Indian J. pure appl. Math., **35**(11) : 1285-1292, November 2004

TWO PROPERTIES OF SEMISTRICTLY PREINVEX FUNCTIONS*

X. M. YANG,

Department of Mathematics, Chongqing Normal University, Chongqing 400047, *People's Republic of China*

X. Q. YANG AND K. L. TEO

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom Kowloon, Hong Kong

(*Received*: 5 *November* 2003; *after final revision*: 26 *July* 2004; *accepted*: 10 *January* 2005)

We consider in this paper the class of semistrictly preinvex functions introduced by Yang and Li (2001, *J. Math. Anal. Appl.*, **258**, 287-308). Two properties of semistrictly preinvex functions are given. These properties include a gradient criterion of semistrictly preinvex functions and a criterion of semistrictly preinvex functions in terms of intermediate-point semistrict and prequasiinvexity. In particular, one of them improves Yang and Li's an interesting result.

Key Words: Semistrictly Preinvex Functions; Gradient, Properties; Prequasiinvexity

1. INTRODUCTION

A significant generalization of convex functions termed preinvex functions was introduced by Weir and Mond[1] and Weir and Jeyakumar[2]. Mohan and Neogy[3] introduced condition C defined as follows:

Condition C. Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$; we say that the function η satisfies the condition C if for any x, y and for any $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda \eta(x, y)) = -\lambda \eta(x, y),$$

$$\eta(x, y + \lambda \eta(x, y)) = (1 - \lambda) \eta(x, y).$$

And they proved that a differentiable function which is invex with respect to η is also preinvex under condition C. Mohan and Neogy also gave an example which shows that condition C may hold for a general class of functions η, rather than just for the trivial case of $\eta(x, y) = x - y$. Later, some important properties of preinvex functions were obtained in[4].

*This research was partially supported by the National Natural Science Foundation of China, NCET and Natural Science Foundation of Chongqing.

Recently, a new class of generalized convex functions was introduced in[5]. These functions are closely related to preinvex functions, and were termed semistrictly preinvex functions. Some properties of semistrictly preinvex functions were established in[5]. In this paper, we consider the class of semistrictly preinvex functions. To improves Yang and Li's an interesting result in[5], we give a new criterion of semistrictly preinvex functions in terms of intermediate-point semistrict preinvexity and prequasiinvexity. On the other hand, we note that many results for monotonicity of the gradient maps have been appeared from 1990's. Based on generalized invex monotonicity and generalized invariant monotonicity had been introduced and some properties have been discussed between generalized invex monotonicity and generalized preinvexity in[6-8]. Thus, motivated both by earlier research works and by the importance of the concept of generalized preinvexity, the second result in this paper is a gradient criterion of semistrictly preinvex functions.

2. NOTATIONS

Weir and Mond[1] and Weir and Jeyakumar[2] first introduced preinvex functions and defined invex sets.

Definition 2.1[1-2] — A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$x, y \in K, \ \lambda \in [0, 1] \Rightarrow y + \lambda \eta (x, y) \in K.$$

Definition 2.2[1-2] — Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Let $f: K \to \mathbb{R}$. We say that f is preinvex if

$$f(y + \lambda \eta (x, y)) \leq \lambda f(x) + (1 - \lambda) f(y), \ \forall x, y \in K, \lambda \in [0, 1].$$

Yang and Li[5] introduced semistrictly preinvex functions as follows:

Definition 2.3[5] — Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Let $f: K \to \mathbb{R}$. We say that f is semistrictly preinvex if $\forall x, y \in K, f(x) \neq f(y)$, we have

$$f(y + \lambda \eta (x, y) < \lambda f(x) + (1 - \lambda) f(y), \ \forall \lambda \in (0, 1).$$

Example 2.1 — This example illustrates that a semistrictly preinvex function is not necessarily a preinvex function. Let

$$f(x) = \begin{cases} -|x| & \text{if } |x| \leq 1; \\ -1 & \text{if } |x| \geq 1, \end{cases}$$

$$\eta(x, y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0; \\ x - y & \text{if } x \leq 0, y \leq 0; \\ x - y & \text{if } x < -1, x > 1; \\ x - y & \text{if } x < -1, y > 1; \\ y - x & \text{if } -1 \leq x, x \leq 0, y \geq 0; \\ y - x & \text{if } -1 \leq y, \leq 0, y \geq 0; \\ y - x & \text{if } 0 \leq x \leq 1, y \leq 0; \\ y - x & \text{if } 0 \leq y \leq 1, x \leq 0. \end{cases}$$

Then, f is a semistrictly preinvex function with respect to η on $\mathbb{R}$. However, by letting $x = 2$, $y = -2$, $\lambda = \frac{1}{2}$, we have

$$f(y + \lambda\, \eta(x, y)) = f\left(-2 + \frac{1}{2}\eta(2, -2)\right)$$
$$= f(0) = 0 > -1 = f(2) = f(-2) = \lambda f(x) + (1 - \lambda) f(y).$$

That is, f is not a preinvex function with respect to the same η.

3. Two Properties of Semistrictly Preinvex Functions

Based on Condition C, we easy give an interesting result as follows:

Lemma 3.1 — Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. If η satisfies condition C, then for any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in [0, 1]$, we have

$$\eta(y + \alpha\,\eta(x, y), y + \beta\,\eta(x, y)) = (\alpha - \beta)\,\eta(x, y).$$

From this identity, we can derive very easily the following useful special case:

$$\eta(y + \alpha\,\eta(x, y), y) = -\eta(y, y + \alpha\,\eta(x, y)) = \alpha\,\eta(x, y).$$

In[5], Yang and Li obtained the following result:

***Theorem* 3.1** — *Let K be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $f : K \to \mathbb{R}$. a preinvex function for the same η on K. If there exists $\alpha \in (0, 1)$ such that for every $x, y \in K, f(x) \neq f(y)$ implies*

$$f(t + \alpha\,\eta(x, y)) < \alpha f(x) + (1 - \alpha) f(y)$$

$$f(y+(1-\alpha)\,\eta\,(x,y)) \;<\; \alpha f(y)+(1-\alpha)f(x),$$

then f is a semistrict preinvex function on K.

Now we generalize the result above as follows:

***Theorem* 3.2** — *Let K be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ where η satisfies condition C, and $f: K \to \mathbb{R}$. a preinvex function for the same η on K. If for every $x, y \in K, f(x) \neq f(y)$, there exists $\alpha \in (0, 1)$ such that*

$$f(y+\alpha\,\eta\,(x,y)) \;<\; \alpha f(x)+(1-\alpha)f(y),$$

then f is a semistrict preinvex function on K.

PROOF: For $x, y \in K$ that satisfy $f(x) \neq f(y)$ and for each $\lambda \in (0, 1)$ based on the conditions stated in theorem, we have

$$f(y+\lambda\,\eta\,(x,y)) \;\leq\; \lambda f(x)+(1-\lambda)f(y). \qquad \dots (1)$$

(*i*) Let $\lambda \leq \alpha$. From Lemma 3.1,

$$y+\frac{\lambda}{\alpha}\,\eta\,(y+\alpha\,\eta\,(x,y),y)=y+\lambda\,\eta\,(x,y).$$

Thus, it follows from (1) and the preinvexity of f that

$$\begin{aligned} f(y+\lambda\,\eta\,(x,y)) &= f\left(y+\frac{\lambda}{\alpha}\,\eta\,(y+\alpha\,\eta\,(x,y),y)\right) \\ &\leq \frac{\lambda}{\alpha}f(y+\alpha\,\eta\,(x,y))+\left(1-\frac{\lambda}{\alpha}\right)f(y) \\ &< \frac{\lambda}{\alpha}[\alpha f(x)+(1-\alpha)f(y)] + \left(1-\frac{\lambda}{\alpha}\right)f(y) \\ &= \lambda f(x)+(1-\lambda)f(y). \end{aligned} \qquad \dots (2)$$

(*ii*) Let $\alpha < \lambda$, that is

$$0 < \frac{1-\lambda}{1-\alpha} < 1.$$

From Lemma 3.1,

$$\begin{aligned} y+\alpha\,\eta\,(x,y)&+\left(1-\frac{1-\lambda}{1-\alpha}\right)\eta\,(x,y+\alpha\,\eta\,(x,y)) \\ &= y+\lambda\,\eta\,(x,y). \end{aligned}$$

According to (1) and the preinvexity of f, we have

$$\begin{aligned} f(y+\lambda\eta(x,y)) &= f\left(y+\alpha\eta(x,y)+\left(1-\frac{1-\lambda}{1-\alpha}\right)\eta(x,y+\alpha\eta(x,y),y)\right) \\ &\le \left(1-\frac{1-\lambda}{1-\alpha}\right)f(x)+\frac{1-\lambda}{1-\alpha}f(y+\alpha\eta(x,y)) \\ &< \left(1-\frac{1-\lambda}{1-\alpha}\right)f(x)+\frac{1-\lambda}{1-\alpha}[\alpha f(x)+(1-\alpha)f(y)] \\ &= \lambda f(x)+(1-\lambda)f(y). \qquad \text{... (3)} \end{aligned}$$

(2) and (3) imply that f is a semistrictly preinvex function on K. □

Remark 3.1 : (*i*) In Theorem 3.2, α depends on x, y, but in Theorem 3.1 α doesn't depend on x, y. Thus Theorem 3.2 extends Theorem 3.1. (*ii*) The proof of Theorem 3.2 is simpler than the proof of Theorem 3.1 (see Theorem 4.2 in[5]).

In[5], we establish a gradient property of semistrictly preinvex functions. In[6], we give relationships between generalized invexity and generalized invariant monotonicity. In this section, motivated by[6], we will give another gradient property of semistrictly preinvex functions.

Condition A: Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Function f is said to satisfy the condition A if for any $x, y \in K$ and $f(x) < f(y)$, the following holds,

$$f(y+\eta(x,y)) < f(y).$$

Theorem 3.3[5] — *Let K be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies condition C, and $f : K \to \mathbb{R}$. a differentiable function that satisfies condition A. Then f is a semistrictly preinvex function for the same η on K if and only if for every $x, y \in K, f(x) \neq f(y)$, we have*

$$f(y) > f(x) + \eta(y,x)^T \nabla f(x).$$

Theorem 3.4 — *Let K be a nonempty invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies condition C, and $f : K \to \mathbb{R}$. a differentiable function that satisfies condition A and $f(x+\eta(y,x)) = f(y)$, $\forall x, y \in K$. Then f is a semistrictly preinvex function for the same η on K if and only if for every pair of points $x, y \in K, f(x) \neq f(y)$, we have*

$$\eta(y,x)^T \nabla f(x) + \eta(x,y)^T \nabla f(y) < 0.$$

PROOF : Suppose that f is a semistrictly preinvex function on K. Let $x, y \in K, f(x) \neq f(y)$. From Theorem 3.3, we have

$$f(y) > f(x) + \eta(y,x)^T \nabla f(x), \quad \dots (4)$$

$$f(x) > f(y) + \eta(x,y)^T \nabla f(y), \quad \dots (5)$$

From (4) and (5), we obtain

$$\eta(y,x)^T \nabla f(x) + \eta(x,y)^T \nabla f(y) < 0.$$

Conversely, suppose that for every pair of points $x, y \in K, f(x) \neq f(y)$, we have

$$\eta(y,x)^T \nabla f(x) + \eta(x,y)^T \nabla f(y) < 0. \quad \dots (6)$$

By Theorem 3.3, we will show that

$$f(y) > f(x) + \eta(y,x)^T \nabla f(x).$$

By Contradiction, suppose that there exists $x, y \in K, f(x) \neq f(y)$ such that

$$f(y) \leq f(x) + \eta(y,x)^T \nabla f(x). \quad \dots (7)$$

From the assumed condition $f(x + \eta(y,x)) = f(y)$, we have

$$f(x + \eta(y,x)) = f(y) \leq f(x) + \eta(y,x)^T \nabla f(x). \quad \dots (8)$$

From the mean-value theorem, we obtain

$$f(x + \eta(y,x)) - f(x) = \eta(y,x)^T \nabla f(x + \bar{\lambda}\eta(y,x)). \quad \dots (9)$$

Thus, (8) and (9) imply

$$\eta(y,x)^T \nabla f(x + \bar{\lambda}\eta(y,x)) \leq \eta(y,x)^T \nabla f(x). \quad \dots (10)$$

(*i*) If $f(x + \bar{\lambda}\eta(y,x)) \neq f(x)$, then from (6) we obtain

$$\eta(x + \bar{\lambda}\eta(y,x), x)^T \nabla f(x) + \eta(x, x + \bar{\lambda}\eta(y,x))^T \nabla f(x + \bar{\lambda}\eta(y,x)) < 0.$$

From Condition C, we have

$$\eta(y,x)^T \nabla f(x) < \eta(y,x)^T \nabla f(x + \bar{\lambda}\eta(y,x)),$$

which contradicts (10).

(*ii*) If $f(x + \bar{\lambda}\eta(y,x)) = f(x)$, we want to show that

$$f(x + \bar{\lambda}\eta(y,x)) = f(x) \neq f(x + \alpha\eta(y,x)),$$

for some $\alpha \in (0, \bar{\lambda})$.

Assume to the contrary that

$$f(x+\bar{\lambda}\eta(y,x))=f(x)=f(x+\alpha\eta(y,x)), \qquad \ldots (11)$$

for any $\alpha \in (0,\bar{\lambda})$.

Let

$$\phi(\alpha)=f(x+\alpha\eta(y,x)), \qquad \forall \alpha \in [0,\bar{\lambda}].$$

Then (11) implies that

$$\phi(\alpha)=const=f(x+\bar{\lambda}\eta(y,x)).$$

Above equality yields

$$0=\phi'(\bar{\lambda})=\eta(y,x)^T\nabla f(x+\bar{\lambda}\eta(y,x)).$$

Hence, $\eta(y,x)^T\nabla f(x+\bar{\lambda}\eta(y,x))=0$, which together with (9) contradicts $f(x)\neq f(y)=f(x+\eta(y,x))$. Thus,

$$f(x+\bar{\lambda}\eta(y,x))=f(x)\neq f(x+\alpha\eta(y,x)), \qquad \ldots (12)$$

for some $\alpha \in (0,\bar{\lambda})$.

Now from (6) and (12), we have

$$\eta(x+\bar{\lambda}\eta(y,x), x+\alpha\eta(y,x)^T\nabla f(x+\alpha\eta(y,x))$$
$$+\eta(x+\alpha\eta(y,x), x+\bar{\lambda}\eta(y,x))^T\nabla f(x+\bar{\lambda}\eta(y,x))<0,$$

and

$$\eta(x, x+\alpha\eta(y,x)^T\nabla f(x+\alpha\eta(y,x))+\eta(x+\alpha\eta(y,x),x)^T\nabla f(x)<0.$$

From Condition C, two inequalities above imply

$$\eta(x+\bar{\lambda}\eta(y,x)\,x)^T\nabla f(x+\alpha\eta(y,x))$$
$$+\eta(x,x+\bar{\lambda}\eta(y,x)^T\nabla f(x+\bar{\lambda}\eta(y,x))<0, \qquad \ldots (13)$$

and

$$\eta(x,x+\bar{\lambda}\eta(y,x))^T\nabla f(x+\alpha\eta(y,x))+\eta(x+\bar{\lambda}\eta(y,x),x)^T\nabla f(x)<0. \qquad \ldots (14)$$

Again from Condition C, (13) + (14) imply

$$\eta(y,x)^T\nabla f(x+\bar{\lambda}\eta(y,x))>\eta(y,x)^T\nabla f(x). \qquad \ldots (15)$$

Combining (9) and (15), we have

$$f(y)>f(x)+\eta(y,x)^T\nabla f(x),$$

which contradicts (7). Thus, for every pair of points $x, y \in K, f(x)\neq f(y)$, we have

$$f(y) > f(x) + \eta(y, x)^T \nabla f(x).$$

ByTheorem 3.3, f is a semistrictly preinvex functior on K. This completes the proof. □

Example 3.1 — Let

$$f(x) = \begin{cases} -1 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

$$\eta(x, y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0, \\ x - y & \text{if } x \leq 0, y \leq 0, \\ 1 - y & \text{if } x < 0, y \geq 0, \\ -1 - y & \text{if } x > 0, y \leq 0. \end{cases}$$

It easily shows that functions f and η satisfying conditions A, C and $f(x + \eta(y, x)) = f(y)$.

REFERENCES

1. T. Weir and B. Mond, *Journal of Mathematical Analysis and Applications*, **136** (1988), 29-38.
2. T. Weir and V. Jeyakumar, *Bulletin of The Australian Mathematical Society*, **38** (1998), 177-89.
3. S. R. Mohan and S. K.Neogy, *Journal of Mathematical Analysis and Applications*, **189** (1995), 901-08.
4. X. M. Yang and D. Li, *Journal of Mathematical Analysis and Applications*, **256** (2001), 229-41.
5. X. M. Yang and D. Li, *Journal of Mathematical Analysis and Applications*, **258** (2001), 287-308.
6. X. M. Yang, X. Q. Yang and K. L. Teo, *Journal of Optimization Theory and Applications*, **117**(3) (2003), 607-25.
7. G. Ruiz-Garzn, R. Osuna-Gmez and A. Rufin-Lizana, *European Journal of Operational Research*, **144**(3) (2003), 501-12.
8. X. M. Yang, X. Q. Yang and K. L. Teo, *European Journal of Operational Research*, **164**(1) (2005), 115-19.