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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION TO CERTAIN ONE-PARAMETER FAMILIES OF INTEGRAL OPERATORS

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The object of the present paper is to derive some properties of certain one-parameter families of integral operators in the unit disc by using the techniques of Briot-Bouquet differential subordination. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

Key Words: Analytic Functions; Integral Operator; Hadamard Product (or Convolution); Subordination

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{... (1.1)}$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. We consider the following one-parameter families of integral operators:

$$P_{\beta}^{\alpha} f(z) = \frac{(\beta+1)^{\alpha}}{\Gamma(\alpha)\, z^{\beta}} \int_0^z t^{\beta-1} \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt \, (\alpha \geq 0;\ \beta > -1;\ z \in E) \qquad \text{... (1.2)}$$

$$Q_{\beta}^{\alpha} f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z t^{\beta-1} \left(1 - \frac{t}{z}\right)^{\alpha-1}$$

$$f(t)\ dt \,(\alpha \geq 0;\ \beta > -1;\ z \in E) \qquad \text{... (1.3)}$$

and

$$J_\beta f(z) = \frac{\beta+1}{z^\beta} \int_0^z t^{\beta-1} f(t)\, dt \; (\beta > -1;\ z \in E), \qquad \ldots (1.4)$$

where Γ denotes the familiar Gamma function and $\binom{\alpha}{\beta} = \frac{\Gamma(\alpha\_1)}{\Gamma(\alpha-\beta+1)\,\Gamma(\beta+1)}$. The operators $P_1^\alpha = P^\alpha, Q_\beta^\alpha$ and J_β were introduced by Jung, Kim and Srivastava[3] and recently Owa[6] studied certain properties of the operators P^α and Q_β^α. For $\alpha \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the operators P^α, Q_1^α and J_β were considered by Bernardi[1,2] and for any real number $\beta > -1$, the operator J_β was used by Owa and Srivastava[7] and by Srivastava and Owa[8].

Using the integral representations of the Gamma and Beta functions, it can be shown that (seel also[3]) for $f \in \mathcal{A}$ given by (1.1),

$$P_\beta^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n}\right)^\alpha a_n z^n$$

$$= \left(\sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n}\right)^\alpha z^n \right) * f(z) \; (\alpha \geq 0;\ \beta > -1), \qquad \ldots (1.5)$$

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n$$

$$= \binom{\alpha+\beta}{\beta} z_2 F_1(1, \beta; \alpha+\beta; z) * f(z) \; (\alpha \geq 0;\ \beta > -1) \qquad \ldots (1.6)$$

and

$$J_\beta f(z) = z + \sum_{n=2}^{\infty} \frac{\beta+1}{\beta+n} a_n z^n \; (\beta > -1) \qquad \ldots (1.7)$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function (c.f. Section 2) and the symbol "*" stands for the Hadamard product or convolution. By virtue of (1.5), (1.6) and (1.7), we see that

$$J_\beta f(z) = P_\beta^1 f(z) = Q_\beta^1 f(z),$$

$$z\left(P_\beta^\alpha f(z)\right)' = (\beta+1) P_\beta^{\alpha-1} f(z) - \beta P_\beta^\alpha f(z) \; (\alpha \geq 1) \qquad \ldots (1.8)$$

and

$$z\left(Q_\beta^\alpha f(z)\right)' = (\alpha+\beta) Q_\beta^{\alpha-1} f(z) - (\alpha+\beta-1) Q_\beta^\alpha f(z) \; (\alpha \geq 1;\ \beta > -1). \qquad \ldots (1.9)$$

In this, we derive some properties of the integral operators P_β^α and Q_β^α by using the techniques of Briot-Bouquet differential subordination. Our results improve the work of Owa[6].

2. Preliminaries

To derive our main results, we require the following lemmas.

Lemma 2.1 — If $-1 \le B < 1$, $B < A$, $\lambda > 0$ and the complex number δ satisfy $\Re(\delta) \ge -\lambda(1-a)/(1-B)$, then the differential equation

$$q(z) + \frac{zq'(z)}{\lambda q(z) + \delta} = \frac{1+Az}{1+Bz} \quad (z \in E)$$

has a univalent solution in E given by

$$q(z) = \begin{cases} \dfrac{z^{\lambda+\delta}(1+Bz)^{\lambda(A-B)/B}}{\lambda \int_0^z t^{\lambda+\delta-1}(1+Bt)^{\lambda(A-B)/B}\, dt} - \dfrac{\delta}{\lambda}, & B \neq 0, \\ \dfrac{z^{\lambda+\delta}\exp(\lambda Az)}{\lambda \int_0^z t^{\lambda+\delta-1}\exp(\lambda At)\, dt} - \dfrac{\delta}{\lambda}, & B = 0. \end{cases} \qquad \text{... (2.1)}$$

If $\phi(z)$ is analytic in E and satisfies

$$\phi(z) + \frac{z\phi'(z)}{\lambda\phi(z) + \delta} \prec \frac{1+Az}{1+Bz} \quad (z \in E)$$

then $\phi(z) \prec q(z) \prec (1+Az)/(1+Bz)$ and $q(z)$ is the best dominant.

The above result can be found in Miller and Mocanu[5].

Lemma 2.2 — Let μ be a positive measure on the unit interval $I = [0, 1]$. Let $g(t, z)$ be a function analytic in E for each $t \in I$, and integrable in t for each $z \in E$ and for almost all $t \in I$, and suppose that $Re\,\{g(t, z)\} > 0$ in E, $g(t, -r)$ is real for real r and $Re\,\{1/g(t, z)\} \ge 1/g(t, -r)$ for $|z| \le r < 1$ and $t \in I$. If $g(z) = \int_I g(t, z)\, d\mu(t)$, then $Re\,\{1/g(z)\} \ge 1/g(-r)$ for $|z| \le r$.

The above lemma is due to Wilken and Feng[10].

For real or complex numbers a, b and c $(c \neq 0, -1, -2, \ldots)$, the hypergeometric function is defined by

$$ {}_2F_1\,(a, b; c; z) = 1 + \frac{a\,.\,b}{1\,.\,c}\,z + \frac{a(a+1)\,.\,b(b+1)}{2!c\,(c+1)}\,z^2 + \dots\,. \qquad \dots (2.2)$$

We note that the series in (2.2) converges absolutely for $z \in E$.

The following identities are well-known[9].

Lemma 2.3 — For real or complex numbers $a, b, c(c \neq 0, -1, -2, \dots)$ and $\text{Re}\,(c) > \text{Re}\,(b) > 0$, we have

$$\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\;{}_2F_1\,(a, b; c; z) \qquad \dots (2.3)$$

$$ {}_2F_1\,(a, b; c; z) = (1-z)^{-a}\;{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \qquad \dots (2.4)$$

$$ {}_2F_1\,(a, b; c; z) = {}_2F_1\,(b, a; c; z) \qquad \dots (2.5)$$

$$ {}_2F_1\left(1, 1; 3; \frac{dz}{dz+1}\right) = \frac{2(1+dz)}{dz}\left\{1 - \frac{\ln\,(1+dz)}{dz}\right\}\;(d \neq 0). \qquad \dots (2.6)$$

3. Main Results

Theorem 3.1 — *If $f \in \mathcal{A}$ satisfies*

$$\frac{P_\beta^{\alpha-2}\,f(z)}{P_\beta^{\alpha-1}\,f(z)} \prec \frac{1+Az}{1+Bz}\;(\alpha \geq 2; \beta > -1; z \in E\,) \qquad \dots (3.1)$$

for arbitrary real numbers, A, B satisfying $-1 \leq B < 1$ and $B < A$, then

$$\frac{P_\beta^{\alpha-1}\,f(z)}{P_\beta^{\alpha}f(z)} \prec \frac{1}{(\beta+1)\,Q\,(z)} = q\,(z) \prec \frac{1+Az}{1+Bz}, \qquad \dots (3.2)$$

where

$$Q\,(z) = \begin{cases} \displaystyle\int_0^1 t^\beta \left(\frac{1+Btz}{1+Bz}\right)^{(\beta+1)\,(A-B)/B} dt, & B \neq 0 \\ \displaystyle\int_0^1 t^\beta \exp\,((\beta+1)\,(t-1)\,Az)\,dt, & B = 0, \end{cases} \qquad \dots (3.3)$$

and $q\,(z)$ is the best dominant of (3.2). Furthermore, if $-1 \leq B < 0, B < A \leq -B/(\beta+1)$, then

$$\text{Re}\left\{\frac{P_\beta^{\alpha-1}\,f(z)}{P_\beta^{\alpha}f(z)}\right\} > \left[{}_2F_1\left(1, \frac{(\beta+1)\,(B-A)}{B}; \beta+2; \frac{B}{B-1}\right)\right]^{-1}\quad (z \in E\,).$$

The result is best possible.

PROOF : Let

$$\phi(z) = \frac{P_{\beta}^{\alpha-1} f(z)}{P_{\beta}^{\alpha} f(z)} \quad (z \in E). \qquad \text{... (3.4)}$$

Then $\phi(z)$ is analytic in E with $\phi(0) = 1$. Using (1.8) in the logarithmic differentiation of (3.4), we have

$$\phi(z) + \frac{z\phi'(z)}{(\beta+1)\phi(z)} = \frac{P_{\beta}^{\alpha-2} f(z)}{P_{\beta}^{\alpha-1} f(z)} \quad (z \in E)$$

which by use of (3.1) and Lemma 2.1 yields

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E),$$

where $q(z)$ is the best dominant of (3.2) and is given by (2.1) for $\lambda = \beta + 1$ and $\delta = 0$. By changing the variables in $q(z)$, we have

$$\phi(z) \prec \frac{1}{(\beta+1)\, Q(z)} \quad (z \in E),$$

where $Q(z)$ is given by (3.3).

Next we show that

$$\inf_{|z|<1} \{\mathrm{Re}\,(q(z))\} = q(-1). \qquad \text{... (3.5)}$$

Setting $a = (\beta+1)(B-A)/B$, $b = \beta + 1$ and $c = \beta + 2$ (so that $c > b > 0$) in (3.3) and by using (2.3), (2.4) and (2.5), we see that for $B \neq 0$

$$Q(z) = (1+Bz)^{a} \int_0^1 t^{b-1} (1+Btz)^{-a}\, dt = \frac{\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(1, a; c; \frac{Bz}{Bz+1}\right). \qquad \text{... (3.6)}$$

To prove (3.5), we show that $Re\{1/Q(z)\} \geq 1/Q(-1), z \in E$. Again (3.3), by (3.6) for $-1 \leq B < 0$, $A < -B/(\beta+1)$ (so that $c > a > 0$) can be rewritten as

$$Q(z) = \int_0^1 g(t, z)\, d\mu(t),$$

where

$$g(t, z) = \frac{1+Bz}{1+(1-t)Bz}$$

and

$$d\mu(t)=\frac{\Gamma(b)}{\Gamma(a)\,\Gamma(c-a)}\,t^{a-1}\,(1-t)^{c-a-1}\,dt$$

is a positive measure on [0, 1].

For $-1\leq B<0$, it may be noted that $Re\ \{g(t,z)\}>0$, $g(t,-r)$ is real for $0\leq r<1$, $t\in[0,1]$ and

$$\text{Re}\left\{\frac{1}{g(t,z)}\right\}=\text{Re}\left\{\frac{1+(1-t)\,Bz}{1+Bz}\right\}\geq\frac{1-(1-t)\,Br}{1-Br}=\frac{1}{g(t,-r)}$$

for $|z|\leq r<1$ and $t\in[0,1]$. Thus by making use of Lemma 2.2 and letting $r\to 1^-$ we get $Re\ \{1/Q(z)\}\geq 1/Q(-1), z\in E$. In the case $A=-B/(\beta+1)$, we get the required result by taking $A\to(-B/(\beta+1))^+$. The result is best possible because of the best dominant property of $q(z)$. This completes the proof of Theorem 3.1.

Putting $A=1-2\gamma$, $B=-1$ and $\beta=1$ in Theorem 3.1, we get

Corollary 3.1 — If $f\in\mathcal{A}$ satisfies

$$Re\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\}>\gamma(\alpha\geq 2; z\in E)$$

for some $\gamma(1/4\leq\gamma<1)$, then

$$Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\}>\left[{}_2F_1\,(1,4\,(1-\gamma);3;1/2)\right]^{-1}\quad(z\in E).$$

The result is best possible.

Remark : Noting that

$$\left[{}_2F_1\,(1,4\,(1-\gamma);3;1/2)\right]^{-1}=\begin{cases}\dfrac{(3-4\gamma)\,(1-2\gamma)}{8(2^{1-4\gamma}+\gamma-1)}, & \gamma\neq\dfrac{1}{4},\dfrac{1}{2},\dfrac{3}{4},\\[2ex] \dfrac{1}{2}, & \gamma=\dfrac{1}{4},\\[2ex] \dfrac{1}{4(2\ln 2-1)}\approx 0.64717, & \gamma=\dfrac{1}{2},\\[2ex] \dfrac{1}{4(1-\ln 2)}\approx 0.81472, & \gamma=\dfrac{3}{4}\end{cases}$$

for $1/4\leq\gamma<1$, we see that Corollary 3.1 improves the corresponding results obtained by Owa[6].

Theorem 3.2 — *If* $f \in \mathcal{A}$ *satisfies*

$$Re\left\{\frac{P_{\beta}^{\alpha-1} f(z)}{P_{\beta}^{\alpha} f(z)}\right\} > \gamma (\alpha \geq 2; \beta > -1; z \in E)$$

for some $\gamma (\gamma < 1)$, then

$$Re\left\{\frac{P_{\beta}^{\alpha-2} f(z)}{P_{\beta}^{\alpha-1} f(z)}\right\} > \gamma (|z| < R(\beta, \gamma)),$$

where

$$R(\beta \gamma) = \begin{cases} \dfrac{1 + (\beta+1)(1-\gamma) - \sqrt{\{1-(\beta+1)\gamma\}^2 + 2(\beta+1)}}{(\beta+1)(1-2\gamma)}, & \gamma \neq \dfrac{1}{2}, \\ \dfrac{\beta+1}{\beta+3}, & \gamma = \dfrac{1}{2}. \end{cases} \quad \text{... (3.7)}$$

The result is best possible.

PROOF : We have

$$\frac{P_{\beta}^{\alpha-1} f(z)}{P_{\beta}^{\alpha} f(z)} = \gamma + (1-\gamma) u(z) \, (z \in E), \quad \text{... (3.8)}$$

where $u(z) = 1 + u_1 z + u_2 z^2 + \ldots$ is analytic and has a positive real part in E. Logarithmic differentiation of (3.8) followed by the use of the identity (1.8) in the resulting equation yields

$$\left\{\frac{P_{\beta}^{\alpha-2} f(z)}{P_{\beta}^{\alpha-1} f(z)}\right\} = (1-\gamma)\left[u(z) + \frac{zu'(z)}{(\beta+1)\{\gamma + (1-\gamma + (1-\gamma) u(z)\}}\right]. \quad \text{... (3.9)}$$

Now, using the well-known[4] estimates

$$\frac{|zu'(z)|}{Re\{u(z)\}} \leq \frac{2r}{1-r^2} \quad \text{and} \quad Re\{u(z)\} \geq \frac{1-r}{1+r} \; (|z| = r < 1)$$

in (3.9), we deduce that

$$\left\{\frac{P_{\beta}^{\alpha-2} f(z)}{P_{\beta}^{\alpha-1} f(z)} - \gamma\right\} \geq (1-\gamma) Re\{u(z)\}\left[1 - \frac{2r}{(\beta+1)\{\gamma(1-r^2) + (1-\gamma)(1-r)^2\}}\right].$$

It is easily seen that the right-hand side of the above expression is positive for $|z| < R(\beta, \gamma)$, where $R(\beta, \gamma)$ is given by (3.7). Thus

$$Re\left\{\frac{P_{\beta}^{\alpha-2}f(z)}{P_{\beta}^{\alpha-1}f(z)}\right\} > \gamma (|z| < R(\beta, \gamma)).$$

To show that the bound $R(\beta, \gamma)$ is best possible, we consider the function $f \in \mathcal{A}$ defined by

$$\left\{\frac{P_{\beta}^{\alpha-1}f(z)}{P_{\beta}^{\alpha}f(z)}\right\} = \gamma + (1-\gamma)\frac{1+z}{1-z} \; (z \in E),$$

where $\alpha \geq 2, \beta > -1$ and $\gamma < 1$. Noting that

$$\frac{P_{\beta}^{\alpha-2}f(z)}{P_{\beta}^{\alpha-1}f(z)} - \gamma = (1-\gamma)\left[\frac{1+z}{1-z} + \frac{2z}{(\beta+1)(1-z)\{1+(1-2\gamma)z\}}\right] = 0$$

for $z = -R(\beta, \gamma)$, we complete the proof of Theorem 3.2.

Theorem 3.3 — *Let* $-1 \leq B < 1, B < A, \alpha \geq 2, \beta > -1$ *and* $\alpha + \beta > 1$ *satisfy*

$$(1-B) + (\alpha+\beta-1)(1-A) \geq 0. \qquad \text{... (3.10)}$$

(*i*) If $f \in \mathcal{A}$ satisfies

$$\frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)} \prec \frac{1+Az}{1+Bz} \; (z \in E) \qquad \text{... (3.11)}$$

then

$$\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)} \prec \frac{1+A^{*}z}{1+Bz} \; (z \in E), \qquad \text{... (3.12)}$$

where $A^{*} = \{(\alpha+\beta-1)A + B\} / (\alpha+\beta)$. Further, if $f \in \mathcal{A}$ satisfies (3.11), then we have

$$\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)} \prec \frac{1}{(\alpha+\beta)(z)} = \tilde{q}(z) \; (z \in E), \qquad \text{... (3.13)}$$

where

$$Q(z) = \begin{cases} \int_0^1 t^{\alpha+\beta-1}\left(\frac{1+Btz}{1+Bz}\right)^{(\alpha+\beta-1)(A-B)/B} dt, & B \neq 0 \\ \int_0^1 t^{(\alpha+\beta-1)} \exp((\alpha+\beta-1)(t-1)Az)\, dt, & B = 0 \end{cases} \qquad \text{... (3.14)}$$

(*ii*) Moreover, if

$$-1 \leq B < 0, B < A \leq \min\left\{\frac{\alpha+\beta-B}{\alpha+\beta-1}, -\frac{2B}{\alpha+\beta-1}\right\},$$

then for $f \in \mathcal{A}$ satisfying (3.11), we have

$$Re\left\{\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}\right\} > \left[{}_2F_1\left(1, \frac{(\alpha+\beta-1)(B-A)}{B}; \alpha+\beta+1; \frac{B}{B-1}\right)\right]^{-1} \quad (z \in E).$$

The result is best possible.

PROOF : Setting

$$\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)} = \phi(z) \quad (z \in E), \qquad \text{... (3.15)}$$

we see that $\phi(z)$ is analytic in the open unit disc E with $\phi(0) = 1$. Since $f \in \mathcal{A}$ satisfies (3.11) using (1.9) in (3.15) easily leads to

$$\psi(z) + \frac{z\psi'(z)}{(\alpha+\beta-1)\psi(z)+1} \prec \frac{1+Az}{1+Bz} \quad (z \in E), \qquad \text{... (3.16)}$$

where $\psi(z) = \{(\alpha+\beta)\psi(z)-1\}/(\alpha+\beta-1)$. Using Lemma 2.1, we deduce that

$$\psi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in E), \qquad \text{... (3.17)}$$

where $q(z)$ is the best dominant of (3.16) and is given by (2.1) for $\lambda = \alpha+\beta-1$ and $\delta = 1$. Again by (3.17), we have

$$\phi(z) \prec \frac{1}{(\alpha+\beta)Q(z)} = \tilde{q}(z) \quad (z \in E),$$

where $Q(z)$ is given by (3.14). This proves (3.12) and (3.13).

Proceeding as in Theorem 3.1, part (*ii*) of Theorem 3.3 follows.

If we take $A = 1 - 2\gamma$ and $B = -1$ in Theorem 3.3, then we have

Corollary 3.2 — If $f \in \mathcal{A}$ satisfies

$$Re\left\{\frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)}\right\} > \gamma(\alpha \geq 2;\ z \in E)$$

for some $\gamma((\alpha+\beta-3)/2(\alpha+\beta-1) \leq \gamma < 1)$, then

$$Re\left\{\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}\right\} > \left[{}_2F_1\,(1, 2\,(\alpha+\beta-1)\,(1-\gamma); \alpha+\beta+1; 1/2\right]^{-1} \;(z \in E).$$

The result is best possible.

Letting $\alpha = 2-\beta$ (resp. $\alpha = 3-\beta$) and $\gamma = 1/2$ in Corollary 3.2, we have the following results which improves the corresponding work of Owa[6].

Corollary 3.3 — If $f \in \mathcal{A}$ satisfies

$$Re\left\{\frac{Q_{\beta}^{-\beta}f(z)}{Q_{\beta}^{1-\beta}f(z)}\right\} > \frac{1}{2}\;(\beta \geq -1;\; z \in E)$$

then

$$.Re\left\{\frac{Q_{\beta}^{1-\beta}f(z)}{Q_{\beta}^{2-\beta}f(z)}\right\} > \frac{1}{4(1-\ln 2)} \approx 0.81472 \;(z \in E).$$

The result is best possible.

Corollary 3.4 — If $f \in \mathcal{A}$ satisfies

$$Re\left\{\frac{Q_{\beta}^{1-\beta}f(z)}{Q_{\beta}^{2-\beta}f(z)}\right\} > \frac{1}{2}\;(\beta : -1; z \in E)$$

then

$$Re\left\{\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)}\right\} > \left[{}_2F_1\,(1, \alpha+\beta-1; \alpha+\beta+1; 1/2\right]^{-1} \;(z \in E).$$

The result is best possible.

Remark : Putting $\gamma = (\alpha+\beta-3)/2(\alpha+\beta-1)$ in Corollary 3.2, we get a result of Owa[6].

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REFERENCES

1. S. D. Bernardi, *Trans. Amer. Math. Soc.*, **135** (1969), 429-46.
2. S. D. Bernardi, *Proc. Amer. Math. Soc.*, **24** (1970), 312-18.
3. I. B. Jung, Y. C. Kim and H. M. Srivastava, *J. Math. Anal. Appl:*, **176** (1993), 138-47.
4. T. H. MacGregor, *Proc. Amer. Soc.*, **14** (1963), 514-20.

5. S. S. Miller and P. T. Mocanu, *J. Diff. Eqs.*, **56** (1985), 297-309.
6. S. Owa, *Georgian Math. J.*, **2** (1995), 538-45.
7. S. Owa and H. M. Srivastava, *Proc. Japan Acad. Ser. A Math. Sci.*, **62** (1986), 125-28.
8. H. M. Srivastava and S. Owa, *J. Math. Anal. Appl.*, **118** (1986), 80-87.
9. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th Edition (Reprinted), Cambridge University Press, Cambridge, 1927.
10. D. R. Wilken and J. Feng, *J. London Math. Soc. (Ser. 2)*, **21** (1980), 287-90.