, \ldots, p,$$

$$v_{n+1}(t) = v_{n+1}(0), \ t \in [-h, 0]$$

$$v_{n+1}(0) = v_{n+1}(T),$$

and

$$w'_{n+1}(t) + M w_{n+1}(t) N w_{n+1}(t-h)$$

$$= f(t, w_n(t), w_n(t-h)) + M w_n(t) + N w_n(t-h), \ t \ne t_k,$$

$$\Delta w_{n+1}(t) + L_k w_{n+1}(t)$$

$$= L_k w_n(t) + I_k(w_n(t)), \ t = t_k,$$

$$w_{n+1}(t) = w_{n+1}(0), \ t\, v \in [-h, 0],$$

$$w_{n+1}(0) = w_{n+1}(T).$$

Finally, we prove that if $x \in [v, w]$ is any solution of PBVP (1.1), then $\rho(t) \le x(t) \le r(t)$, $t \in [-h, T]$. To this end, we assume, without loss of generality, that $v_n(t) \le x(t) \le w_n(t)$ for some n, since $v_0(t) \le x(t) \le w_0(t)$. From property (b), we can get

$$v_{n+1}(t) \le x(t) \le w_{n+1}(t), \ t \in [-h, T].$$

Hence we can conclude that

$$v_n(t) \leq x(t) \leq w_n(t), \text{ for all } n,$$

passing to the limit as $n \to \infty$, we obtain

$$\rho(t) \leq x(t) \leq r(t), \qquad t \in [-h, T].$$

Which completes the proof.

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