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BOUNDEDNESS OF COMPONENTS OF FATOU SETS OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Non-existence of unbounded components of Fatou sets of composition of finitely many functions entire and meromorphic in the complex plane studied under some stuitable conditions following[15].

Key Words: Fatou Set; Julia Set; Iteration; Hyperbolic Domain

1. INTRODUCTION AND RESULTS

Let $f(z)$ be a transcendental meromorphic function in the complex plane **C**. The nth iteration of $f(z)$ is denoted by $f^n(z)$, i.e. $f^n(z) = f(f^{n-1}(z))$, $n = 1, 2, \ldots$. Then $f^n(z)$ is well defined for $z \in \mathbf{C}$ outside a (possible) countable set of consisting of the poles of $f^k(z), k = 1, 2, \ldots, n-1$. Define the Fatou set $F(f)$ of $f(z)$ as

$$F(f) = \left\{ z \in \overline{\mathbf{C}} : \left\{ f^n(z) \right\} \text{ is well defined and normal in a neighbourhood of } z \right\}$$

and $J(f) = \overline{\mathbf{C}} \setminus F(f)$ is the Julia set of $f(z)$. $F(f)$ is open and $J(f)$ is closed and perfect. It is well-known that $F(f)$ is completely invariant under f, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$. Let U be a connected component of $F(f)$. For each n, there is a component U_n of $F(f)$ such that $f^n(U) \subseteq U_n$. If for some $n \geq 1, f^n(U) \subseteq U$, then U is called a periodic component and such the smallest integer n is the period of periodic component U. In particular, a periodic component of period one is also called invariant. If for some n, U_n is periodic, but U is not periodic, then U is called preperiodic; U is called a Baker domain of period p, if U is periodic, $f^{pn}(z) \to a \in \partial U \cup \{\infty\}$ in U as $n \to \infty$ and $f^p(z)$ is not defined at $z = a$; U is called wandering if $U_n \cap U_m = \phi$ for all $n \neq m$.

The discussion of boundedness of the components of $F(f)$ of a transcendental entire function $f(z)$ of order less that $n\ \frac{1}{2}$ attracted interests of many researchers, see[3,10,12,14,15] etc.. Zheng[15] investigated the subject for the case of meromorphic function and proved the following:

Theorem A — *Let $f(z)$ be a transcendental meromorphic function. If we have*

$$\limsup_{r \to +\infty} \frac{L(r,f)}{r} = +\infty,$$

where $L(r,f) = \min\{|f(z)| : |z| = r\}$, then the Fatou set, $F(f)$ of f has no unbounded preperiodic or periodic domains

In particular, f has no Baker domains.

Let $f(z)$ be a meromorphic function in **C**. $T(r,f)$ denotes the Nevanlinna characteristic function of $f(z)$ and $\delta(\infty,f)$ the deficiency of f at ∞ (see[7]). The growth order and lower order of $f(z)$ are defined respectively by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

The following is an immediate consequence of Theorem A.

Corollary 1 — Let f be a transcendental meromorphic function with lower order $\mu < 1/2$. Assume that $\delta(\infty,f) > 1 - \cos \pi\mu$. Then every preperiodic or periodic components of $F(f)$ is bounded.

In particular, f has no Baker domains.

Corollary 1 was given in[8] for the case of entire functions, but the proof of the key lemma, that is Lemma 4, and so Lemma 7 there is incomplete, we shall discuss this in Section 2.

We don't know whether all the wandering domains of $f(z)$ in Theorem A are bounded. For a transcendental meromorphic function $f(z)$ in **C**, we denote by $M(r,f)$ and $L(r,f)$ the maximum modulus and minimum modulus of $f(z)$ on $|z| = r$ respectively. In this paper, we prove the following results.

Theorem 1 — *Let $f_j(z)$ $(j = 1, 2, \ldots, m; m \geq 1)$ be transcendental entire functions and such that for some number $h > 1$ and all the sufficiently large r, there exists a $r_j \in (r, r^h)$ satisfying*

$$L(r_j, f_j) > M(r, f_j)^h, j = 1, 2, \ldots, m.$$

Set

$$g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z).$$

Then $F(g)$ has no unbounded components.

The condition of Theorem 1 can be achieved and for this we have the following consequence.

Corollary 2 — Let $f_j(z)$ ($j = 1, 2, \ldots, m; m \geq 1$) be transcendental entire functions with finite order and such that for some $\alpha \in (0, 1)$ and $d > 1$ and all the sufficiently large r, there exists a $r_j \in (r, r^d)$ satisfying

$$L(r_j, f_j) > M(r, f_j)^{\alpha}, j = 1, 2, \ldots, m, \qquad \ldots (1)$$

and for some small $\varepsilon > 0$, there exists a $\tilde{r}_j \in (r, r^d)$ such that

$$\log M(\tilde{r}_j, f_j) > r^{\varepsilon}, j = 1, 2, \ldots, m.$$

Set

$$g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z).$$

Then $F(g)$ has no unbounded components.

It is easy to see that transcendental entire functions with the positive lower order and the order less than 1/2 satisfy the conditions of Corollary 2. Generally from the proof of Barry[4], Theorem, it follows that if a transcendental entire function f has the lower order $\mu < 1/2$ and for some $\mu < \beta < 1/2$ and all the sufficiently large r, there exists a $\tilde{r} \in (r, r^d)$ such that

$$\log M(\tilde{r}, f) < \tilde{r}^{\beta},$$

then f satisfies the property (1) with $\alpha = \cos \pi\beta$.

Corollary 3 — Let $f_j(z)$ ($j = 1, 2, \ldots, m; m \geq 1$) be transcendental entire functions of growth order and lower order lying in $\left(0, \frac{1}{2}\right)$. Set

$$g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z).$$

Then $F(g)$ has no unbounded components.

Corollary 3 was proved by Wang[12] for $m = 1$. For meromorphic functions, then we have the following result.

***Theorem* 2** — *Let $f_j(z)$ ($j = 1, 2, \ldots, m; m \geq 1$) be transcendental meromorphic functions with the property that for a $d > 1$, arbitrary large $\rho > 0$ and all the sufficiently large r, there exists a $r_j \in (r, r^d)$ such that*

$$\log L(r_j, f_j) > \rho \log r, j = 1, 2, \ldots, m. \qquad \ldots (2)$$

Set

$$g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z).$$

If U is wandering component of $F(g)$ and there is a point $z_0 \in U$ satisfying

$$\log^+ \log^+ \left| f_j \circ \cdots \circ f_1(g^{n-1}(z_0)) \right| = O(n), j = 1, \ldots, m, \ n \to \infty, \qquad \ldots (3)$$

then U is bounded.

It is to be noted that we define the Fatou set of composition of finitely many meromorphic functions by the same method as that for a meromorphic function. Theorem 2 was proved by Zheng[14] for the case of one entire function with order less than 1/2. From Ostrowskii[9], a transcendental meromorphic function f with order $\sigma < 1/2$ satisfies the condition of Theorem 2 if $\delta(\infty, f) > 1 - \cos \pi\sigma$. Theorems 1 and 2 can be applied to consideration of the skew-product associated with finitely many meromorphic functions. We shall do this in Section 3.

2. The Proof of Theorems

First of all, we need to recall some properties on the hyperbolic domains, see[1,5]. An open set W in C is called hyperbolic if $\mathbf{C} \setminus W$ contains at least two points. Let U be a hyperbolic doamin in C $\rho_U(z)$ is the density of the hyperbolic metric on U and $\lambda_U(z_1, z_2)$ stands for the hyperbolic distance between z_1 and z_2 on U, i.e.

$$\lambda_U(z_1, z_2) = \inf_{\gamma \in U} \int_\gamma \rho_U(z) |dz|,$$

where γ is a Jordan curve connecting z_1 to z_2 in U. If U is simply-connected and $d(z, \partial U)$ is the euclidean distance between $z \in U$ and ∂U, then for any $z \in U$,

$$\frac{1}{2d(z, \partial U)} \le \rho_U(z) \le \frac{2}{d(z, \partial U)}. \qquad \ldots (4)$$

Let $f: U \to V$ be analytic, where U and V are hyperbolic domains. By the Principle of the Hyperbolic Metric, we have

$$\lambda_V(f(z_1), f(z_2)) \le \lambda_U(z_1, z_2), \quad \text{for} \quad z_1, z_2 \in U. \qquad \ldots (5)$$

Proof of Theorem 1 — Suppose that $F(g)$ has an unbounded component U by contradiction. Then U is simply-connected (see[2]). Take a point $z_0 \in U$. Then for a sufficiently large $R_0 > |z_0|$, there is a $\tilde{R}_1 \in \left(R_0, R_0^h\right)$ such that

$$|f_1(z)| \geq L(\tilde{R}_1, f_1) \geq M(R_0, f_1)^h > M(R_0, f_1), |z| = \tilde{R}_1.$$

We draw a Jordan curve γ connecting z_0 to a point on $\{z : |z| = \tilde{R}_1\}$ such that $\gamma \subset U \cap \{z : |z| \leq \tilde{R}_1\}$. Put $R_1 = M(R_0, f_1)$. Then

$$f_1(\gamma) \cap \{z : |z| = R_1^h\} \neq \phi \quad \text{and} \quad f_1(\gamma) \cap \{z : |z| = R_1\} \neq \phi.$$

There exists a $\tilde{R}_2 \in \left(R_1, R_1^h\right)$ such that

$$\begin{aligned} |f_2(z)| \geq L\left(\tilde{R}_2, f_2\right) &\geq M(R_1, f_2)^h = M(M(R_0, f_1), f_2)^h \\ &\geq M(R_0, f_2 \circ f_1)^h, \ |z| = \tilde{R}_2. \end{aligned}$$

Put $R_2 = M(R_1, f_2)$. Then

$$f_2 \circ f_1(\gamma) \cap \{z : |z| = R_2^h\} \neq \phi \text{ and } f_2 \circ f_1(\gamma) \cap \{z : |z| = R_2\} \neq \phi.$$

Thus there is a point $z_2 \in \gamma$ satisfying

$$|f_2 \circ f_1(z_2)| \geq M(R_0, f_2 \circ f_1)^h.$$

Inductively, set $R_{m-1} = M(R_{m-2}, f_{m-1})$ and

$$f_{m-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| = R_{m-1}^h\} \neq \phi$$

and

$$f_{m-1} \circ \cdots \circ f_1(\gamma) \cap \{z : |z| = R_{m-1}\} \neq \phi.$$

There exists a $\tilde{R}_m \in \left(R_{m-1}, R_{m-1}^h\right)$ such that

$$\begin{aligned} |f_m(z)| &\geq L(\tilde{R}_m, f_m) \geq M\left(R_{m-1}, f_m\right)^h \\ &= M(M(R_{m-2}, f_{m-1}), f_m)^h \\ &\geq \ldots\ldots\ldots\ldots \\ &\geq M(R_0, f_m \circ f_{m-1} \circ \cdots \circ f_1)^h \\ &= M(R_0, g)^h, \ |z| = \tilde{R}_m. \end{aligned}$$

This implies that there is a point $z_{m1} \in \gamma$ such that

$$\left| g(z_{m1}) \right| > M(R_0, g)^h > \left| g(z_0) \right|^h,$$

and setting $R_{m1} = M(R_{m-1}, f_m)$, we have

$$g(\gamma) \cap \left\{ z : |z| = R_m^h \right\} \neq \phi \text{ and } g(\gamma) \cap \left\{ z : |z| = R_m \right\} \neq \phi.$$

Repeating the process above inductively, there is a point $z_{mn} \in \gamma$ such that

$$\left| g^n(z_{mn}) \right| > M(r_{n-1}, g)^h \geq M(R_0, g^n)^h > \left| g^n(z_0) \right|^h, \qquad \text{... (6)}$$

where

$$r_n = M(r_{n-1}, g), r_0 = R_0, \ n = 1, 2, \ldots.$$

Since $g^n(U) \subseteq U_n$, U_n is a simply-connected unbounded component of $F(g)$, it follows from (4) that for any $a \in \partial U_n$, we have

$$\rho_{U_n}(z) \geq \frac{1}{2d(z, \partial U_n)} \geq \frac{1}{2|z-a|} \geq \frac{1}{2(|z|+|a|)}.$$

Then we have

$$\lambda_{U_n}(g^n(z_0), g^n(z_{mn})) \geq \int_{|g^n(z_0)|}^{|g^n(z_{mn})|} \frac{dr}{2(r+|a|)}$$

$$\geq \frac{1}{2} \log \frac{|g^n(z_{mn})| + + a|}{|g^n(z_0)| + |a|}. \qquad \text{... (7)}$$

Set $A = \max \{\lambda_U(z_0, z) : z \in \gamma\}$. Clearly $A \in (0, \infty)$. From (5), noting $z_{mn} \in \gamma \subset U$, we have

$$\lambda_{U_n}(g^n(z_0), g^n(z_{mn})) \leq \lambda_U(z_0, z_{mn}) \leq A. \qquad \text{... (8)}$$

Therefore, combining (6), (7) and (8) gives

$$\left| g^n(z_0) \right|^h < M(R_0, g^n)^h < \left| g^n(z_{mn}) \right| + |a| \leq \left(\left| g^n(z_0) \right| + |a| \right) e^{2A}.$$

This is impossible as $n \to \infty$, because a and e^{2A} are constants and

$$\left| g^n(z_0) \right| \to +\infty \ (n \to +\infty).$$

Theorem 1 follows.

Proof of Corollary 2 — First of all we take a number $\beta > d$ with $\alpha\beta \geq 1$ and $\varepsilon\beta > \sigma$, where σ is the biggest one of the orders of $f_j, j = 1, 2, ..., m$. From the hypotheses of Corollary 2, for all the sufficiently large r, there exists a $\tilde{r}_j \in (r^{\beta}, r^{\beta^2})$ satisfying

$$\log M(\tilde{r}_j, f_j) > r^{\varepsilon\beta} > r^{\zeta} \log M(r, f_j) > \beta^4 \log M(r, f_j), \qquad \text{... (9)}$$

where $0 < \zeta < \varepsilon d - \sigma$. And there exists a $r_j \in (r^{\beta^2}, r^{\beta^3})$ such that

$$\begin{aligned} \log L(r_j, f_j) &> \alpha \log M(r^{\beta^2}, f_j) \\ &> \alpha \log M(\tilde{r}_j, f_j) \\ &> \alpha\beta^4 \log M(r, f_j) \\ &\geq \beta^3 \log M(r, f_j),\ j = 1, ..., m, \end{aligned}$$

where we used (9). Then the conditions of Theorem 1 holds with $h = \beta^3$, and so Corollary 2 follows.

Before discussing Corollary 3, we recall the following notation. Let $E(1, r)$ stand for the part of E contained in the interval $(1, r)$, where E is a Lebesgue-measurable set on the positive real axis. The lower and upper logarithmic densities of E are defined by

$$\underline{\log \text{dens}\, E} := \liminf_{r \to \infty} \frac{1}{\log r} \int_{E(1,r)} \frac{dt}{t}$$

$$\overline{\log \text{dens}\, E} := \limsup_{r \to \infty} \frac{1}{\log r} \int_{E(1,r)} \frac{dt}{t}.$$

Corollary 3 follows immediately from Corollary 2 and the following result:

Let $f(z)$ be a transcendental entire function of the (lower) order $(\mu)\,\rho$ less than $\frac{1}{2}$ The set

$$E = \{r : \log L(r, f) \geq \cos(\pi\alpha) \log M(r, f)\}, \qquad \text{... (10)}$$

where $(\mu)\,\rho < \alpha < 1/2$ has positive (upper) lower logarithmic density.

It is easy to see that if E in (10) has positive lower logarithmic density, then for some $d > 1$ and all the sufficiently large r there is a $\tilde{r} \in (r, r^d)$ such that

$$L(\tilde{r}, f) > M(\tilde{r}, f)^{\cos(\pi\alpha)}$$

But we cannot deduce the above result in E in (10) has only positive upper logarithmic density. Observe the set

$$E = \bigcup_{n=0}^{\infty} (r_n, r_n^2),$$

where $r_0 = 1, r_{n+1} = \exp r_n^2$. A simply calculation implies that $\overline{\log \text{dens}}\, E \geq 1/2$ and for arbitrary sufficiently large d, we have $\left(r_n^2, r_n^{2d}\right) \subset (0, +\infty) \setminus E$, when n is chosen sufficiently large. Therefore, the proof of Lemma 4 in (8) is wrong. Thus we raise a question: Does Corollary 3 still hold without the assumption of the order less than 1/2 ?

Proof of Theorem 2 — According to (3), there exists an $A > 1$ such that

$$\log \left| f_j \circ \cdots \circ f_1 (g^{n-1} (z_0)) \right| < A^n, j = 1, \ldots, m, \quad n \in \mathbf{N}. \qquad \ldots (11)$$

We assume a positive number ρ such that $B = \rho/d > A$ and a $R_0 > \max \{e^A, |z_0|\}$, such that for all $r \geq R_0$, there is a $r_j \in (r, r^d)$ satisfying

$$\log L (r_j, f_j) > \rho \log r, j = 1, 2, \ldots, m. \qquad \ldots (12)$$

Suppose that U is unbounded. We will derive a contradiction from (12). We draw a Jordan curve γ connecting z_0 to one of points on $\left\{z : |z| = R_0^h\right\}$ with $\gamma \subset U \cap \left\{z : |z| \leq R_0^h\right\}$. From the property (3) there is a $r_0 \in \left\{R_0, R_0^h\right\}$ such that

$$\log \left| f_1 (z) \right| \geq \log L (r_0, f_1) > \rho \log R_0 = dB \log R_0, |z| = r_0.$$

Set $R_1 = R_0^B$. Then $R_1 > e^A > |f_1 (z_0)|$, and

$$f_1 (\gamma) \cap \left\{z : |z| \leq R_1\right\} \neq \phi \quad \text{and} \quad f_1 (\gamma) \cap \left\{z : |z| = R_1^d\right\} \neq \phi.$$

Furthermore there is a $r_1 \in \left(R_1, R_1^d\right)$ such that

$$\log |f_2 (z)| \geq \log L (r_1, f_2) > \rho \log R_1, \quad |z| = r_1.$$

Set $R_2 = R_1^B$. Then $R_2 > e^A > |f_2 \circ f_1 (z_0)|$, and

$$f_2 \circ f_1 (\gamma) \cap \left\{z : |z| = R_2\right\} \neq \phi \text{ and } f_2 \circ f_1 (\gamma) \cap \left\{z : |z| \leq R_2^d\right\} \neq \phi.$$

Inductively, set $R_{m-1} = R_{m-2}^B$ and we have

$$f_{m-1} \circ f_1(\gamma) \cap \left\{ z : |z| \le R_{m-1} \right\} \neq \phi$$

and

$$f_{m-1} \circ \dots \circ f_1(\gamma) \cap \left\{ z : |z| = R_{m-1}^d \right\} \neq \phi.$$

There is a $r_{m-1} \in \left(R_{m-1}, R_{m-1}^d \right)$ such that

$$\log \left| f_m(z) \right| \ge \log L(r_{m-1}, f_m) > \rho \log R_{m-1}, \quad |z| = r_{m-1}.$$

Set $R_{m1} = R_{m-1}^B$. We have $R_{m1} > e^A > |g(z_0)|$. And hence

$$g(\gamma) \cap \left\{ z : |z| = R_m \right\} \neq \phi \text{ and } g(\gamma) \cap \left\{ z : |z| = R_m^d \right\} \neq \phi,$$

and there exists a $z_{m1} \in \gamma$ such that

$$\log |g(z_{m1})| > d \log R_{m1}.$$

Repeating the same process as above by replacing R_0 with R_{m1} and noting that $R_{m1} = R_0^{B^m} > \exp A^2$, we can find a point $z_{mn} \in \gamma$ such that

$$\log \left| g^n(z_{mn}) \right| > d \log R_{mn} = dB^{mn} \log R_0$$
$$> A^n > \log \left| g^n(z_0) \right|, \quad n = 1, 2, \dots . \qquad \dots (13)$$

Since U is an unbounded wandering domain, $J(f)$ has an unbounded component, say Γ, and $g^n : U \to \mathbf{C} \setminus \Gamma, n = 1, 2, \dots$ are analytic. By noting that $\rho_{\mathbf{C} \setminus \Gamma}(z)\, d(z, \Gamma) \ge \frac{1}{2}$ for each $z \in \mathbf{C} \setminus \Gamma$ and $z_{mn} \in \gamma$, we have

$$\max \{\lambda_U(z_0, z) : z \in \gamma\} \ge \int_{|g^n(z_0)|}^{|g^n(z_{mn})|} \rho_{\mathbf{C} \setminus \Gamma}(z) |dz|$$

$$\ge \frac{1}{2} \int_{|g^n(z_0)|}^{|g^n(z_{mn})|} \frac{1}{|z| + |a|} |dz|$$

$$= \frac{1}{2}\log\frac{|g^n(z_{mn})|+|a|}{|g^n(z_0)|+|a|}.$$

From (13) this is impossible. Theorem 2 follows.

3. The Skew-Product Map and Semigroup

Let m be a positive integer. We denote by Σ_m the one sided word space, that is

$$\Sigma_m = \{1, ..., m\}^{\mathbf{N}} = \{1, ..., m\} \times \{1, ..., m\} \times ...$$

and by $\sigma: \Sigma_m \to \Sigma_m$ the shift map, that is

$$\sigma: (w_1, w_2, ..., w_n, ...) \to (w_2, w_3, ..., w_{n+1}, ...).$$

Let $f_j(z)$ be meromorphic functions in $\mathbf{C}$, $j = 1, 2, ..., m, m \geq 1$. Define a map by

$$\tilde{f}: \Sigma_m \times \mathbf{C} \to \Sigma_m \times \overline{\mathbf{C}}, (w, z) \to (\sigma w, f_{w_1}(z)).$$

$\tilde{f}$ is called the skew-product map associated with the generator system $\{f_1, ..., f_m\}$, see[11]. Now, we give the definition of the Fatou set of the skew-product map. Set for $w \in \Sigma_m$

$$F_w = \{z \in \mathbf{C}: \left\{f_{w_n} \circ \cdots \circ f_{w_1}(z)\right\}_n$$

is well defined and normal in a neighbourhood of $z\}$.

$J_w = \overline{\mathbf{C}} \setminus F_w$. Write $\tilde{F}_w = \{w\} \times F_w$ and and $\tilde{J}_w = \{w\} \times J_w$. The Fatou set $\tilde{F}(\tilde{f})$ and the Julia set $\tilde{J}(\tilde{f})$ of the skew-product map $\tilde{f}$ are defined respectively by

$$\tilde{F}(\tilde{f}) = \bigcup_{w \in \Sigma_m} (\{w\} \times F_w) = \bigcup_{w \in \Sigma_m} \tilde{F}_w,$$

and

$$\tilde{J}(\tilde{f}) = \bigcup_{w \in \Sigma_m} (\{w\} \times J_w) = \bigcup_{w \in \Sigma_m} \tilde{J}_w,$$

Every component of $\tilde{F}_w$ is of the form $\{w\} \times U$, where U is a domain in which $\left\{f_{w_n} \circ \cdots \circ f_{w_1}(z)\right\}_n$ is normal and $w_n \in \{1, 2, ..., m\}$, $n = 1, 2, ...$. $\{w\} \times U$ is said to be bounded if U is bounded. It is easy to see that each component of $\tilde{F}(\tilde{f})$ lies in one of $\tilde{F}_w$ for some

$w \in \Sigma_m$. Observing these facts, from Theorem 1, Corollary 3 and their proofs, we can obtain the following result.

***Theorem* 3** — *Let $f_j(z)$ ($j = 1, 2, ..., m; m \geq 1$ satisfy the assumption of Theorem 1. If $\tilde{f}$ is the skew-product map associated with the generator system $\{f_1, ..., f_m\}$, then $\tilde{F}(\tilde{f})$ has only bounded components.*

Corollary 4 — Under the assumption of Corollary 3, the skew-product map $\tilde{f}$ associated with the generator system $\{f_1, ..., f_m\}$ has only bounded components of the Fatou set $\tilde{F}(\tilde{f})$.

We denote by

$$G = \langle f_1, f_2, ..., f_m \rangle$$

the transcendental meromorphic semigroup finitely generated by $\{f_1, f_2, \cdots f_m\}$ with semigroup operation being composition of functions. $F(G)$ denotes the Fatou set of G, that is

$$F(G) = \{z \in \overline{\mathbf{C}} : G \text{ is well defined and normal in a neighbourhood of } z\}.$$

$J(G)$ denotes the Julia set of G, that is $J(G) = \overline{\mathbf{C}} \setminus F(G)$. $F(G)$ is open and $J(G)$ is non-empty and closed.

Noting the following fact

$$F(G) = \bigcap_{w \in \Sigma_m} F_w$$

we obtain the results about non-existence of unbounded components of $F(G)$ corresponding to Theorem 3 and Corollary 4. From Theorem 2, we have

***Theorem* 4** — *Let $f_j(z)$ ($j = 1, 2, ..., m; m \geq 1$) satisfy the assumption of Theorem 2. Set*

$$G(z) = \langle f_1(z), f_2(z), ..., f_m(z) \rangle.$$

If U is a wandering component of $F(g)$ and there is a point $z_0 \in U$ satisfying

$$\log^+ \log^+ \left| f_{w_n} \circ \cdots \circ f_{w_1}(z_0) \right| = O(n), \text{ for some } w \in \Sigma_m,$$

then U is bounded.

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