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CLASSICAL STRING MODEL OF MAGNETIC FIELD LINES IN RELATIVISTIC MAGNETOHYDRODYNAMICS

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It is found from Maxwell's equations that the magnetic field lines are good analogues of relativistic strings. The Lorentz force per unit length of a magnetic tube is interpretable as the Magnus force acting on each individual magnetic tube. In order to present the large scale treatment of magnetic flux quantization, the electromagnetic field is described in terms of the dilatonic and axionic fields via the Kalb-Ramond gauge potential 2-form. If the dilatonic and axionic fields are assumed to bear a definite relation, the dynamical equation characterizing axionic field leads to the vanishing of the Lorentz force. The string-like interaction of the magnetic field lines with the perfectly conducting fluid generates string waves with a velocity less than that of light speed. The conservation of magnetic flux leads to a geometrical condition in the form of the Weingarten identity which ensures the existence of a family of "magnetic world sheets". In the process of collapse, a compact spacelike cross-section of a magnetic tube terminates into a trapped surface if the magnetic energy grows faster along the fluid flow lines than that along the magnetic field lines.

Key Words: Maxwell Equation; Magnetohydrodynamics

1. INTRODUCTION

The work of Yodzis[1] initiated potentially interesting applications of relativistic magnetohydrodynamics (RMHD) to the problems of galactic cosmogony, gravitational collapse, and pulsar theory. It is now generally believed that collapsed stellar objects such as dwarf stars and neutron stars are very likely to possess strong magnetic fields of the order of 10^8-$10^{12} \cdot G$. The discovery of pulsars as single rotating neutron stars[2] has stimulated considerable intrinsic interest in the effects brought about by ultrastrong magnetic fields. In a neutron star the gravitational binding energy per particle can be a tenth of the rest energy. Thus a fully consistent relativistic magnetohydrodynamic model is required to learn the evolution of a rotating neutron star's interior. A significant step in this direction has been the work of Bekenstein and Oron[3] in which interior hydromagnetic structure of a rotating neutron star's interior is coupled to gravitational interaction. Invoking electrons theory of the conductivity of matter, it has been argued by Bakenstein and Oron[4] that the electric field vanishes in a frame comoving with the fluid if the electric current is to be maintained finite in the limit of infinite electrical conductivity. Consequently, the magnetic field is frozen into the fluid.

The frozen-in magnetic field implies that the magnetic field lines are material lines[5]. The idea of magnetic field lines as vibrating elastic strings with a tension derived from the electromagnetic stress tensor appeared in Alfvén's pioneering work[6] was the result of his intellectual creation. Its relativistic formulation seems possible in a manner analogous to that of the classical theory of relativistic strings[7]. In a recent work of Thompson and Blaes[8], it is shown that a Lagrangian description of the frozen-in magnetic field leads to the action associated with Nambu-Goto strings[9-10]. It is further shown, by applying the technique of bosonization to the Fermi fields on each quantum of magnetic flux, that the electric current is expressible in terms of an axion field. This work gives us impetus to investigate the string model of the magnetic field lines.

This paper is devoted to the investigation of the string model of the magnetic field lines. We display the mechanical properties associated with the magnetic flux tubes at a macroscopic level. The present analysis is designed on the lines of general classical string model of vortex defect[11]. We shall follow Carter[12] in our approach. The paper is organized as follows. In Section 2, we recapitulate Maxwell's equations. In Section 3, we introduce string interpretation of the magnetic field lines. From Maxwell's second equation, we derive an equation which resembles with that of global cosmic string in an axion field background. In Section 4, we relate magnetic flux quantization with the conservation of circulation of the electromagnetic potential around a topological defect in some region of spacetime. In order to incorporate the macroscopic effect of magnetic flux quantization, we describe the electromagnetic field in terms of dilatonic and axionic fields via the Kalb-Ramond gauge potential 2-form. If the dilatonic and axionic fields are assumed to bear a definite relation, the dynamical equation characterising axionic field leads to the vanishing of the Lorentz force. This, in turn, provides an equation that describes the mechanical properties of stringlike behaviour of the magnetic field lines. In Section 5, we prove that the vanishing of Greenberg's vector is just equivalent to the Weingarten identity for a 2-dimensional timelike submanifold of spacetime manifold V_4. This identity guarantees the existence of a family of "magnetic world sheets". In Section 6, we describe stringlike interaction of the magnetic field lines with the perfectly conducting fluid. Section 7 is devoted to investigate the effects brought by the magnetic field on the convergence of light rays. It is shown that a compact spacelike cross-section of a magnetic tube forms a trapped surface if the magnetic energy grows faster along the fluid flow lines than that along the magnetic field lines.

2. MAXWELL'S EQUATIONS

We begin with a brief resume of Maxwell's equations. We consider a highly conducting fluid embedded in a 4-dimensional spacetime manifold V_4 with a pseudo-Riemannian metric tensor g_{ab} of Lorentzian signature +2. We denote the fluid 4-velocity vector by u^a which satisfies the normalization condition

$$u^a u_a = -1. \qquad \text{... (2.1)}$$

Inside a conducting fluid, the electromagnetic field is described by the electric field — magnetic induction tensor (or Faraday tensor) H_{ab} and the electric induction — magnetic field tensor G_{ab}. The electromagnetic field inside the fluid evolves in accordance with Maxwell's equations[13]

$$H_{[ab;c]} = 0 \quad \text{or} \quad \overset{*}{H}{}^{ab}_{\ ;b} = 0, \qquad \text{... (2.2a)}$$

$$G^{ab}_{\ ;b} = J^a, \qquad \text{... (2.2b)}$$

where semi-colon (;) is used to denote the covariant derivative with respect to the metric tensor

g_{ab}. The square bracket indicates index anti-symmetrization. An overhead asterisk (*) denotes the dual, for example, $\overset{*}{H}{}^{ab} = \frac{1}{2}\eta^{abcd} H_{cd}$. Here η^{abcd} represents the Levi-Civita totally anti-symmetric tensor density. The electric current J^a is decomposed along and orthogonal to u^a as

$$J^a = \rho_e u^a + g^a, \qquad \text{... (2.3)}$$

where ρ_e is the proper electric charge density. The first term on the right hand side of (2.3) represents the convection current and the second term is the conduction current.

If the conducting fluid is assumed to be of an infinite electrical conductivity, the vanishing of the electric field is essential for the electric current to remain finite[4,13]. As a consequence of it, the infinitely conducting fluid obeys the following constitutive relation[13]

$$H_{ab} = \mu G_{ab}, \qquad \text{... (2.4)}$$

where μ denotes the magnetic permeability. The electric field E_a and the magnetic induction B_a measured by an observer comoving with the fluid are, respectively, given by

$$E_a = H_{ab} u^b, \qquad \text{... (2.5a)}$$

$$B_a = \overset{*}{H}_{ab} u^b. \qquad \text{... (2.5b)}$$

The magnetic field H_a and the electric induction D_a are, respectively, expressible as

$$H_a = \overset{*}{G}_{ab} u^b, \qquad \text{... (2.6a)}$$

$$D_a = G_{ab} u^b, \qquad \text{... (2.6b)}$$

with constitutive relations

$$B_a = \mu H_a, \qquad D_a = \lambda E_a, \qquad \text{... (2.7)}$$

where λ is the dielectric permitivity. Since $E_a = 0$ in the limit of an infinite electrical conductivity, it follows from (2.5)-(2.7) that

$$\overset{*}{H}{}^{ab} = u^a B^b - u^b B^a, \qquad \text{... (2.8a)}$$

$$G^{ab} = \eta^{abcd} H_c u_d. \qquad \text{... (2.8b)}$$

With the help of (2.4), (2.8a) and 2.8b) are easily arranged to give

$$H_{ab} \overset{*}{H}{}^{ac} = 0. \qquad \text{... (2.9)}$$

This is known as MHD condition.

3. String Model of Magnetic Field Lines

This section is devoted to the string model of the magnetic field lines. We closely follow Stachel[14] and Carter[12] in our aproach to derive the string equation of motion from Maxwell's second equation (2.2b).

It is apparent from (2.8a) that $\overset{*}{H}{}^{ab}$ is the skew-symmetric product of a pair of linearly independent vectors (u^a, B^a). The vectors (u^a, B^a) span a 2-dimensional vector space which is usually referred to as a blade of the bivector $\overset{*}{H}{}^{ab}$. Because of (2.9) $\overset{*}{H}{}^{ab}$ is a simple bivector of second rank. Since u^a is timelike and B^a is spacelike, the simple bivector $\overset{*}{H}{}^{ab}$ is timelike. Such a simple timelike bivector $\overset{*}{H}{}^{ab}$ has been called a "magnetic blade[15]". Since the contraction of (2.2a) with H_{ca} yields the necessary and sufficient Frobenius condition [16] for $\overset{*}{H}{}^{ab}$ to be 2-surface forming, there exists a family of timelike 2-surfaces spanned by the fluid flow lines and the magnetic field lines[27]. This means that the magnetic blades mesh together to form a well-defined family of timelike 2-surfaces formed by the fluid flow lines and the magnetic field lines. We denote this family by $\{M\}$. Each member M is a 2-dimensional timelike world sheet embedded in a 4-dimensional spacetime manifold V_4. From now on, we shall call each M a "magnetic world sheet". It is known that a string is a spacelike curve and it traces out a world sheet when it is in motion[7]. Now if the magnetic field is frozen into the fluid, the fluid particles once locked to a magnetic field line will remain so throughout the motion. This in turn implies that each "magnetic world sheet" can be regarded as a string. Thus Maxwell's first eq. (2.2a) leads us to conclude that each magnetic field line, in the case of frozen-in magnetic field, can be thought of as a string.

Since $\overset{*}{H}{}^{ab}$ is a simple timelike bivector, its dual H^{ab} is a simple spacelike bivector. Such simple spacelike bivector field generates 2-dimensional vector space orthogonal to $\{M\}$ at each point of 4-dimensional spacetime manifold V_4. This means that there is a congruence (or a family) of spacelike 2-subspaces orthogonal to $\{M\}$. Since through each spacelike 2-subspace there passes magnetic flux orthogonally (i.e., each spacelike 2-subspace is orthogonally threaded by the magnetic field lines), each such 2-subspace can be regarded as a 2-dimensional orthogonal cross-section of a magnetic tube. We denote each such cross-section by the symbol S and its family by $\{S\}$. Thus there is a congruence of cross-sections $\{S\}$ of a magnetic tube.

These magnetic tube's cross-sections $\{S\}$ may be 2-surface forming. This can be seen as follows. If we invoke Frobenius condition[16] for H^{ab} to be 2-surface foming, we find from (2.2b), (2.3) and (2.4) that the proper charge density ρ_e vanishes and the magnetic field is orthogonal to the conduction current. The existence of such spacelike 2-surfaces has been pointed out by Carioli[18] in the context of {2 + 2} formulation of RMHD. But the existence of such spacelike 2-surfaces imposes stringent constraints on RMHD flows, viz., the rotation of both the fluid flow lines and the magnetic field lines must vanish. Henceforth, we shall not assume that the family of cross-sections $\{S\}$ are 2-surface forming.

In order to achieve string interpretation of the magnetic field lines, we follow the approach initiated by Carter[12] in the context of the theory of embedding in the background spacetime manifold of metric g_{ab} and introduce a unit timelike bivector $\overset{=}{\varepsilon}{}^{ab}$ associated with a "magnetic world sheet"

$$\overset{=}{\varepsilon}{}^{ab} = u^a n^b - u^b n^a, \qquad \text{... (3.1a)}$$

with the property that

$$\overset{=}{\varepsilon}{}^{ab} \overset{=}{\varepsilon}{}_{ab} = -2, \qquad u^a n_a = 0, \qquad \text{... (3.1b)}$$

where n^a denotes the unit spacelike vector field whose integral curves represent the magnetic field lines. An overhead parallelism (=) is used to indicate the effect of projection into M. The fundamental projection tensor of M is expressible as

$$\overset{=}{g}{}^a_{\ b} = \overset{=}{\varepsilon}{}^{ac} \overset{=}{\varepsilon}{}_{cb} = n^a n_b - u^a u_b. \qquad \text{... (3.2)}$$

The dual of $\overset{=}{\varepsilon}{}^{ab}$ is denoted by $\overset{*}{\varepsilon}{}^{ab}$. It is defined by

$$\overset{*}{\varepsilon}{}^{ab} = \frac{1}{2} \eta^{abcd} \overset{=}{\varepsilon}{}^{ca} = \eta^{abcd} u_c n_d \qquad \text{... (3.3)}$$

with the following properties

$$\overset{*}{\varepsilon}{}^{ab} \overset{*}{\varepsilon}{}_{ac} = \hat{\gamma}^b_c, \qquad \overset{*}{\varepsilon}{}^{ab} \overset{=}{\varepsilon}{}_{ac} = 0, \qquad \overset{*}{\varepsilon}{}^{ab} \overset{*}{\varepsilon}{}_{ab} = 2, \qquad \text{... (3.4)}$$

where

$$\hat{\gamma}^b_c = \delta^b_c + u^b u_c - n^b n_c. \qquad \text{... (3.5)}$$

It follows from (3.2) and (3.5) that

$$\delta^b_c = \overset{=}{g}{}^b_c + \hat{\gamma}^b_c, \qquad \text{... (3.6)}$$

where $\hat{\gamma}^b_c$ is the fundemntal projection tensor of the magnetic tube's cross-section S. An overhead caret (∧) is used to indicate the effect of projection into S. Thus we observe from (3.6) that the spacetime metric tensor g_{ab} is split into two orthogonal non-degenerate submetrics $\overset{=}{g}{}_{ab}$ and $\hat{\gamma}_{ab}$.

A straightforward but simple calculation gives rise to the following relation

$$\overset{*}{\varepsilon}_{ca}\,\overset{*}{\varepsilon}{}^{ab}_{;b} = K_c\,, \qquad \text{... (3.7)}$$

where

$$K_c = \overset{o}{n}_c - \dot{u}_c + n_c\,\dot{u}_a\,n^a + u_c\,\overset{o}{n}_a\,u^a. \qquad \text{... (3.8)}$$

Here an overhead circumflex (o) and dot ($\cdot$) are, respectively, used to indicate the directional derivatives along the unit vector field n^a and the fluid velocity field u^a. We call K_c the extrinsic curvature vector of the "magnetic world sheet" M in the terminology of Carter and Langlois[19].

In view of (3.3) and (2.8b), setting $G^{ab} = -H\,\overset{*}{\varepsilon}{}^{ab}$ and substituting it into (2.2b), we find that

$$H_{,b}\,\overset{*}{\varepsilon}{}^{ab} + H\,\overset{*}{\varepsilon}{}^{ab}_{;b} = -J^a, \qquad \text{... (3.9)}$$

where H denotes the magnetic field intensity. Contracting (3.9) with $\overset{*}{\varepsilon}_{ca}$ and making use of the first of (3.4) and (3.7), we have

$$H\,K_c + Z_c = H_{,b}\,\hat{\gamma}^b_c\,, \qquad \text{... (3.10)}$$

where

$$Z_c = B^{-1}\,J^a\,H_{ac} = \overset{*}{\varepsilon}_{ca}\,\mathcal{J}^a. \qquad \text{... (3.11)}$$

It is interesting to note that (3.10) bears a striking resemblance with that of the equation of motion for a global cosmic string in an axion field background[20,21]. Z_c represents the Lorentz force per unit length of a magnetic tube acting on each individual magnetic tube. This force is interpretable as a magnetic analogue of Joukowski force (or Magnus force) that has been recognized in the context of vortex tube[19]. The magnetic field intensity H acts as a tension in the string interpretation of the magnetic field lines. One point worth making here is that (3.10) is also derivable from $T^{ab}_{(m);b} = -H^{ab}J_b$, where $T^{ab}_{(m)}$ denotes the stress-energy density of the megnetic field and $H^{ab}J_b$ is the Lorentz force. Since the evolution of the magnetic field obeys the equation of motion of relativistic strings and the magnetic field possesses definite stress-energy density $T^{ab}_{(m)}$, the magnetic field lines are good analogues of relativistic strings in their own right.

4. THE TIE-UP WITH CIRCULATION OF THE ELECTROMAGNETIC POTENTIAL AND AXIONIC FIELD

In this section, we begin with the conservation of circulation associated with the electromagnetic potential around a closed contour dragged with the fluid. Maxwell's first equation (2.2a) is identically satisfied if H_{ab} is expressed in terms of the electromagnetic potnetial A_b as

$$H_{ab} = 2\,A_{[b\,;\,a]}. \quad \text{... (4.1)}$$

The circulation associated with the electromagnetic potential A_a is defined by

$$K = \oint_C A_a\, dx^a \quad \text{... (4.2)}$$

where C denotes a spacelike closed curve drawn around a magnetic tube imbedded in the fluid.

Following Bekenstein[22] the directional derivative of the circulation along the fluid velocity u^a can be cast as

$$\dot{K} = \oint_S \left\{ L_u\, H_{ab} \right\} d\,\sigma^{ab}, \quad \text{... (4.3)}$$

where L_u denotes the Lie derivative with respect to u^a and S is the cross-section of a magnetic tube. $d\sigma^{ab}$ is an area element on S bounded by the contour C. It is evident from (4.3) that the circulation is conserved if and only if

$$L_u\, H_{ab} = 0. \quad \text{... (4.4)}$$

This condition is equivalently expressible as

$$L_u\, \overset{*}{H}{}^{ab} + u^c_{;c}\, \overset{*}{H}{}^{ab} = 0. \quad \text{... (4.5)}$$

Interesting (2.8a) in (4.5) and contracting the resulting equation with u_a, we find that

$$D_{\text{con.}}\, B^a + u^b_{;b}\, B^a = 0, \quad \text{... (4.6)}$$

where $D_{\text{con.}}$ denotes the convected derivative first introduced by Oldroyd[23] for the derivation of constitutive equations in the context of continuum mechanics and was later modified and extended by Carter and Quintana[24]. Eq. (4.6) is a relativistic analogue of Zorawski's criterion[25] for the conservation of the flux associated with any vector field across any material surface. As a consequence (4.6) provides a relativistic version of Alfvén's theorem[25] of magnetic flux conservation through a closed contour C comoving with the fluid. Thus the magnetic flux conservation through a cross-section S of a magnetic tube is the direct outcome of the conservation of circulation associated with the electromagnetic potential around a closed contour C that bounds S. Consequently, the quantization of magnetic flux in a tube of cross-section S must be related to such circulation around some topological defect that triggers in S. The usual quantization condition for the conserved circulation around a topological defect should be a multiple of the standard Bohr unit $2\pi h$. The formation of topological defect[26] is a very common feature of phase transitions in a condensed matter. Such phenomenon may arise in a neutron star's interior in which the superconducting protons are expected to form a Type II superconductor. The magnetic flux can penetrate into Type II superconductor, but only in the form of this flux tubes carrying quantized magnetic flux. The large scale description of such phenomenon require to express the electromagnetic potential A_a, as being periodic with period 2π, in terms of an axionic field $\mathcal{H}^{abc}$, at least provisionally, according to the following prescription[19].

$$A_a = \frac{1}{\lfloor 3} \eta_{abcd} \mathcal{H}^{bcd}. \qquad \text{... (4.7)}$$

We now invert (4.7) as

$$\mathcal{H}^{abc} = \eta^{abcd} A_d. \qquad \text{... (4.8)}$$

An interpretation attached to (4.8) is that $\mathcal{H}^{abc}$ will just be the dual of the electromagnetic potential A_a.

It follows from (4.1) and (4.8) that

$$\mathcal{H}^{abc}_{;a} = \overset{*}{H}{}^{bc}. \qquad \text{... (4.9)}$$

It is evident from (4.9) that the divergence of the axionic field represents the dual of the Faraday 2-form H_{ab}. It was Moffat[27] who first pointed out that the magnetic helicity conservation is a topological consequence of the magnetic flux conservation. This means that the magnetic helicity must have to be related to an axionic field. Following Carter and Khalatnikov[28], the magnetic helicity vector h^a may be defined as

$$h^a = \frac{1}{2} \eta^{abcd} A_b H_{cd}. \qquad \text{... (4.10)}$$

Substitution of (4.7) into (4.10) yields

$$h^a = \mathcal{H}^{abc} H_{bc}. \qquad \text{... (4.11)}$$

Taking divergence of (4.11) and making use of (4.9) and (2.2a), we find that

$$h^a_{;a} = \overset{*}{H}{}^{bc} H_{bc}. \qquad \text{... (4.12)}$$

Invoking MHD condition (2.9) in (4.12), we have

$$h^a_{;a} = 0 \qquad \text{... (4.13)}$$

which describes the conservation of the magnetic helicity. We note in passing that the axionic field introduced as above leads to the conservation of the magnetic helicity. Such an axionic formulation leads us to conjecture G^{ab} as the dual of the Kalb-Ramond[29] type of gauge potnetial 2-form B_{ab}, i.e.

$$G^{ab} = \frac{1}{2} \eta^{abcd} B_{cd}. \qquad \text{... (4.14)}$$

This, in view of (2.2b), gives the electric current J^a that can be expressed as

$$J^a = \frac{1}{\lfloor 3} \eta^{abcd} N_{bcd}, \qquad \text{... (4.15)}$$

where N_{abc} is called "dilatonic" field and is described according to the following specification.

$$N_{abc} = 3 B_{[bc\,;a]}, \qquad \text{... (4.16)}$$

which satisfies the closure property

$$N_{[abc\,;d]} = 0. \qquad \text{... (4.17)}$$

This is the Poincaré integrability[30] condition for the local existence of the Kalb-Ramond gauge potential 2-form B_{ab}. Furthermore, the closure condition (4.17), when applied to (4.15), guarantees the conservation of the electric current

$$J^a_{;a} = 0. \qquad \text{... (4.18)}$$

Thus N_{abc} is the electric current 3-form.

We assume that "dilatonic" and axionic fields associated with the electromagnetic fields, as introduced are not independent but bear a relation of the following type

$$N_{abc} = \phi^2 \mathcal{H}_{abc}, \qquad \text{... (4.19)}$$

where ϕ is the dilatonic field amplitude.

Inverting (4.15), we find that

$$N_{abc} = \eta_{abcd} J^d. \qquad \text{... (4.20)}$$

It follows from (4.9), (4.19) and (4.20) that

$$\mathcal{H}_{dbc} \mathcal{H}^{abc}_{;a} = 2\phi^{-2} H_{de} J^e. \qquad \text{... (4.21)}$$

But in the axionic formulation, the dynamical equation of motion[30] is

$$\mathcal{H}^{dbc} \mathcal{H}^{abc}_{;a} = 0. \qquad \text{... (4.22)}$$

Substitution of (4.22) into (4.21) gives

$$H_{ab}\ J^{b}=0 \qquad \text{... (4.23)}$$

This is the condition that has been obtained by Thompson and Blaes[8] in axionic formulation of RMHD and this leads us to believe that we are on the right track with respect to our assumption (4.19). It may then be concluded that in the axionic formulation the dynamical eq. (4.22) turns out to be equivalent to the force-free condition (i.e., the vanishing of the Lorentz force) as is exhibited by (4.23). Thus the force-free condition displays the mechanical properties of string like behaviour of the magnetic field lines in the local axionic field.

5. Geometrical Condition for the Existence of "Magnetic World Sheet"

As is discussed in the preceding section that the field associated with the string behaviour of the magnetic field lines is an axionic field because of quantized magnetic flux. Besides it, there is another aspect of the magnetic flux conservation which provides a geometrical condition for the existence of a family of "magnetic world sheets". In order to avchieve this condition we follow Carter[12] in our approach and introduce the second fundamental tensor $K_{ab}^{\cdot\cdot c}$ of a "magnetic world sheet" M as

$$K_{ab}^{\cdot\cdot c} = \overset{=}{g}{}^{c}_{d\Vert a}\ \overset{=}{g}{}^{d}_{b} = \overset{=}{g}{}^{c}_{d;e}\ \overset{=}{g}{}^{e}_{a}\ \overset{=}{g}{}^{d}_{b}, \qquad \text{... (5.1)}$$

where double vertical bar ($\Vert$) is used to indicate the tangential covariant derivative on a "magnetic world sheet" M. Substitution of (3.2) into (5.1) gives

$$K_{ab}^{\cdot\cdot c} = \Big\{ n_a n_b \Big(\overset{o}{n}{}^{c} + u^{c}\ \overset{o}{n}_{d} u^{d} \Big) + u_a u_b \Big(\dot{u}^{c} - n^{c}\ \dot{u}_{d} n^{d} \Big)$$
$$- u_a n_b \Big(\dot{n}^{c} - u^{c}\ \dot{u}_{d} n^{d} \Big) - n_a u_b \Big(\overset{o}{u}{}^{c} + n^{c}\ \overset{o}{n}_{d} u^{d} \Big) \Big\}, \qquad \text{... (5.2)}$$

from which it can be deduced that

$$K_{[ab]}^{\cdot\cdot c} = u_{[a}n_{b]}\ N^{c}, \qquad \text{... (5.3)}$$

where N^c denotes Greenberg's vector[31] and is given by

$$N^{c} = \hat{\gamma}^{c}_{\cdot a} \Big(\overset{o}{u}{}^{a} - \dot{n}^{a} \Big). \qquad \text{... (5.4)}$$

Setting $B^b = Bn^b$ in (4.6) and contracting the resulting equation with $\hat{\gamma}^{c}_{b}$, we obtain

$$\hat{\gamma}^{c}_{a} \Big(\overset{o}{u}{}^{a} - \dot{n}^{a} \Big) = 0. \qquad \text{... (5.5)}$$

This result already appears in[17] in which Greenberg's theory of spacelike congruence is applied to the congruence of magnetic field lines to prove that the vanishing of Greenberg's vector (i.e., Greenberg's transport law) is a necessary and sufficient condition for both the fluid flow lines and magnetic field lines to be 2-surface forming. Extending the associated understanding of this work, Tsamparlis and Mason[32] have shown that the vanishing of Greenberg's vector is a necessary and sufficient condition for the congruence of magnetic field lines to be material lines. On account of (4.6) and (5.5), the magnetic field lines are material lines if the magnetic flux is conserved in a magnetic tube. This means that a magnetic tube imbedded in a perfectly conducting fluid is a material tube.

Substitution of (5.5) into (5.3) yields

$$K_{[ab]}^{\cdot\cdot c} = 0, \qquad \text{... (5.6)}$$

which is the Weingarten identity[12]. This identity guarantees the existence of a family of "magnetic world sheets" $\{M\}$. It is evident from (4.6), (5.4) and (5.5) that the magnetic flux conservation leads to the Weingarten identity which is a necessary and sufficient condition for the existence of a family of "magnetic world sheets $\{M\}$. Thus the Weingarten identity is the required geometrical condition which ensures the existence of $\{M\}$ due to the magnetic flux conservation.

6. String-Like Interaction of Magnetic Field Lines

The conservation of magnetic flux implies that the magnetic field lines are the material lines. This means that fluid particles once locked to a magnetic field line will remain so throughout the motion. The magnetic field lines behave as vibrating elastic strings. The dynamical interaction of the magnetic field lines with a perfectly conducting fluid can be thought of as the fluid particles locked with the magnetic field lines are in a state of vibration. Such dynamical interaction may be described in a manner analogous to that of the extrinsic motion[32] of the string. The extrinsic motion is given by[33].

$$\bar{\bar{T}}^{ab} = K_{ab}^{\cdot\cdot c} = F^c, \qquad \text{... (6.1)}$$

where F^c denotes the Lorentz force. $\bar{\bar{T}}^{ab}$ is the part of the stress-energy tensor tangential to the "magnetic world sheet" M. Since the string behaviour of the magnetic field lines is characterized by the vanishing of the Lorentz force, i.e. $F^c = 0$, eq. (6.1) reduces to

$$\bar{\bar{T}}^{ab} \; K_{ab}^{\cdot\cdot c} = 0. \qquad \text{... (6.2)}$$

The stress-momentum-energy tensor of a perfectly conducting fluid interacting with the magnetic field is expressible in the form

$$T^{ab} = T^{ab}_{(f)} + T^{ab}_{(m)} \qquad \text{... (6.3)}$$

in which $T^{ab}_{(f)}$ is the contribution due to the fluid and $T^{ab}_{(m)}$ is the magnetic field contribution. The fluid contribution of the stress-momentum-energy tensor is

$$T^{ab}_{(f)} = (\rho + p)\, u^a u^b + p\, g^{ab}, \qquad \text{... (6.4)}$$

where ρ and p are to be interpreted as the proper energy density and the pressure. The magnetic field contribution of the stress-momentum-energy tensor[13] is

$$T^{ab}_{(m)} = \mu H^2 u^a u^b + \frac{1}{2} \mu H^2 g^{ab} - \mu H^a H^b. \qquad \text{... (6.5)}$$

Substitution of (6.4) and (6.5) into (6.3) gives

$$T^{ab} = U u^a u^b - T n^a n^b + p^* \hat{\gamma}^{ab}, \qquad \text{... (6.6)}$$

where

$$U = \rho + \frac{1}{2} \mu H^2, \qquad T = \frac{1}{2} \mu H^2 - p,$$

$$p^* = p + \frac{1}{2} \mu H^2. \qquad \text{... (6.7)}$$

It is now useful to recast (6.6) in the form

$$T^{ab} = \bar{\bar{T}}^{ab} + \hat{T}^{ab} \qquad \text{... (6.8)}$$

in which $\bar{\bar{T}}^{ab} = U u^a u^b - T n^a n^b$ and $\hat{T}^{ab} = p^* \hat{\gamma}^{ab}$. One point worth making here is that $\bar{\bar{T}}^{ab}$ is tangential to the "magnetic world sheet".

Invoking the defining expression of $\bar{\bar{T}}^{ab}$ in (6.2) and making use of (5.2), we find that

$$U \hat{\gamma}^c_a \dot{u}^a - T \hat{\gamma}^c_a \overset{\circ}{n}{}^a = 0. \qquad \text{... (6.9)}$$

This describes the dynamical interaction of the magnetic field lines with the fluid. Eq. (6.9), due to (3.8), reduces to the form

$$C_E^2 K^a = \left(1 - C_E^2\right) \hat{\gamma}^a_b \dot{u}^b, \qquad \text{... (6.10)}$$

where $C_E = \sqrt{\dfrac{T}{U}}$ and C_E is interpretable as the speed of propagation of string waves relative to the rest frame of the fluid. On account of (6.7), we have $C_E = \sqrt{\dfrac{\frac{1}{2}\mu H^2 - p}{\rho + \frac{1}{2}\mu H^2}}$. This shows that a hydromagnetic system triggered with extrinsic perturbations will remain stable if $T > 0$, i.e. $\frac{1}{2}\mu H^2 > p$. The speed of propagation obeys $C_E \leq 1$ if $\rho + p \geq 0$.

7. Effect of Magnetic Field on Light Rays

In studies of gravitational collapse, a useful local property is the trapped surface[34] which provides the practical definition of a black hole and is the key ingradient in the singularity theorems[35]. These results motivate to investigate the effects brought by the magnetic field on the convergence of light rays. We now confine our attention to such investigation in this section.

The shape tensor[33] $\hat{K}_{ab}^{\cdot\cdot c}$ for a magnetic tube's cross-section S can be defined as

$$\hat{K}_{ab}^{\cdot\cdot c} = \hat{\gamma}^{c}_{d|a}\,\hat{\gamma}^{d}_{b} = \hat{\gamma}^{c}_{d;e}\,\hat{\gamma}^{e}_{a}\,\hat{\gamma}^{d}_{b}, \quad \text{... (7.1)}$$

where single vertical bar (|) is used to denote the tangential covariant derivative on a cross-section S. Substitution of (3.5) into (7.1) gives

$$\hat{K}_{ab}^{\cdot\cdot c} = u^{c}\,u_{e;d}\,\hat{\gamma}^{e}_{b}\,\hat{\gamma}^{d}_{a} - n^{c}\,n_{e;d}\,\hat{\gamma}^{e}_{b}\,\hat{\gamma}^{d}_{a}, \quad \text{... (7.2)}$$

The kinetic quantities associated with the fluid 4-velocity vector fluid u^{a} are defined by Ehlers decomposition[36] of the covariant derivative of u^{a} :

$$u_{a;b} = \sigma_{ab} + \frac{1}{3}\,\theta\,\gamma_{ab} + \omega_{ab} - \dot{u}_{a}\,u_{b}, \quad \text{... (7.3)}$$

where σ_{ab}, θ, ω_{ab} and $\dot{u}_{a}$ denote, respectively, the rate of shear tensor, the volume expansion, the vorticity tensor, and the acceleration vector. $\gamma_{ab} = g_{ab} + u_{a}\,u_{b}$ is the projection tensor.

The kinematical quantities associated with the spacelike congruence of magnetic field lines generated by the unit vector field n^{a} are defined by Greenberg's decomposition [31] of the covariant derivative of n^{a} :

$$n_{a;b} = \hat{\sigma}_{ab} + \frac{1}{2}\,\hat{\theta}\,\hat{\gamma}_{ab} + \hat{\omega}_{ab} + \overset{o}{n}_{a}\,n_{b} - \dot{n}_{a}\,u_{b} + (\dot{u}_{c}\,n^{c})\,u_{a}\,u_{b}$$
$$+ \left(\overset{o}{n}_{c}\,u^{c}\right) u_{a}\,n_{b} + u_{a}\,u_{c;b}\,n^{c}, \quad \text{... (7.4)}$$

where $\hat{\sigma}_{ab}$, $\hat{\theta}$ $\hat{\omega}_{ab}$ denote, respectively, the rate of shear tensor, the area expansion and the vorticity tensor.

Substitution of (7.3) and (7.4) into (7.2) yields

$$\hat{K}_{ab}^{\ \ c} = u^{c}\Big\{\sigma_{ab} - n_{a}\,\sigma_{be}\,n^{e} - \sigma_{ea}\,n^{e}\,n_{b} + \sigma_{ed}\,n^{e}\,n^{d}\,n_{a}\,n_{b}$$
$$+ \frac{1}{3}\,\theta\,\hat{\gamma}_{ab} + \omega_{ba} + n_{a}\,\omega_{eb}\,n^{e} - \omega_{ea}\,n^{e}\,n_{b}\Big\}$$
$$- n^{c}\left\{\hat{\sigma}_{ab} + \frac{1}{2}\,\hat{\theta}\,\gamma_{ab} + \hat{\omega}_{ba}\right\}. \quad \text{... (7.5)}$$

The skew-symmetric part of the shape tensor is obtainable from (7.5) in the form

$$\hat{K}_{ab}^{\cdot\cdot c} = \hat{\omega}_{ab} n^c - \omega_{\cdot ab\cdot} u^c - n^d \omega_{d[a} {}^n{}_{b]}. \qquad \text{... (7.6)}$$

Contracting (7.5) on a and b. We obtain

$$\hat{K}^{\cdot\cdot c} = \left\{ \frac{2}{3}\theta - \sigma_{ab} n^a n^b \right\} u^c - \hat{\theta} n^c. \qquad \text{... (7.7)}$$

where $\hat{K}^c$ represents the extrinsic curvature vector of a magnetic tube's cross-section S.

We now choose two null normal vectors k_+^a and k_-^a to a magnetic tube's cross-section S in such a way that k_+^a is the outward directed future pointing null vector and k_-^a is the outward directed past pointing null vector in a manner analogous to that of Guven and Murchadha[37]. These two null vectors can be expressed as

$$k_\pm^a = \frac{1}{\sqrt{2}} (u^a \pm n^a) \qquad \text{... (7.8)}$$

with normalization

$$k_+^a k_{-a} = -1. \qquad \text{... (7.9)}$$

The divergence of $k_\pm^a$ is given by

$$\Theta_\pm = \frac{1}{\sqrt{2}} \left(u_{a;b} \pm n_{a;b} \right) \hat{\gamma}^{ab} \qquad \text{... (7.10)}$$

which, due to (7.3) and (7.4), assume the form

$$\Theta_\pm = \frac{1}{\sqrt{2}} \left[\left(\frac{2}{3}\theta - \sigma_{ab} n^a n^b \right) \pm \hat{\theta} \right], \qquad \text{... (7.11)}$$

where $\Theta_\pm$ are called optical parameters[12] of beams of light rays.

Contracting (4.6) with H_b and making use of (7.3) in the resulting equation, we find that

$$\frac{2}{3}\theta - \sigma_{ab} n^a n^b = -\frac{1}{2}\left(\ln \frac{1}{2} \mu H^2 \right). \qquad \text{... (7.12)}$$

It is known[15] from (2.2a) that

$$\overset{o}{B} + B\hat{\theta} = 0 \qquad \text{... (7.13)}$$

which may equivalently be expressed as

$$\hat{\theta} = -\frac{1}{2}\left(\ln \frac{1}{2} \mu H^2 \right)^{o}. \qquad \text{... (7.14)}$$

Substitution of (7.12) and (7.14) into (7.11) yields

$$\Theta_{\pm} = -\frac{1}{2^{1/2}\mu H^2}\left[\left(\frac{1}{2}\mu H^2\right)^{\odot} + \left(\frac{1}{2}\mu H^2\right)\right], \qquad \text{... (7.15)}$$

$$\Theta_{\pm} = -\frac{1}{2^{1/2}\mu H^2}\left[\left(\frac{1}{2}\mu H^2\right)^{\odot} - \left(\frac{1}{2}\mu H^2\right)\right], \qquad \text{... (7.16)}$$

in which Θ_+ represents the expansion of the outgoing light rays and Θ_- is the expansion of the ingoing light rays. It is evident from (7.15) that $\Theta_+ < 0$ on S as the magnetic energy increases on a "magnetic world sheet" M. In other words, the outgoing light rays converge as the magnetic energy increases on a "magnetic world sheet". Similarly, (7.16) shows that $\Theta_- \gtrless 0$ on S according as the magnetic energy grows faster (or slower) along the magnetic field lines than that along the fluid flow lines. This means that the ingoing light rays diverge or converge according as the magnetic energy grows faster (or slower) along the magnetic field lines than that along the fluid flow lines. It is known that a closed spacelike 2-surface is a trapped surface if $\Theta_+ < 0$ and $\Theta_- < 0$ everywhere[38]. In accordance with this definition, a magnetic tube's cross-section S will be trapped if the magnetic energy grows faster along the fluid flow lines than that along the magnetic field lines. it is worth noticing that the differing signs of Θ_+ and Θ_- ensure that S is orientable and determine a preferred orientation with a direction normal to S being called inward if it points towards the neighbouring trapped surfaces and outward if it points in the opposite sense.

It follows from (7.7) and (7.11) that

$$-\frac{1}{2}\hat{K}^a\,\hat{K}_a = \Theta_+\Theta_-, \qquad \text{... (7.17)}$$

which, due to (7.15) and (7.16), assumes the form

$$\hat{K}^a\,\hat{K}_a = \frac{1}{(\mu H^2)^2}\left[\left\{\left(\frac{1}{2}\mu H^2\right)^{o}\right\}^2 - \left\{\left(\frac{1}{2}\mu H^2\right)\right\}^2\right]. \qquad \text{... (7.18)}$$

This shows that $\hat{K}^a$ is spacelike (timelike or null) according as $\left(\frac{1}{2}\mu H^2\right)^{\odot} \gtrless \left(\frac{1}{2}\mu H^2\right)$. A special case of interest is the marginally trapped surface[12] for which $\hat{K}^a$ is null. Hence the cross-section S forms a marginally trapped surface if the variation of the magnetic energy along the magnetic field lines becomes equal to that along the fluid flow lines.

In the following, we relate our discussion with the Hawking mass. A tedious but straightforward calculation provides the Gauss equation in the form

$$\hat{R}_{ab} = R_{hk}\, \hat{\gamma}_a^h\, \hat{\gamma}_b^k + \left(R_{ehkf}\, u^e u^k - R_{ehkf}\, n^e n^k \right) \hat{\gamma}_a^h\, \hat{\gamma}_b^f$$

$$+ \hat{K}_e\, \hat{K}_{ab}^{\cdot\cdot e} - \hat{K}^c{}_{be}\, \hat{K}_{ac}^{\cdot\cdot e} + 2\, \hat{K}_{[cb]}^{\cdot\cdot e}\, \hat{K}^c{}_{ae}, \quad \text{... (7.19)}$$

where $\hat{R}_{ab}$ represents the Ricci tensor of the magnetic tube's cross-section S and R_{ab} is the Ricci tensor of the spacetime manifold V_4.

Following Carter[33], we define the extrinsic conformation tensor $C_{ab}^{\ c}$ of a magnetic tube's cross-section S as

$$C_{ab}^{\cdot\cdot c} = \hat{K}_{ab}^{\cdot\cdot c} - \frac{1}{2}\, \hat{\gamma}_{ab}\, \hat{K}^c. \quad \text{... (7.20)}$$

If a magnetic tube's cross-section S is assumed to be conformally invariant, its conformation tensor vanishes. As a consequence, (7.20) reduces to

$$\hat{K}_{ab}^{\cdot\cdot c} - \frac{1}{2}\, \hat{\gamma}_{ab}\, \hat{K}^c. \quad \text{... (7.21)}$$

This satisfies the Weingarten identity

$$\hat{K}_{[ab]}^{\cdot\cdot c} = 0 \quad \text{... (7.22)}$$

whence conformally invariant family of magnetic tube's cross-sections $\{S\}$ meshes together to form a well-defined spacelike 2-surface through each point of the spacetime manifold V_4.

Contracting (7.19) with $\hat{\gamma}^{ab}$ and making use of (7.21), we obtain

$$\hat{R} = R + 2\left(R_{ab}\, u^a u^b - R_{ab}\, n^a n^b \right)$$

$$- 2R_{abcd}\, u^a u^c n^b n^d + \frac{1}{2}\, \hat{K}^a\, \hat{K}_a. \quad \text{... (7.23)}$$

It is now possible to recast (7.23), appealing to the Einstein field equation and using (6.6), (7.8) and (7.17), in the form

$$R = 4\, T_{ab}\, k_+^a\, k_-^b + 2k_M + (3p - \rho) - \Theta_+\, \Theta_-, \quad \text{... (7.24)}$$

where $k_M = - R_{abcd}\, u^a u^c n^b n^d$ is the sectional curvature of a "magnetic world sheet" M.

It is known from recent investigations[37] that the formation of trapped surfaces is closely related to the quasi-local mass. In spherical symmetry, the quasi-local mass is just the Hawking mass[37]. Flanagan[38] has pointed out in his hoop conjecture formulation of black hole's horizon that the definition of the quasi-local mass advanced by Hawking[39] is more appropriate than Arnowitt-Deser-Misner[40] (ADM) definition of mass. Thus, we consider the Hawking mass.

$$m_H = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_S (\hat{R} + \Theta_+ \Theta_-)\, d\sigma, \quad \dots (7.25)$$

where $A = \int_S d\sigma$ is the area of a magnetic tube's cross-section S.

Inserting (7.24) in (7.25), we have

$$m_H = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_S \left[4T_{ab}\, k_+^a k_-^b + 2k_m + (3p - \rho) \right] d\sigma. \quad \dots (7.26)$$

The first term on the right hand side of (7.26) is positive by the dominant energy condition[35]; he second term is positive by the chronology condition[35] (i.e. there are no closed timelike curves); the third term is non-negative, i.e. $\rho \le 3p$ in the extremely dense matter. Thus it is apparent from (7.26) that $m_H > 0$.

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For example, the convected derivative for a general tensor with mixed components S^{a}_{b} is given by

$$D_{\mathrm{con.}} S^{a}_{b} = \dot{S}^{a}_{b} + S^{a}_{e}\left(\dot{u}_{b} u^{e} + u^{e}_{;b}\right) + \ldots\ldots\ldots\ldots - S^{e}_{b}\left(\dot{u}_{e} u^{a} + u_{;e}\right) - \ldots\ldots\ldots\ldots$$

making use of this definition for B^{a}, one may write

$$D_{\mathrm{con.}} B^{a} = \dot{B}^{a} - B^{e}\dot{u}_{e} u^{a} - u^{a}_{;e} B^{e}.$$

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