Indian J. pure appl. Math., **35**(10) : 1179-1183, October 2004

THE PROPERTY OF GOOD DECOMPOSITION IN HARDY FIELD

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(*Received* 9 *May* 2003; *after final revision* 15 *March* 2004; *accepted* 28 *October* 2004)

We study the operation of subtraction in the class of increasing regularly varying functions belonging to Harrdy field and we prove that this class has the property of good decomposition.

Key Words: Regularly Varying Functions; Slowly Varying Functions; Normalized Functions; Hardy Field

In[5] Karamata introduced the class of regularly varying functions in the following way: a positive measurable function R, defined on some neighbourhood of infinity, is said to be regularly varying with index $\rho \in \mathbb{R}$ if, for every $s > 0$,

$$\lim_{t \to +\infty} \frac{R(st)}{R(t)} = s^{\rho}. \qquad \text{... (1)}$$

When $\rho = 0$, functions that satisfy (1) are called slowly varying. The following canonical representation for regularly varying functions with index ρ is proved in[5]:

$$R(t) = t^{\rho} c(t) \exp\left\{ \int_a^t \frac{\varepsilon(u)}{u}\, du \right\}, \quad (x \geq a) \qquad \text{... (2)}$$

for some $a > 0$, where c and ε are measurable and such that $c(x) \to c \in (0, +\infty)$ and $\varepsilon(x) \to 0$ as $x \to +\infty$. Regularly varying functions having constant c-function are called normalised. Normalised regularly varying function R with index ρ satisfies

$$\lim_{x \to +\infty} \frac{xR'(x)}{R(x)} = \rho. \qquad \text{... (3)}$$

For detailed expositions of the theory of regularly varying functions see[1] or[8].

For slowly varying functions the sum, product and quotient are always slowly varying, but the difference need not be slowly varying. Even if we restrict ourselves to nondecreasing functions, there exist examples of slowly varying L and F such that the difference $G = L - F$ is not slowly varying. However, it is possible to find subsets of slowly varying functions with better behaviour with respect to the operation of subtraction. Such classes were investigated in[3,4,9&10]. To study this, we need the following definition.

Definition 1 — Let $\mathcal{M}$ be a class of functions. A slowly varying function $L \in \mathcal{M}$ is said to have the property of good decomposition in the class $\mathcal{M}$ if, whenever we decompose L into a sum $L = F + G$ of two functions $F, G \in \mathcal{M}$, then F and G are necessarily slowly varying.

In[3], the class $\mathcal{M}$ consists of nondecreasing functions. In[4] $\mathcal{M}$ is the class of increasing convex functions, while L is additively slowly varying, i.e. $L(x) = l(e^x)$, where l is (multiplicatively) slowly varying, as in (1). In[3&4] we characterized the classes having the property of good decomposition in the following way.

***Theorem* 3** — *Let L be a nondecreasing slowly varying function such that $L(t) \to +\infty$, as $t \to +\infty$. Then L has the property of good decomposition in the class of nondecreasing functions if and only if*

$$\limsup_{t \to +\infty} \, (L(st) - L(t)) = M(s) \qquad \text{... (4)}$$

is finite for every $s > 1$.

The class satisfying the condition (4) is called $O\Pi^+$. This class is rather small and it does not allow functions with faster rate of growth than $\log t$. In order to allow functions with faster rate of growth than $\log t$, stronger monotonicity assumptions had to be imposed on the class $\mathcal{M}$, as in the following theorem.

***Theorem* 4** — *An increasing convex additively slowly varying function L has the property of good decomposition in the class of increasing convex functions if and only if, for every $y \in \mathbb{R}$, there exists a constant $C = C(y)$ (not depending on x) such that*

$$0 \le \Delta_y^2 L(x) = L(x + 2y) - 2L(x + y) + L(x) \le Cx, \qquad \text{... (5)}$$

for x large enough.

The class satisfying the condition (5) is called $O\Pi_2^+$ and it contains functions with the rate of growth not faster than x^3, which means when passing from additive slow variation to the usual, multiplicative, slow variation, that functions cannot grow faster than $(\log t)^3$.

These results about slowly varying functions yield some corollaries about regularly varying functions, since every regularly varying function R can be written as $R(t) = t^{\rho} L(t)$, where ρ is the index and L is a slowly varying function. In the case of regularly varying functions, Definition 1 is modified accordingly. The theorems above extend to regularly varying functions, see[9].

In this paper we consider functions belonging to the Hardy field. Hardy field $\mathcal{H}$ (see[6&7]) is defined as a set of germs of real-valued functions on positive half lines in $\mathbb{R}$ that is closed under differentiation and that form a field under the usual addition and multiplication of germs.

Functions from Hardy field have the following properties: they are ultimately continuous together with their derivatives and strictly nomnotone (except for constant functions), they are of constant sign and $\lim_{x \to +\infty} f(x)$ exists. Every two functions from Hardy field are comparable, because for every $f, g \in \mathcal{H}, f/g \in \mathcal{H}$ too, and therefore there exists the limit $\lim_{x \to +\infty} f(x)/g(x)$. This means that Hardy field is well ordered with respect to the relation "asymptotically larger" (or smaller) $\preceq$.

Every element of Hardy field $\mathcal{H}$ is a normalized (see[3]) Karamata function (regularly or rapidly varying), because from the properties of Hardy field it follows that, for every function $f \in \mathcal{H}$, the limit below exists (possibly infinite):

$$\lim_{x \to +\infty} \frac{xf'(x)}{f(x)} = \rho. \qquad \dots (6)$$

Put $\varepsilon(x) = xf'(x)/f(x)$. When ρ from (6) is finite, then f is a regularly varying function and by integrating this equality the canonical representation (2) can be easily obtained.

An example of Hardy field is the class of logarithmico-exponential functions, introduced by Hardy[2]. A function belonging to this class is a real function defined for all values of x greater than some definite value, by a finite combination of the ordinary algebraic symbols ($+, -, \times, \div, \sqrt[n]{\ }$) and functional symbols log and exp operating on the variable x and real constants. This is a standard class in asymptotic analysis.

In[6], the fact that the difference of two regularly varying functions need not be regularly varying was noted in connection with solutions of linear differential equations of second order with functional coefficients from Hardy field - namely if two solutions of a certain equation are regularly varying, their linear combination is again a solution, but it is interesting to know whether it is also regularly varying.

In the theorem below the class $\mathcal{M}$ from Definition 1 is the class $\mathcal{H}^+$ of positive ultimately increasing functions from $\mathcal{H}$. Thus regularly varying functions from $\mathcal{H}^+$ must have index $0 \le \rho < +\infty$.

Theorem 1 — *Ler R be a regularly varying function of index* $\rho \ge 0$ *from* $\mathcal{H}^+$. *Then R has the property of good decomposition in* $\mathcal{H}^+$.

PROOF : We have to prove that whenever we decompose a regularly varying function $R \in \mathcal{H}^+$ into a sum of two functions.

$$R(z) = F(x) + G(x), \qquad \ldots (7)$$

$F, G \in \mathcal{H}^+$, then F and G are always regularly varying. From (7), since every two functions are comparable in $\mathcal{H}$, we have that

$$\lim_{x \to +\infty} \frac{F(x)}{R(x)} = c \geq 0, \quad \text{and} \quad \lim_{x \to +\infty} \frac{G(x)}{R(x)} = d \geq 0, \qquad \ldots (8)$$

where $c + d = 1$. In the case when both $c > 0$ and $d > 0$, the functions F and G are both regularly varying with the same index ρ, because they are asymptotically equivalent to regularly varying functions $cR(x)$ and $dR(x)$, respectively (see[2], Prop. 7 for the fact that a function asymptotically equal to a regularly varying function is itself regularly varying with the same index).

If $c = 0$, then $d = 1$. Then, because of (8), G is regularly varying with index ρ, and we have only to prove that F is regularly varying too. When $c = 0$ then $F(x) = o(R(x))$ and if F is regularly varying, its index must be $\leq \rho$.

Suppose the contrary, that F from (7) is not regularly varying and if it were, then its index k is such that $k > \rho$. We use the fact that in $\mathcal{H}^+$ every function is normalized, see[6]. Hence, since $F \in \mathcal{H}^+$, then relation (6) implies

$$\lim_{x \to +\infty} \frac{x F'(x)}{F(x)} = k, \qquad \ldots (9)$$

with $\rho < k \leq +\infty$.From (9) it follows that there exist an $\varepsilon > 0$ and a Δ so that for all $x \geq \Delta$

$$\frac{xF'(x)}{F(x)} \geq \rho + \varepsilon.$$

It follows that, for x large enough,

$$(\ln F(x))' \geq (\ln x^{\rho + \varepsilon})'.$$

This means that the function $\ln F(x) - \ln x^{\rho + \varepsilon}$ is strictly increasing, i.e. the function $F(x)/x^{\rho + \varepsilon}$ is strictly increasing for x large enough. But since

$$\frac{F(x)}{x^{\rho + \varepsilon}} \leq \frac{R(x)}{x^{\rho + \varepsilon}},$$

it follows that $R(x)/x^{\rho + \varepsilon}$ does not tend to zero, which is impossible since R is regularly varying of index ρ. Therefore, the limit in (9) is $\leq \rho$, which means that F is regularly varying.

The case when $c = 1$ and $d = 0$ can be treated in the same way. □

No decreasing function from $\mathcal{H}$ has the property of good decomposition in the class of decreasing functions from $\mathcal{H}$. This can be shown in the following way: every decreasing regularly varying $R \in \mathcal{H}$ can be written as a sum $R(x) = F(x) + G(x) = (R(x) - e^{-x}) + e^{-x}$, where the component $G(x) = e^{-x}$ is not regularly varying.

Theorem 1 has the following consequences which can be proved by adapting the proof of Theorem 1.

Corollary 1 — If the difference of two regularly varying functions from $\mathcal{H}$ is positive and bounded away from zero, then it is regularly varying.

PROOF : We suppose that

$$\lim_{x \to +\infty} (R(x) - G(x)) = \lim_{x \to +\infty} F(x) = l > 0.$$

When l is finite, then F is slowly varying because it tends to a constant. Note that the case when F is decreasing is possible here. The case when $l = +\infty$, can be treated in the same way as in the proof of the Theorem 1. □

Corollary 2 — If the difference of two regularly varying functions from $\mathcal{H}$ is positive increasing, then it is regularly varying.

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