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ON STANDARD IDEALS IN WEIGHTED CONVOLUTION ALGEBRAS ON SUBSEMIGROUPS OF $\mathbb{R}^+$

H. V. DEDANIA

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, *Gujarat, India*
E-mail: hvdedania@yahoo.com

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Let S be a unital, dense, difference, subsemigroup of the positive real line. It is a well-known problem to ask whether there exists a local weight ω on S such that the convolution of Banach algebra $l^1(S, \omega)$ has only standard closed ideals. Thomas[3] proved that, if the weight ω satisfies some extra condition, then every primary element of $l^1(S, \omega)$ generates a standard closed ideal. In this article, we shall prove that, if the weight ω satisfies some mild condition, then there do exist secondary elements in $l^1(S, \omega)$ which generate standard closed ideals.

Key Words : Semigroup; Weight; Radical and Commutative Banach Algebra

Throughout, let S be a unital subsemigroup of the positive real line $\mathbb{R}^+$. It is a difference subsemigroup of $\mathbb{R}^+$ if $S = G \cap \mathbb{R}^+$ for some subgroup G of $\mathbb{R}$. A weight on S is a map $\omega : S \to (0, \infty)$ such that $\omega(s + t) \leq \omega(s)\, \omega(t)$ $(s, t \in S)$. Then the Banach space $\ell^1(S, \omega)$ consisting of all functions $f : S \to \mathbb{C}$ such that $\|f\| := \sum_{s \in S} |f(s)|\, \omega(s) < \infty$ is a unital, commutative Banach algebra with the usual convolution product.

A weight ω on S is

(1) local if $\lim_{n \to \infty} \omega(ns)^{1/n} = 0$ $(s \in S \setminus \{0\})$;

(2) locally bounded if, for any $0 < a < b < \infty$, $\sup\{\omega(s) : s \in [a, b] \cap S\} < \infty$;

(3) locally positive if, for any $0 < a < \infty$, $\inf\{\omega(s) : s \in [0, a] \cap S\} > 0$.

It is clear that ω is local if and only if $l^1(S, \omega)$ is a local Banach algebra, i.e., it has exactly one maximal closed ideal; namely $M_{[0]} = \{f \in l^1(S, \omega) : f(0) = 0\}$. For an element $f \in l^1(S, \omega)$, let $\text{supp } f = \{s \in S : f(s) \neq 0\}$ and $\alpha(f) = \inf\{s \in S : f(s) \neq 0\}$. An element

$f \in l^1(S, \omega)$ is a primary element if $\alpha(f) \in S$ and $f(\alpha(f)) \neq 0$; otherwise it is a secondary element.

For $f \in l^1(S, \omega)$, let $I_f = \overline{f * l^1(S, \omega)}$. Then I_f is the smallest closed ideal of $l^1(S, \omega)$ containing f. For $d \geq 0$, define

$$M_{[d]}(S, \omega) = \left\{ f \in l^1(S, \omega) : f(s) = 0 \; (s \in [0, d) \cap S) \right\};$$

$$M_{(d)}(S, \omega) = \left\{ f \in l^1(S, \omega) : f(s) = 0 \; (s \in [0, d) \cap S) \right\};$$

$$M_{[\infty]}(S, \omega) = M_{(\infty)}(S, \omega) = \{0\}.$$

Then these are closed ideals of $l^1(S, \omega)$. These are standard closed ideals; the order closed ideals of $l^1(S, \omega)$, if any, are non-standard closed ideals.

A well-known problem on the algebra $l^1(S, \omega)$ is the following: Does there exist a local weight ω on S such that $l^1(S, \omega)$ has only standard closed ideals? In particular, does there exist a local weight ω on S such that I_f is a standard closed ideal of $l^1(S, \omega)$ for each $f \in l^1(S, \omega)$? When $S = \mathbb{Z}^+$, the problem is successfully solved by[3,4]. When S is dense in $\mathbb{R}^+$ (in particular, $S = \mathbb{Q}^+$), the picture is not clear. However, Thomas[5] proved that, if ω satisfies some extra condition, then I_f is a standard closed ideal $l^1(S, \omega)$ for each primary element $f \in l^1(S, \omega)$. Then Domar[2] strengthened this result for the same types of elements. However, nothing has been said about the secondary elements of $l^1(S, \omega)$. It will be immediate from our result that there do exist secondary elements f of $l^1(S, \omega)$ such that I_f is a standard closed ideal. Though our method is constructive, it is unlikely that this can work to solve this problem completely.

First we prove three lemmas which will be needed in the proof of our main Theorem.

Lemma 1 — Let S be a unital, dense, difference subsemigroup of $\mathbb{R}^+$. Let ω be a locally bounded, local weight on S. Let $f \in l^1(S, \omega)$. Then the following are equivalent:

(*i*) I_f is a standard closed ideal of $l^1(S, \omega)$;

(*ii*) there exist a decreasing sequence $\{r_n\}$ in $[\alpha(f), \infty) \cap S$ which converges to $\alpha(f)$, a sequence $\{\varepsilon_n\}$ in $(0, \infty)$ which converges to 0, and a sequence $\{f_n\}$ in $l^1(S, \omega)$ such that $\| \delta_{r_n} - f_n * f \| \leq \varepsilon_n \; (n \in \mathbb{N})$.

PROOF : (*i*) implies (*ii*). This is clear.

(*ii*) implies (*i*). Let $f \in l^1(S, \omega)$ and let $d = \alpha(f)$. Fix $t \in (d, \infty) \cap S$. Then there is n_0 such that $t - r_n \geq 0$ $(n \geq n_0)$. Set $m_t := \sup\{\omega(t - r_n) : n \geq n_0\}$. Since ω is locally bounded, $m_t < \infty$. Then, for all $n \geq n_0$,

$$\| \delta_t - \delta_{t-r_n} * f_n * f \| = \| \delta_{t-r_n} * (\delta_{r_n} - f_n * f) \|$$

$$\leq \omega(t - r_n) \| \delta_{r_n} - f_n * f \| \leq m_t \, \varepsilon_n.$$

Hence, $\delta_t \in I_f$. Since $t \in (d, \infty) \cap S$ is arbitrary and I_f is a closed ideal, $M_{(d)}(S, \omega) \subseteq I_f$.

Now if f is a secondary element, then $I_f \subseteq M_{(d)}(S, \omega)$ is always true; and so $I_f = M_{(d)}(S, \omega)$. Now suppose that f is a primary element. In this case, define $g(d) = 0$ and $g(s) = f(s)$ otherwise. Then $g \in M_{(d)}(S, \omega) \subseteq I_f$. Then $f(d)\,\delta_d = f - g \in I_f$. Hence $M_{[d]}(S, \omega) \subseteq I_f$. But $I_f \subseteq M_{[d]}(S, \omega)$ is always true; and so $I_f = M_{[d]}(S, \omega)$.

Lemma 2 — Let S be a unital, dense, difference subsemigroup of $\mathbb{R}^+$ and let ω be a local weight on S. If $f \in l^1(S, \omega)$ has finite support, then there exists $p \in l^1(S, \omega)$ such that $p * f = \delta_{\alpha(f)}$.

PROOF : Let $f \in l^1(S, \omega)$ have a finite support, say supp $f = \{r_1, \ldots, r_n\}$ such that $r_1 < \ldots < r_n$. Then $\alpha(f) = r_1$. Set $g(s) = f(s + r_1)$ $(s \in S)$. Since S is a difference semigroup, $g \in l^1(S, \omega)$ such that $g(0) = f(r_1) \neq 0$ and $f = \delta_{r_1} * g$. Since $l^1(S, \omega)$ is a local Banach algebra, g is invertible. Let $p \in l^1(S, \omega)$ such that $p * g = \delta_0$. Then $p * f = p * \delta_{r_1} * g = \delta_{r_1} = \delta_{\alpha(f)}$. □

Lemma 3 — Let S be a unital, dense, difference subsemigroup of $\mathbb{R}^+$. Let ω be a locally bounded weight on S. Then ω is locally positive.

PROOF : Suppose, if possible, ω is not locally positive. Then there exists $a \in S$ such that $\inf\{\omega(s) : s \in [0, a] \cap S\} = 0$. Choose a sequence $\{s_n\}$ in $[0, a] \cap S$ such that $\omega(s_n) \to 0$ as $n \to \infty$. Let $b \in S \setminus \{0\}$. Then $\{b + s_n\}$ is contained in $[b, a + b] \cap S$. Since ω is locally bounded, there exists a finite number M such that $\omega(s) \leq M$ $(s \in [b, a + b] \cap S)$; in particular,

$\omega(a+b-s_n) \leq M$ $(n \in \mathbb{N})$. Then $0 < \omega(a+b) \leq \omega(a+b-s_n)\ \omega(s_n) \leq M\ \omega(s_n) \to 0$ as $n \to \infty$. This is a contradiction. Thus ω is a locally positive weight on S.

Theorem 1 — *Let* S be a unital, dense, difference subsemigroup of $\mathbb{R}^+$. Let ω be a locally bounded, local weight on S. Let $E \subseteq S$ be a countable set. Then there is $f \in l^1(S, \omega)$ such that I_f is a standard closed ideal and supp $f = E$.

PROOF : Let $E = \{r_n : n \in \mathbb{N}\}$, and let $d = \inf\{r_n : n \in \mathbb{N}\}$.

Case (i) $d \in E$: Set $F := \{s-d : s \in E\}$. Then $0 \in F \subseteq S$. Then there exists an invertible element $g \in l^1(S, \omega)$ such that supp $g = F$. Since g is invertible, $g * l^1(S, \omega) = l^1(S, \omega)$. Take $f = \delta_d * g \in l^1(S, \omega)$. Then $I_f = I_{\delta_{d * g}} = \delta_d * g * l^1(S, \omega) = \delta_d * l^1(S, \omega) = M_{[d]}(S, \omega)$ and supp $f = E$.

Case (ii) $d \notin E$: Choose a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ and a positive sequence $\{\varepsilon_k\}$ such that: (1) $r_1 = r_{n_1} > r_{n_2} > \to d$; (2) $r_{n_k} \leq r_i$ $(1 \leq i < n_k, k \in \mathbb{N})$; (3) $\varepsilon_k\, \omega(r_{n_k}) \to 0$ as $k \to \infty$. For convenience we set $n_0 = 0$. Since ω is locally bounded and since S is a difference semigroup, the weight ω is locally positive due to Lemma 3. Hence $\varepsilon_k \to 0$. So we may assume that $\varepsilon_k \leq 1$ $(k \in \mathbb{N})$.

Choose a complex number $\alpha_1 \in \mathbb{C}^\bullet$ $(= \mathbb{C} \setminus \{0\})$. Set $f_1 = \alpha_1\, \delta_{r_1}$. Then there exists $p_1 \in l^1(S, \omega)$ such that $p_1 * f_1 = \delta_{r_{n_1}} = \delta_{r_1}$ due to Lemma 2. Next, for each $n_1 < i \leq n_2$, choose $\alpha_i \in \mathbb{C}^\bullet$ such that

$$\sum \left\{ |\alpha_i|\, \omega(r_i) : n_1 < i \leq n_2 \right\}$$

$$\leq 2 \left| \alpha_{n_2} \right| \omega\left(r_{n_2} \right) \leq \min \left\{ \frac{\varepsilon_1\, \omega(r_{n_1})}{\| p_1 \|}, \frac{|\alpha_{n_1}|\, \omega(r_{n_1})}{2} \right\}.$$

Set $f_2 = \sum \left\{ \alpha_i\, \delta_{r_i} : 1 \leq i \leq n_2 \right\}$. Since the support of f_2 is finite, there exists $p_2 \in l^1(S, \omega)$ such that $p_2 * f_2 = \delta_{r_{n_2}}$ due to Lemma 2 again. Suppose we have chosen $\alpha_1, \ldots, \alpha_{n_k} \in \mathbb{C}^\bullet$ such that, for each $2 \leq j \leq k$, the following is satisfied:

$$2 \left| \alpha_{n_j} \right| \omega\left(r_{n_j} \right) \leq \min \left\{ \frac{\varepsilon_{j-1}\, \omega(r_{n_{j-1}})}{\| p_{j-1} \|}, \frac{|\alpha_{n_{j-1}}|\, \omega(r_{n_{j-1}})}{2} \right\},$$

and

$$\sum \left\{ |\alpha_i| \, \omega(r_i) : n_{j-1} < i \leq n_j \right\} \leq 2 \left| \alpha_{n_j} \right| \omega\left(r_{n_j} \right),$$

where $p_j * f_j = \delta_{r_{n_j}}$. Now, for each $n_k < i \leq n_{k+1}$, choose $\alpha_i \in \mathbb{C}^\bullet$ such that

$$\sum \left\{ |\alpha_i| \, \omega(r_i) : n_k < i \leq n_{k+1} \right\}$$

$$\leq 2 \left| \alpha_{n_{k+1}} \right| \omega\left(r_{n_{k+1}} \right) \leq \min \left\{ \frac{\varepsilon_k \, \omega\left(r_{n_k} \right)}{\| p_k \|}, \frac{\left| \alpha_{n_k} \right| \omega\left(r_{n_k} \right)}{2} \right\}.$$

Set $f_{k+1} = \sum \left\{ \alpha_i \, \delta_{r_i} : 1 \leq i \leq n_{k+1} \right\}$. Since the support of f_{k+1} is finite, there exists $p_{k+1} \in l^1(S, \omega)$ such that $p_{k+1} * f_{k+1} = \delta_{r_{n_{k+1}}}$ because of Lemma 2.

Finally define $f = \sum \left\{ \alpha_i \, \delta_{r_i} : i \in \mathbb{N} \right\}$. Then supp $f = E$ and

$$\| f \| = \sum_{i=1}^{\infty} |\alpha_i| \, \omega(r_i) = \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} |\alpha_i| \, \omega(r_i)$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \alpha_{n_k} \right| \omega\left(r_{n_k} \right) = 2 \left| \alpha_{n_1} \right| \omega\left(\alpha_{n_1} \right) \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty.$$

Hence $f \in \ell^1(S, \omega)$. Then, for all sufficiently large k,

$$\left\| \delta_{r_{n_k}} - p_k * f \right\| = \| p_k * f_k - p_k * f \| \leq \| p_k \| \, \| f_k - f \|$$

$$= \| p_k \| \sum_{j=n_{l_{k+1}}}^{\infty} |\alpha_j| \, \omega(r_j) = \| p_k \| \sum_{j=k}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} |\alpha_i| \, \omega(r_i)$$

$$\leq \| p_k \| \sum_{j=k}^{\infty} 2 \left| \alpha_{n_{j+1}} \right| \omega\left(r_{n_{j+1}} \right) \leq 3 \| p_k \| \left| \alpha_{k+1} \right| \omega\left(r_{n_{k+1}} \right)$$

$$\leq \frac{3}{2} \, \varepsilon_k \, \omega\left(r_{n_k} \right).$$

Hence, by Lemma 1, it follows that I_f is a standard closed ideal. □

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