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PERIODIC SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS

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The Set of ω-periodic solutions is found for delay differential equations with piecewise continuous arguments.

Key Words: Delay Differential Euqation; Piecewise Continuous Argument; Periodic Solution

1. INTRODUCTION

The delay differential equations with piecewise continuous arguments were initiated by Cooke and Wiener[1] and Shah and Wiener[2]. These equations can be applied to certain "sequential-continuous" models of disease dynamics as treated in[7]. As mentioned in these references, the strong interest in such equations motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. The theorems of delay differential equations with piecewise continuous arguments is the subjects of many investigations. See for example[1-15].

In[4], periodic solution of the following differential equations with constant deviating argument

$$y'(t) + ay(t) + by([t-1]) = 0, t \geq 0 \qquad \text{... (*)}$$

is investigated, where a and b are constants, $[\cdot]$ is greatest-integer function. Since the solution by $y(0) = D_0$ and $y(-1) = D_{-1}$, can be given explicitly[4], the set of all can be found. When more general equations of the form

$$y'(t) + a(t)y(t) + b(t)y([t-1]) = 0, t \geq 0, \qquad \text{... (1)}$$

and the equations with piecewise constant delay of the form

$$y'(t) + a(t)\, y(t) + c(t)\, y(t - [t]) = 0, \ t \geq 0 \qquad \text{... (2)}$$

are considered, where $a(t)$, $b(t)$ and $c(t)$ are continuous real functions with positive integral periodic ω. The problem arises as what are the corresponding set of ω-periodic solutions.

In this paper, we consider following equation which more general than (1) and (2),

$$y'(t) + a(t)\, y(t) + b(t)\, y([t-1]) + c(t)\, y(t - [t]) = 0, \ t \geq 0, \qquad \text{... (3)}$$

where $a(t)$, $b(t)$ and $c(t)$ are continuous real functions with positive integral periodic ω. We introduce a new technique to prove set of ω-periodic solutions of (3) and certain linear algebra equation are isomorphic to obtain some necessary and sufficient conditions for (3) has not nontrivial ω-periodic solution and the set of ω-periodic solutions of (3) is an infinite aggregate. Because, we using same definition of a solution for (1) and (3), then results of (3) is hold for (1). Noting that (2) differ with (3) on a solution definition, but by using same technique we get the results of (2) as (3), further, we lead off a necessary and sufficient condition for every solution of (2) is with period ω.

Here, we always let Z be the set of positive integers and $\omega \in Z$.

2. On Equations with Piecewise Constant Argument and Piecewise Constant Delay

In (1) (or ((3)), $a(t)$, $b(t)$ and $c(t)$ are continuous real functions on $[-1, \infty)$, which are periodic with common period ω. We adopt the definition for a solution of (1) or (3) stated in[4]. By a solution of (1) (or (3)) we mean a function $y(t)$ which is defined on the set $\{-1, 0\} \cup [0, \infty)$ and which satisfies the conditions:

(i) $y(0) = D_0$ and $y(-1) = D_{-1}$ (where D_0 and D_{-1} are initial data);

(ii) $y(t)$ is continuous on $[0, \infty)$;

(iii) The derivative $y'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$, where one-side derivatives exists;

(iv) Equation (1) (or (3)) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral endpoints.

The set of all ω-periodic solutions of (3) will be denoted by Ω_ω. In order to determine, we will let

$$\alpha_n = \exp\left(- \int_{n-1}^{m} a(s)\, ds \right), \ n \in Z, \qquad \text{... (4)}$$

$$\beta_n = - \int_{n-1}^{n} b(s) \exp\left(- \int_{s}^{n} a(v)\, dv \right) ds, \ n \in Z, \qquad \text{... (5)}$$

$$\gamma_n = - \int_{n-1}^{n} c(s) \exp\left(- \int_{s}^{n} a(v)\, dv - \int_{0}^{s-n+1} (a(u) + c(u))\, du \right) ds, \; n \in Z, \quad \text{... (6)}$$

or

$$\delta_n = \int_{n-1}^{n} c(s) \exp\left(- \int_{s}^{n} a(v)\, dv \right)$$

$$\left(\int_{0}^{s-n+1} b(v) \exp\left(- \int_{v}^{s-n+1} (a(u) + c(u)\, du \right)\right) dv\, ds, n \in Z. \quad \text{... (7)}$$

Since $a(t)$, $b(t)$ and $c(t)$ are ω-periodic, it easy to see that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are also ω-periodic sequencies. Consider the periodic difference boundary value problem

$$\begin{cases} z_0 = z_\omega, \\ \delta_k z_\omega + \gamma_k z_1 + \beta_k z_{k-1} + \alpha_k z_k - z_{k+1} = 0, k = 1, 2, ..., \omega, \\ z_1 = z_{\omega+1} \end{cases} \quad \text{... (8)}$$

or equivalently, the system of ω linear equations

$$\begin{cases} (\gamma_1 + \alpha_1) z_1 - z_2 + (\delta_1 + \beta_1) z_\omega = 0, \\ \dots \\ \delta_k z_\omega + \gamma_k z_1 + \beta_k z_{k-1} + \alpha_k z_k - z_{k+1} = 0, \\ \dots \\ (\gamma_\omega - 1) z_1 + \beta_\omega z_{\omega-1} + (\delta_\omega + \alpha_\omega) w_\omega = 0. \end{cases} \quad \text{... (9)}$$

By using the matrices, (9) can be denoted

$$A_\omega \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{... (10)}$$

it is easy to see that, when $\omega \geq 3$,

$$A_\omega = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_1 \\ \gamma_2 & 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_2 \\ \gamma_3 & 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{\omega-1} & 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_{\omega-1} \\ \gamma_\omega & 0 & 0 & 0 & \cdots & \cdots & 0 & \delta_\omega \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha_1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & \beta_1 \\ \beta_2 & \alpha_2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \beta_3 & \alpha_3 & -1 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{\omega-1} & \alpha_{\omega-1} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & \beta_\omega & \alpha_\omega \end{bmatrix}, \quad \text{... (11)}$$

$$A_2 = \begin{bmatrix} \gamma_1 + \alpha_1 & \delta_1 + \beta_1 - 1 \\ \gamma_2 + \beta_2 - 1 & \delta_2 + \alpha_2 \end{bmatrix}, \ \omega = 2, \quad \text{... (12)}$$

and

$$A_1 = (\alpha_1 + \beta_1 + \gamma_1 + \delta_1 - 1), \ \omega = 1. \quad \text{... (13)}$$

Let the solution space of (10) be denoted by Ψ_ω.

***Theorem* 1** — *The solution spaces Ω_ω and Ψ_ω are isomorphic.*

PROOF : Let $y(t)$ be a ω-periodic solution of (3). It is easy to see that

$$y'(t) + (a(t) + c(t))\, y(t) + b(t)\, y([t-1]) = 0, \ 0 \leq t < 1. \quad \text{... (14)}$$

so that

$$\left(y(t) \exp\left(\int_0^t (a(u) + c(u)\, du \right)\right)' + y(-1)\, b(t)$$

$$\exp\left(\int_0^t (a(u) + c(u))\, du \right) = 0, \quad 0 \le t < 1. \qquad \text{... (15)}$$

Integrating (15) from 0 to t, we have

$$y(t) \exp\left(\int_0^t (a(u) + c(u))\, du \right) - y(0) + y(-1) \int_0^t b(v)$$

$$\exp\left(\int_0^v (a(u) + c(u))\, du \right) dv = 0, \quad 0 \le t < 1. \qquad \text{... (16)}$$

Thus,

$$y(t) = y(0) \exp\left(- \int_0^t (a(u) + c(u))\, du \right) - y(-1) \int_0^t b(v)$$

$$\exp\left(-\int_v^t (a(u) + c(u))\, du \right) dv = 0, \quad 0 \le t < 1. \qquad \text{... (17)}$$

From (3), we see that for $n = 0, 1, 2, \ldots,$

$$\left(y(t) \exp\left(\int_n^t a(s)\, ds \right)\right)' + y(n-1)\, b(t) \exp\left(\int_n^t a(v)\, dv \right)$$

$$+ y(t-n)\, c(t) \exp\left(\int_n^t a(v)\, dv \right) = 0, \quad n \le t < n+1.$$

it is following that, for $n \le t < n + 1$

$$y(t) \exp\left(\int_n^t a(s)\, ds \right) - y(n) + y(n-1) \int_n^t b(t) \exp\left(\int_n^s a(v)\, dv \right) ds$$

$$+ \int_n^t y(s-n)\, c(s) \exp\left(\int_n^s a(v)\, dv \right) ds = 0. \qquad \text{... (18)}$$

That is for $n \le t < n + 1$

$$y(t) = y(n) \exp\left(-\int_n^t a(s)\,ds\right) - y(n-1)\int_n^t b(s)\exp\left(-\int_s^t a(v)\,dv\right)$$

$$-\int_n^t y(s-n)\,c(s)\exp\left(-\int_s^t a(v)\,dv\right)ds. \qquad \ldots (19)$$

By (17) and (19), for we have

$$y(t) = y(n)\exp\left(-\int_n^t a(s)\,ds\right) - y(n-1)\int_n^t b(s)\exp\left(-\int_s^t a(v)\,dv\right)ds$$

$$-y(0)\int_n^t c(s)\exp\left(-\int_s^t a(v)\,dv - \int_0^{s-n}(a(u)+c(u)\,du\right)ds \qquad \ldots (20)$$

$$+y(-1)\int_n^t c(s)\exp\left(-\int_s^t a(v)\,dv\right)\left(\int_0^{s-n} b(v)\exp\left(-\int_v^{s-n}(a(u)+c(u))\,du\right)dv\right)ds.$$

Since $\lim_{n \to (n+1)^-} y(t) = y(n+1)$, we see further that for $n = 0, 1, 2, \ldots,$

$$y(n+1) = \alpha_{n+1}\,y(n) + \beta_{n+1}\,y(n-1) + \gamma_{n+1}\,y(0) + \delta_{n+1}\,y(-1). \qquad \ldots (21)$$

If we now let $z_k = y(k-1)$ for $k = 1, 2, \ldots$ and $z_0 = z_\omega$ then $\{z_k\}_{k=0}^{\infty}$ is a ω-periodic sequence and from (21), we see that the column vector $(z_1, z_2, \ldots, z_\omega)^T$ is a solution of (10), that is $(z_1, z_2, \ldots, z_\omega) \in \Psi_\omega$.

Conversely, let $(z_1, z_2, \ldots, z_\omega)^T \in \Psi_\omega$. Define $z_0 = z\omega$ and extending the finite sequence $\{z_1, z_2, \ldots, z_\omega\}$ into the unique infinite ω-periodic sequence $\{z_k\}_{k=0}^{\infty}$. Let $y(n) = z_{n+1}$ for $n = -1, 0, 1, \ldots$, and let the function $y(t)$ on each interval $[n, n+1) \subset [0, \infty)$ be defined by (20). Then it is not difficult to check that this function $y(t)$ is a ω-periodic solution of (3). In other words, we have found a one to one and onto mapping from Ω_ω to Ψ_ω. The linearity of this mapping is easy to see. The proof of Theorem 1 is complete.

***Theorem* 2** — *(3) has not nontrivial ω-periodic solution if and only if*

$$\det A_\omega \neq 0. \qquad \ldots (22)$$

The set of all ω-periodic solutions of (3) is an infinite aggregate if and only if

$$\det A_\omega = 0. \quad \text{... (23)}$$

PROOF : The results are immediate corollary of Theorem 1.

Corollary 1 — The set of all ω-periodic solutions of (3) is an infinite aggregate if one of the following two conditions is satisfied:

(*i*) for $k = 1, 2, ..., \omega$,

$$\alpha_k + \beta_k + \gamma_k + \delta_k = 1; \quad \text{... (24)}$$

(*ii*) for $k = 1, 2, ..., \omega$

$$\alpha_k + \beta_k + 1, \sum_{k=1}^{\omega} \gamma_k = 0 \text{ and } \sum_{k=1}^{\omega} \delta_k = 0. \quad \text{... (25)}$$

PROOF : It is easy to see that, if (*i*) or (*ii*) is satisfied the matrix A_ω is singular. By using Theorem 2, the proof Corollary 1 is complete.

Now we consider (1), it easy to see (1) is (3) when $c(t) = 0$ on $[-1, \infty)$ together for $n \in Z$, let $\alpha_n, \beta_n, \gamma_n$ and δ_n as (4)-(5), relatively. We see that α_n and β_n which are independent of $c(t)$ but then $\gamma_n = 0$ and $\delta_n = 0$. Set B_ω is a matrix as following:

$$B_\omega = \begin{bmatrix} \alpha_1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & \beta_1 \\ \beta_2 & \alpha_2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \beta_3 & \alpha_3 & -1 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{\omega-1} & \alpha_{\omega-1} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & \beta_\omega & \alpha_\omega \end{bmatrix}, \quad \omega \geq 3,$$

... (26)

$$B_2 = \begin{bmatrix} \alpha_1 & \beta_1 - 1 \\ \beta_2 - 1 & \alpha_2 \end{bmatrix}, \quad \omega = 2, \quad \text{... (27)}$$

of

$$B_1 = (\alpha_1 + \beta_1 - 1), \ \omega = 1. \quad \text{... (28)}$$

As an immediate corollary of Theorem 1, in the following result is true.

Corollary 2 — (*i*) has not nontrivial ω-periodic solution if and only if

$$\det B_\omega \neq 0; \qquad \text{... (29)}$$

The set of all ω-periodic solutions of (3) is an infinite aggregate if and only if

$$\det B_\omega = 0. \qquad \text{... (30)}$$

Example 1 — Consider the following equation

$$y'(t) - (\sin \pi t)\, y(t) + (2\pi \sin 2\pi t)\, y([t-1]) + (\sin \pi t)\, (t - [t]) = 0, \; t \geq 0, \qquad \text{... (31)}$$

where $a(t) = -\sin \pi t$, $b(t) = 2\pi \sin 2\pi t$ and $c(t) = \sin \pi t$ are continuous real functions with period 2. It is easy to verify that

$$\alpha_1 = \exp\left(\frac{2}{\pi}\right), \beta_1 = -4\pi(\pi+1) + 4\pi(\pi-1)\exp\left(\frac{2}{\pi}\right),$$

$$\gamma_1 = 1 - \exp\left(\frac{2}{\pi}\right), \delta_1 = 4\pi(\pi+1) + 4\pi(1-\pi)\exp\left(\frac{2}{\pi}\right),$$

$$\alpha_2 = \exp\left(-\frac{2}{\pi}\right), \beta_2 = -4\pi(\pi-1) + 4\pi(\pi+1) + 4\pi(\pi+1)\exp\left(-\frac{2}{\pi}\right),$$

$$\gamma_2 = 1 - \exp\left(-\frac{2}{\pi}\right), \delta_2 = -4\pi(\pi-1) - 4\pi(1+\pi)\exp\left(-\frac{2}{\pi}\right).$$

It follows that $\det A_2 = 0$, by using Theorem 2, we know The set of all 2-periodic solutions of (31) is an infinite aggregate.

3. On Equations with Piecewise Constant Delay

In (2), $a(t)$ and $c(t)$ are continuous real functions on $[0, \infty)$, which are with period ω. We adopt the definition for a solution of (2) stated in[1]. By a solution of (1) (or (3)) we mean a function $y(t)$ which is defined on $[0, \infty)$ and which satisfies the conditions:

(*i*) $y(0) = D_0$ (where D_0 is initial data);

(*ii*) $y(t)$ is continuous on $[0, \infty)$;

(*iii*) The derivative $y'(t)$ exists at each point $t \in [0, \infty)$ with the possible exception of the points $[t] \in [0, \infty)$, where one-side derivatives exists;

(*iv*) Equation (2) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral endpoints.

The set of all ω-periodic solutions of (2) will be denoted by $\overline{\Omega}_\omega$. In order to determine

$\overline{\Omega}_{\omega}$, for $n \in Z$, let α_n and γ_n as (4) and (6), relatively. Consider the periodic difference boundary value problem or equivalently the system of ω linear equations.

$$\begin{cases} z_0 = z_{\omega}, \\ \gamma_k z_1 + \alpha_k z_k - z_{k+1} = 0, k = 1, 2, ..., \omega, \\ z_1 = z_{\omega+1}, \end{cases} \quad \text{... (32)}$$

$$\begin{cases} (\gamma_1 + \alpha_1) z_1 - z_2 = 0, \\ \cdots \\ \gamma_k z_1 + \alpha_k z_k - z_{k+1} = 0, \\ \cdots \\ (\gamma_{\omega} - 1) z_1 + \alpha_{\omega} z_{\omega} = 0 \end{cases} \quad \text{... (33)}$$

by using the matrices, (33) can be denoted

$$C_{\omega} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{\omega-1} \\ z_{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \text{... (34)}$$

it is easy to see that,

$$C_{\omega} = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 & 0 \\ \gamma_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{\omega-1} & 0 & \cdots & 0 & 0 \\ \gamma_{\omega} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} \alpha_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{\omega-1} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_\omega \end{bmatrix}, \ \omega \geq 3, \qquad \dots (35)$$

$$C_2 = \begin{bmatrix} \gamma_1 + \alpha_1 & -1 \\ \gamma_2 - 1 & \alpha_2 \end{bmatrix}, \ \omega = 2, \qquad \dots (36)$$

or

$$C_1 = (\gamma_1 + \alpha_1 - 1), \ \omega = 1. \qquad \dots (37)$$

Let the solution space of (34) be denoted by $\overline{\Psi}_\omega$.

Lemma 1 — The solution spaces $\overline{\Omega}_\omega$ and $\overline{\Psi}_\omega$ are isomorphic.

PROOF : Let $y(t)$ be a ω-periodic solution of (2). It is easy to see that

$$y'(t) + (a(t) + c(t))\, y(t) = 0, \ 0 \leq t < 1, \qquad \dots (38)$$

so that

$$y(t) = y(0) \ \exp\left(- \int_0^t (a(u) + c(u))\, du \right), \ 0 \leq t < 1. \qquad \dots (39)$$

From (2), we see that for $n = 0, 1, 2, \dots,$

$$\left(y(t) \ \exp\left(\int_s^t a(s)\, ds \right) \right)' + y(t-n) \exp\left(\int_n^t a(s)\, ds \right) = 0, \ n \leq t < n+1, \qquad \dots (40)$$

it is following that, for $n = 0, 1, 2, \dots,$

$$y(t) = y(n) \exp\left(-\int_s^t a(s)\, ds \right) - \int_n^t y(s-n)\, c(s)$$

$$\exp\left(-\int_s^t a(v)\, dv \right) ds, \ n \leq t < n+1. \qquad \dots (41)$$

By (39) and (41), we have

$$y(t) = y(n) \exp\left(-\int_n^t a(s)\, ds \right) - y(0) \int_n^t c(s)$$

$$\exp\left(-\int_s^t a(v)\, dv - \int_0^{s-n} (a(u) + c(u))\, du \right) ds, \; n \le t < n + 1. \quad \ldots (42)$$

Let $t \to \infty$, we see further that for $n = 0, 1, 2, \ldots,$

$$y(n+1) = \alpha_{n+1}\, y(n) + \gamma_{n+1}\, y(0). \quad \ldots (43)$$

If we now let $z_k = y(k-1)$ for $k = 1, 2, \ldots,$ and $z_0 = z_\omega$, then $\left\{ z_k \right\}_{k=1}^{\infty}$ is a ω-periodic sequence and form (43) we see that the column vector $(z_1, z_2, \ldots, z_\omega)^T \in \overline{\Psi}_\omega$. Conversely, let $(z_1, z_2, \ldots, z_\omega)^T \in \overline{\Psi}_\omega$. Define $z_0 = z\omega$ and extending the finite sequence $\left\{ z_1, z_2, \ldots, z_\omega \right\}$ into the unique infinite ω-periodic sequence $\left\{ z_k \right\}_{k=1}^{\infty}$. Let $y(n) = z_{n+1}$ for $n = 0, 1, 2, \ldots,$ and let the function $y(t)$ on each interval $[n, n+1) \subset [0, \infty)$ be defined by (42). Then it is not difficult to check that this function $y(t)$ is a ω-periodic solution of (2). In other words, we have found a one to one and onto mapping from $\overline{\Omega}_\omega$ to $\overline{\Psi}_\omega$. The linearity of this mapping is easy to see. The proof of Lemma 1 is complete.

Lemma 2 — (2) has a nontrivial ω-periodic solution if and only if det $C_\omega = 0$.

PROOF : By Lemma 1 we know that the result is true. The proof of Lemma 1 is complete.

Lemma 3 — If (2) has a nontrivial ω-periodic solution, then every solution of (2) is with period ω.

PROOF : Let $y(t)$ be a nontrivial ω-periodic solution of (2), and $x(t)$ be a solution of (2). First of all we assert that $y(0) \neq 0$. If $y(0) = 0$, then form (39) we see that $y(t) \equiv 0, 0 \le t < 1$, so $y(1) = 0$. By using (42), (43) and by induction, for $n = 0, 1, 2, \ldots,$ we know that $y(t) \equiv 0$, on $[n, n+1)$, lead to a contradiction. Hence $y(0) \neq 0$. Let $x(0) = \lambda\, y(0)$, by (39), we have

$$x(t) = x(0) \exp\left(-\int_0^t (a(u) + c(u)\, du \right) = \lambda\, y\,(0))$$

$$\exp\left(-\int_0^t (a(u) + c(u))\, du \right) = \lambda\, y(t), \; 0 \le t < 1, \quad \ldots (44)$$

we see further that

$$x(1) = \lambda y(1), \quad \text{... (45)}$$

and relation (42) and (43), by induction for $n = 0, 1, 2, \ldots$, we have $x(t) = \lambda y(t)$, on $[n, n+1)$. Hence $x(t)$ is with period ω. The proof of Lemma 2 is complete.

Using Lemmas 2 and 3 we obtain the following theorem.

Theorem 3 — *Every solution of (2) is with period ω if and only if det $C_\omega = 0$.*

Example 2 — Consider the following equation

$$y'(t) - (\sin \pi t)\, y(t) + (\sin \pi t)\, y(t - [t]) = 0, \; t \geq 0 \quad \text{... (46)}$$

where $a(t) = -\sin \pi t$ and $c(t) = \sin \pi t$ are continuous real functions with period 2. It is easy to verify that

$$\alpha_1 = \exp\left(\frac{2}{\pi}\right), \; \gamma_1 = 1 - \exp\left(\frac{2}{\pi}\right), \; \alpha_2 = \exp\left(-\frac{2}{\pi}\right), \; \gamma_2 = 1 - \exp\left(-\frac{2}{\pi}\right).$$

It follows that by using Theorem 3, we know every solution of (2) is with period 2.

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