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ON THE EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR THE p-LAPLACIAN*

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In this paper, we study the vector p-Laplacian,

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u + f(t,u) + h(t) \text{ a.e } t \in [0,T] \\ u(0) = u(T),\ u'(0) = u'(T),\ 1 < p < \infty. \end{cases} \quad \dots (*)$$

We prove that there exists $\bar{\lambda} > 0$ such that for all $\lambda \in [-\bar{\lambda}, 0)$ problem (*) has at least three distinct solutions under some suitable conditions. On the other hand, we prove problem (*) has at least one solution for any $0 \le \lambda < \lambda_1$ under some suitable conditions. The method is critical point theory.

Key Words : One Dimensional p-Laplacian, Periodic Solution; Critical Point Theory

1. INTRODUCTION

In this paper, we study a class of p-Laplacian equations of the form

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u + f(t,u) + h(t) \text{ a.e } t \in [0,T] \\ u(0) = u(T),\ u'(0) = u'(T),\ 1 < p < \infty. \end{cases} \quad \dots (1.1)$$

where $T > 0$ and $f : [0, T] \times R \to R$ satisfies the following assumption: f is measurable in t for every $x \in R$ and continuous in x for a.e. $t \in [0, T]$.

Recently there has been an increasing interest for periodic problems driven by the one-dimensional p-Laplacian. We refer to the works of Del Pino-Manasevich-Murua[4], Fabry-Fayyad[5], Gao[6] and Dang-Oppenheimer[7]. In all these works the approach is degree theoretical and they establish the existence of one solution. In this paper, we will show that problem (1.1) has at least three distinct solutions as $\lambda \to 0^-$.

Our study is based on a bifurcation result by Mawhin and Schmitt[2], related to the two-point boundary value problem

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$$-u''-\lambda u=f(x,u)+h,\ u(0)=u(\pi)=0.$$

By assuming that f is bounded and satisfying a sign condition, they obtained the desired result. Our objective is to extend this result to periodic p-Laplacian. If λ is sufficiently near to 0 from the left, where 0 is the first eigenvalue of the corresponding linear problem, then (1.1) has at least three solutions. If $0 \le \lambda < \lambda_1$, then problem (1.1) has at least one solution.

In order to state the Mawhin-Schmitt problem in the context of $W_T^{1,p}$, we recall that the eigenvalue problem for the p-Laplacian, see[3].

$$\begin{cases} \left(|u'|^{p-2}u'\right)' + \lambda |u|^{p-2}u = 0 \text{ in } (0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0 \end{cases} \quad \text{... (1.2)}$$

has as solutions the numbers

$$\lambda_n = \left(\frac{2\pi_p n}{T}\right)^p, \quad n=0,1,2,3,\dots$$

here

$$\pi_p = 2(p-1)^{1/p} \int_0^1 (1-t^p)^{-1/p}\,dt.$$

Throughout the paper, let $I=[0,T]$, we will set $C=C(I,R)$, $C^1=C^1(I,R)$, $C_T^1 = \{u\in C^1 \mid u(0)=u(T), u'(0)=u'(T)\}$, $L^p=L^p(I,R)$ and $X=W_T^{1,p}=\left\{u\in W^{1,p}; u(0)=u(T)\right\}$.

The norm in C will be defined by $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$, the norm in C_T^1 will be defined by $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, the norm in L^p by

$$\|u\|_p = \left(\int_0^T |u(t)|^p\,dt\right)^{\frac{1}{p}},$$

the norm in X by $\|u\|_X = \|u\|_p + \|u'\|_p$. We can easily prove that for each $p>1$, the embedding of X in C is compact.

Each $u\in L^1(I)$ can be written as $u(t)=\bar{u}+\tilde{u}(t)$, with

$$\bar{u} := \frac{1}{T}\int_0^T u(t)\,dt, \quad \int_0^T \tilde{u}(t)\,dt = 0.$$

We will use Sobolev' inequality

$$\|\tilde{u}\|_\infty \leq T^{\frac{1}{q}} \|u'\|_p \text{ for each } u \in X;$$

here $\frac{1}{p}+\frac{1}{q} = 1$ and Wirtinger's inequality

$$\|\tilde{u}\|_p^p \leq T^p \|\tilde{u}'\|_p^p \text{ for each } u \in X.$$

Our main results are as follows:

Theorem 1.1 — *Assume that:*

($H1$) *for almost all $t \in I$, all $x \in R$, we have*

$$|f(t,x)| \leq a(t)|x|^{r-1} + b(t),$$

with $a, b \in L^{r'}(I)$ where $1 \leq r < p$ and $\frac{1}{r}+\frac{1}{r'} = 1$;

$$(H2) \quad \int_0^T F(t,x) \to +\infty \quad \text{as} \quad |x| \to +\infty,$$

here

$$F(t,x) = \int_0^x f(t,u)\,du.$$

$$(H3) \quad h \in L^q(I) \text{ such that } \int_0^1 h(t)\,dt = 0.$$

Then there exists $\bar{\lambda} > 0$ such that for all $\lambda \in [-\bar{\lambda}, 0)$ problem (1.1) has at least three distinct solutions.

If $h(t) \equiv 0$, we have the following result:

Theorem 1.2 — *Assume that ($H1$) and*

$$(H4) \quad \lim_{|x| \to \infty} f(t,x)\,x - pF(t,x) = -\infty \text{ uniformly for almost all } t \in I.$$

Then problem (1.1) has at least one solution for any $0 \leq \lambda < \lambda_1$.

The proof of the theorems are given in Section 3. In Section 2, we present some preliminary results on the variational setting of the p-Laplacian equations in X and the related Palais-Smale compactness condition.

2. Preliminaries

It follows from assumption ($H1$) that the function φ on X given by

$$\varphi_\lambda(u) = \frac{1}{p}\|u'\|_p^p - \frac{\lambda}{p}\|u\|_p^p - \int_0^T [F(t, u(t)) + h(t)\,u(t)]\,dt$$

is continuously differentiable and weakly lower semicontinuous on X (the proof is similar to that for $p = 2$, see[1]). Moreover, one has

$$\langle \varphi_\lambda'(u), v\rangle = \int_0^T |u'|^{p-2}\,u'\,v'\,dt$$

$$-\lambda\int_0^T |u|^{p-2}\,uv\,dt - \int_0^T f(t,u)\,v\,dt - \int_0^T hv\,dt,$$

for all $u, v \in X$. It is easy to see that the solutions of problem (1.1) correspond to the critical points of φ_λ.

Let us denote

$$I(u) = \frac{1}{p}\|u'\|_p^p$$

and

$$G_\lambda(u) = \frac{\lambda}{p}\|u\|_p^p + \int_0^T [F(t, u(t)) + h(t)\,u(t)]\,dt, \qquad \forall u \in X.$$

Proposition 2.1 — $I' : X \to X^*$ is a mapping of type (S_+) (see[8]), i.e. any sequence $\{u_n\}$ in X satisfying $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} \langle I'(u_n), u_n - u\rangle \le 0$$

contains a convergent subsequence.

PROOF : Assume that $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle I'(u_n), u_n - u\rangle \le 0$. Then we obtain

$$\limsup_{n\to\infty} \langle I'(u_n) - I'(u), u_n - u\rangle \le 0.$$

This together with the monotone property of I' implies that

$$\lim_{n\to\infty} \langle I'(u_n) - I'(u), u_n - u\rangle = \le 0,$$

i.e.,

$$\int_0^T \left\langle |u_n'|^{p-2} u_n' - |u'|^{p-2} u', u_n' - u' \right\rangle dt \to 0, \quad n \to \infty.$$

On the other hand, for all $x, y \in R$ the following inequalities hold

$$\left\langle |x|^{p-2} x - |y|^{p-2} y, x - y \right\rangle \geq \left(\frac{1}{2}\right)^{p-1} |x-y|^p, \quad \text{when } p \geq 2,$$

$$(|x| + |y|)^{2-p} \left\langle |x|^{p-2} x - |y|^{p-2} y, x - y \right\rangle$$

$$\geq (p-1)|x-y|^2, \text{ when } 1 < p \leq 2.$$

Then, by (2.1), one has that u_n' converges to u' in $[0, T]$ in measure. Thus there exists a subsequence (without loss of generality assume it is the whole sequence) with

$$u_n'(t) \to u'(t) \text{ for a.e. } t \in [0, T], \; n \to \infty,$$

and so

$$\frac{1}{p} \left| u_n'(t) - u'(t) \right|^p \to 0 \text{ for a.e. } t \in [0, T], \; n \to \infty. \quad \dots (2.2)$$

Also one has

$$\langle I'(u_n), u_n - u \rangle \to 0 \quad \text{as} \quad n \to \infty.$$

In addition, one has

$$\langle I'(u_n), u_n - u \rangle = \int_0^T |u_n'|^p dt - \int_0^T |u_n'|^{p-2} u_n' u' dt$$

$$\geq \int_0^T |u_n'|^p dt - \int_0^T |u_n'|^{p-1} |u'| dt$$

$$\geq \int_0^T |u_n'|^p dt - \int_0^T \left[\frac{p-1}{p} |u_n'|^p + \frac{1}{p} |u'|^p\right] dt$$

$$= \frac{1}{p} \int_0^T (|u_n'|^p - |u'|^p) dt.$$

Now the lower semi-continuity of I yield

$$\lim_{n\to\infty} \frac{1}{p} \int_0^T |u_n'|^p \, dt \geq \frac{1}{p} \int_0^T |u_n'|^p \, dt.$$

Thus,

$$\frac{1}{p} \int_0^T |u_n'|^p \, dt \to \frac{1}{p} \int_0^T |u'|^p \, dt \text{ as } n \to \infty.$$

As a result the sequence $\left\{ \frac{1}{p} \left| u_n' \right|^p \right\}$ is equi-absolutely continuous on $[0, T]$. From the following inequality

$$\frac{1}{p} \left| u_n'(t) - u'(t) \right|^p \leq \frac{2^p}{p} \left(\left| u_n' \right|^p + |u'|^p \right)$$

we have that

$$\left\{ \frac{1}{p} \int_0^T \left| u_n'(t) - u'(t) \right|^p dt \right\} \text{ is equi-absolutely continuous on } [0, T]. \quad \text{... (2.3)}$$

Now (2.2), (2.3) and the convergence theorem of Vitali (see[8] [pp. 1017] guarantees that

$$\int_0^T \frac{1}{p} \left| u_n'(t) - u'(t) \right|^p \, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $u_n' \to u'$ in L^p.

Also $u_n \to u$ in X, implies that $u_n \to u$ in L^p. Hence $u_n \to u$ in X, and we are finished.

Since the sum of one mapping of type (S_+) with a weakly-strongly continuous mapping is still of type (S_+), we obtain the following result.

Proposition 2.2 — $\varphi_\lambda' : X \to X^*$ is a mapping of type (S_+).

Proposition 2.3 — Suppose each sequence $\{u_n\}$ in X such that

$\{\varphi_\lambda(u)_n\}$ is bounded and $\varphi_\lambda'(u_n) \to 0$

as $n \to \infty$ has a bounded subsequence. Then φ_λ satisfied the *PS*-condition.

PROOF : Assume that $\{u_n\} \subset X$, $\{\varphi_\lambda(u_n)\}$ is bounded and $\varphi_\lambda'(u_n) \to 0$. Now we know that $\{u_n\}$ contains a bounded subsequence, for simplicity, we still denote it by $\{u_n\}$. Also since X is reflexive we can extract a subsequence (we still denote it by $\{u_n\}$) such that $u_n \to u$ in X. Since $\varphi_\lambda'(u_n) \to 0$, one has

$$\langle \varphi_\lambda'(u_n), u_n - u \rangle \to 0 \text{ as } n \to \infty.$$

Since φ_λ' is a mapping of type (S_+), we have that $u_n \to u$ in X. Hence φ_λ satisfies the PS-condition.

Next we do some remarks about the Palais-Smale condition for φ_λ. We recall that φ_λ is said to satisfy the Palais-Smale condition at level c, $(\mathrm{PS})_c$, if every sequence $\{u_n\}$ for which

$$\varphi_\lambda(u_n) \to c, \quad \left|\left| \varphi_\lambda'(u_n) \right|\right|_{X^*} \to 0 \quad \text{as} \quad n \to \infty$$

possesses a convergent subsequence. When φ_λ satisfies $(\mathrm{PS})_c$ for all $c \in R$, we simply say that φ_λ satisfies the (PS) condition. In the proof of Theorem 1.2, we use a weaker version of the (PS) condition due to Cerami. We say that φ_λ satisfies the Palais-Smale-Cerami condition, (PSC), if every sequence $\{u_n\}$ for which

$$\varphi_\lambda(u_n) \text{ is bounded, } (1 + \| u_n \|_X) \| \varphi_\lambda'(u_n) \|_{X^*} \to 0 \quad \text{as} \quad n \to \infty$$

possesses a convergent subsequence.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 : We divide the proof in several steps.

Step 1 : From $(H1)$, we have for almost all $t \in I$ and $x \in R$, that

$$| F(t,x) | \leq a_1(t) + c_1(t) |x|^r \qquad \text{... (3.1)}$$

with $a_1 \in L^1$ and $c_1 \in L^{r'}(I)$, also

$$\int_0^T h(t) | u(t) | \, dt \leq \| h \|_q \| u \|_p \leq C_1 \| u \|_X. \qquad \text{... (3.2)}$$

So, for $\lambda < 0$, we have

$$\varphi_\lambda(u) = \frac{1}{p} \, || u' ||_p^p - \frac{\lambda}{p} || u ||_p^p - \int_0^T [F(t, u(t)) + h(t) \, u(t)] \, dt$$

$$\geq \frac{1}{p} \, || u' ||_p^p - \frac{\lambda}{p} \, || u ||_p^p - C_1 \| u \|_X - C_2 \| u \|_p^r - C_3$$

$$\geq C_4(\lambda) \, || u ||^p - C_1 \, || u ||_X - C_2 \| u \|_X^r - C_3$$

for some $C_4(\lambda), C_1, C_2, C_3 > 0$ (recall that $\lambda < 0$). Since $1 \leq r < p$ we deduce that φ_λ is coercive.

Consequently, φ_λ is bounded from below and satisfies the *PS* condition.

Now consider the direct sum decomposition

$$X = R \oplus V, \text{ with } V = \left\{ v \in X : \int_0^T v(t)\,dt = 0 \right\}.$$

So we can write that $u = \bar{u} + \tilde{u}$ $\bar{u} \in R$, and $\tilde{u} \in V$.

Using the Wirtinger inequality, for all $v \in V$, we have that

$$\varphi_\lambda(v) = \frac{1}{p} \| v' \|_p^p - \frac{\lambda}{p} \| v \|_p^p - \int_0^T [F(t, v(t) + h(t)\, v(t)]\, dt$$

$$\geq \frac{1}{p}(1 - \lambda T^{p)}) \| v' \|_p^p - C_5 \| v' \|_p^r - C_6 \| v' \|_p - C_7$$

for some $C_5, C_6, C_7 > 0$. Thus φ_λ is coercive on V uniformly for $\lambda < 0$. So we can find $m \in R$ such that

$$\varphi_\lambda(v) \geq m, \qquad \forall \lambda < 0, \quad \forall v \in V.$$

Step 2 : From $(H2)$, we have for $x \in R$,

$$\varphi_\lambda(x) = -\frac{\lambda |x|^p}{p} - \int_0^T F(t, x)\, dt.$$

Choosing $x_0^+ > 0$ sufficiently large, we get from $(H2)$ that

$$\int_0^T F(t, x_0^+)\, dt > -2m,$$

so that

$$\varphi_\lambda(x_0^+) < -\frac{\lambda |x_0^+|^p}{p} + 2m.$$

Then for λ sufficiently near to 0, there exists $\bar{\lambda} < 0$, such that for all $\lambda \in [\bar{\lambda}, 0)$, $\varphi_\lambda(x_0^+) < m$. Then same conclusion holds for a $x_0^- < 0$.

Step 3 : Put

$$U^+ \stackrel{\text{df}}{=} \left\{ x \in X : \int_0^T x(t)\, dt > 0 \right\},$$

$$U^{-} \overset{\text{df}}{=} \left\{ x \in X : \int_0^T x(t)\, dt < 0 \right\}.$$

Then from Step 2, for $\lambda \in [\bar{\lambda}, 0)$,

$$-\infty < \inf_{U^{\pm}} \varphi_\lambda < m.$$

Next let

$$c_\lambda^{\pm} \overset{\text{df}}{=} \inf_{U^{\pm}} \varphi_\lambda.$$

Note that by virtue of the coercivity of φ_λ, the numbers $c_\lambda^{\pm}$ are finite and because of the above inequality, for $\lambda \in [\bar{\lambda}, 0)$, one has $c_\lambda^{\pm} < m$. Moreover, since φ_λ satisfies PS condition, we can find $u_\lambda^{\pm} \in \overline{U}^{\pm}$ such that $\varphi_\lambda\left(u_\lambda^{\pm}\right) = c_\lambda^{\pm}$. If $u_\lambda^{\pm} \in \partial \overline{U}^{\pm} = V$, then $c_\lambda^{\pm} \geq m$, a contradiction since $\lambda \in [\bar{\lambda}, 0)$. Hence $u_\lambda^{\pm} \in U^{\pm}$ and so $\varphi_\lambda'\left(u_\lambda^{\pm}\right) = 0$.

Because $\varphi_\lambda|_V \geq m > \varphi_\lambda\left(x_0^{\pm}\right)$, we can use the saddle point theorem of Rabinowitz and obtain $v_\lambda \in X$ such that $\varphi_\lambda'(v_\lambda) = 0$ and $\varphi_\lambda(v_\lambda) \geq m > \varphi_\lambda\left(u_\lambda^{\pm}\right)$, with $\lambda \in [\bar{\lambda}, 0)$. Hence $v_\lambda \neq u_\lambda^{\pm}$.

PROOF OF THEOREM 1.2

Step 1 : Let $\left\{u_n\right\}_{n=1}^{\infty} \subseteq X$ be a sequence such that

$$\varphi_\lambda(u_n) \to c \quad \text{and} \quad (1 + \| u_n \|_X) \| \varphi_\lambda(u_n) \|_{X^*} \to 0 \quad \text{as} \quad n \to \infty. \qquad \text{... (3.3)}$$

Then, we have

$$|\langle \varphi_\lambda'(u_n), u_n \rangle - p\varphi_\lambda(u_n) + pc| \leq \| \varphi_\lambda'(u_n) \|_{X^*}) \| u_n \|_X + | p\varphi_\lambda(u_n) - pc|$$

$$\leq (1 + \| u_n \|_X) \| \varphi_\lambda'(u_n) \|_{X^*} + p\, | \varphi_\lambda(u_n) - c| \to 0$$

as $n \to \infty$. So

$$p\varphi_\lambda(u_n) - \langle \varphi_\lambda'(u_n), u_n \rangle = -p \int_0^T F(t, u_n)\, dt + \int_0^T f(t, u_n)\, u_n\, dt$$

$$= \int_0^T [f(t, u_n) u_n - pF(t, u_n)]\, dt \to pc, \qquad \text{... (3.4)}$$

as $n \to \infty$.

We now claim that $\{u_n\}_{n \geq 1} \subseteq X$ is bounded. Suppose that the claim is not true. By passing to subsequence if necessary, we may assume that $\| u_n \|_X \to \infty$. Set $v_n = \frac{u_n}{\| u_n \|_X}$, $n \geq 1$. We may assume that $v_n \to v$ in X and $v_v \to v$ in $C(I)$ (recall that X is embedded compactly in $C(I)$).

From (3.3), there exists $M_1 > 0$ such that for all $n \geq 1$,

$$\varphi_\lambda(u_n) \leq M_1.$$

So

$$\frac{\varphi_\lambda(u_n)}{\| u_n \|_X^p} \leq \frac{M_1}{\| u_n \|_X^p}$$

and

$$\frac{1}{p} \| v_n' \|_p^p - \frac{\lambda}{p} \| v_n \|_p^p - \int_0^T \frac{F(t, u_n(t))\, dt}{\| u_n \|_X^p} \leq \frac{M_1}{\| u_n \|_X^p}.$$

By (3.1), we have

$$\int_0^T \frac{F(t, u_n(t))\, dt}{\| u_n \|_X^p} \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently,

$$\| v' \|_p^p \leq \lambda \| v \|_p^p.$$

If $v = 0$, then $\| v' \|_p \to 0$, hence $v_n \to 0$ in X, a contradiction since $\| v_n \|_X = 1$, for all $n \geq 1$. Thus the set $I_0 = \{t \in I : v(t) \neq 0\}$ is nonempty. We have just seen that $| I_0 | > 0$ (here by $| \cdot |$ we denote the Lebesgue measure on R). Moreover, note that for all $t \in I_0$ we have $| u_n(t) | \to \infty$. We have

$$\int_0^T [f(t, u_n) u_n - pF(t, u_n)]\, dt = \int_{I_0} [f(t, u_n) u_n - p F(t, u_n)]\, dt + \int_{I_0^C} [f(t, u_n) u_n - pF(t, u_n)]\, dt.$$

By virtue of hypothesis ($H4$), we can find $M_2 > 0$ such that for almost all $t \in I$, all $|x| \geq M_2$, we have

$$f(t, x)\, x - pF(t, x) \leq -1.$$

On the other hand, we see that for all $t \in I$, all $|x| \leq M_2$, we have

$$f(t, x)\, x - pF(t, x) \leq \beta_1 \text{ for some } \beta_1 > 0.$$

Therefore it follows that for almost all $t \in I$, all $x \in R$, we have

$$f(t, x)\, x - pF(t, x) \leq \beta_1. \qquad \text{... (3.5)}$$

Using Fatou's Lemma and hypothesis ($H4$), we obtain

$$\lim_{n \to \infty} \int_{I_0} [f(t, u_n) u_n - pF(t, u_n)]\; dt = -\infty.$$

Also from (3.5), we have

$$\int_{I_0^C} [f(t, u_n) u_n - pF(t, u_n)]\, dt \leq \beta_1 T.$$

Therefore, we deduce that

$$\lim_{n \to \infty} \int_0^T [f(t, u_n) u_n - pF(t, u_n)]\, dt = -\infty. \qquad \text{... (3.6)}$$

Comparing, (3.4) and (3.6), we reach a contradiction. This contradiction implies that $\{u_n\}_{n \geq 1} \subseteq X$ is bounded. By $(1 + \| u_n \|_X) \| \varphi_\lambda'(u_n) \|_{X^*} \to 0$ as $n \to \infty$ we have $\| \varphi_\lambda'(u_n) \|_{X^*} \to 0$ as $n \to \infty$. Also since X is reflexive, we can extract a subsequence (we still denote it by $\{u_n\}$) such that $u_n \to u$ in X. Since $\varphi_\lambda'(u_n) \to 0$, one has

$$\langle \varphi_\lambda'(u_n), u_n - u \rangle \to 0 \text{ as } n \to \infty.$$

Since φ_λ is a mapping of type (S_+), we have that $u_n \to u$ in X. Hence φ_λ satisfies the PSC-condition.

Step 2 : $\varphi_\lambda(x) \to -\infty$ as $|x| \to \infty$, $x \in R$. We have

$$\frac{d}{dx}\left(\frac{F(t,x)}{|x|^p}\right) = \frac{|x|^p f(t,x) - p|x|^{p-2} x F(t,x)}{|x|^{2p}}$$

$$= \frac{xf(t,x) - pF(t,x)}{|x|^{p+1}}.$$

By virtue of hypothesis $(H4)$, we see that for $\beta_2 > 0$ we can find $M_3 > 0$ such that for almost all $t \in I$, all $x > M_3$, we have

$$xf(t,x) - pF(t,x) \le -\beta_2.$$

So,

$$\frac{d}{dx}\left(\frac{F(t,x)}{x^p}\right) \le \frac{\beta_2}{x^{p+1}} \quad \text{for} \quad x > M_3.$$

Integrating from $x > 0$ to $+\infty$, and noitng that $\dfrac{F(t,x)}{x^p} \to 0$ as $x \to \infty$, we get

$$F(t,x) \ge \frac{\beta_2}{p} \text{ for almost all } t \in I, x > M_3.$$

So if $x \in R, x \ge M_3$, we have

$$\varphi_\lambda(x) = \frac{\lambda}{p}|x|^p - \int_0^T F(t,x)\,dt$$

$$\le -\frac{\lambda}{p}|x|^p - \frac{\beta_2 T}{p}.$$

Because $\beta_2 > 0$ are arbitrary and $\lambda \ge 0, p > 1$, we conclude that $\varphi_\lambda(x) \to -\infty$ as $x \to +\infty$. Similarly we show that $\varphi_\lambda(x) \to -\infty$ as $x \to -\infty$.

Let

$$D = \left\{ v \in X : \int_0^T |v(t)|^{p-2}\, v(t)\,dt = 0 \right\}.$$

We prove that D is nonempty. Now if λ is a eigenvalue and φ is an associated eigenfunction of (1.2), then integration the differential equation in (1.2) over $[0, T]$ gives

$$\int_0^T |\varphi(t)|^{p-2}\varphi(t)\,dt = 0.$$

So, D is nonempty. Moreover, D is a closed cone.

Step 3 : $\varphi(v) \to +\infty$ as $\|v\| \to \infty$ with $v \in D$.

As in the proof of Theorem 1.1, one has

$$|F(t,x)| \le a_1(t) + c_1(t)|x|^r,$$

with $a_1 \in L^1(I)$ and $c_1 \in L^{r'}(I)$. If $v \in D$., we have

$$\varphi_\lambda(v) = \frac{1}{p}\|v'\|_p^p - \frac{\lambda}{p}\|v\|_p^p - \int_0^T F(t,v)\,dt$$

$$\ge \frac{1}{p}\|v'\|_p^p - \frac{\lambda}{p}\|v\|_p^p - C_1\|v\|_p^r - C.$$

From Mawhin[3] [Corollary 9.3], we know that $\lambda_1\|v\|_p^p \le \|v'\|_p^p$ for all $v \in D$. Therefore, we have

$$.\varphi_\lambda(v) \ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|v'\|_p^p - C_1\|v\|_p^r - C.$$

By $0 \le \lambda < \lambda_1$, we have that $1 - \dfrac{\lambda}{\lambda_1} > 0$. It follows that $\varphi_{\lambda|V}$ is coercive.

Because of Step 2 and Step 3, we can find $x_0 \in R, x_0 > 0$ such that $\varphi_\lambda(\pm x_0) < \inf_D \varphi_\lambda$. Let $E = [-x_0, x_0]$ and $E_1 = \{-x_0, x_0\}$. Evidently $E_1 \cap D = \phi$. Let $\theta \in C(E, X)$ and $\theta(\pm x_0) = \pm x_0$. We prove that each θ intersects D.

Let $\pi : X \to R$ be a continuous map, defined by

$$\pi(u) = \frac{1}{|x_0|^{p-2}T}\int_0^T |u|^{p-2}u\,dt.$$

Then $u \in D$ if and only if $\pi(u) = 0$. Next, we have to show that $0 \in \pi(\theta(E))$.

For $t \in [0, 1]$, $x \in E$ define

$$\theta_t(x) = t\,\pi(\theta(x)) + (1-t)\,x.$$

Note that $\theta_t \in C^0(R;R)$ defines a homotopy of $\theta_0 = id$ with $\gamma_1 = \pi \circ \theta$. Moreover,

$\theta_{t\,|\,\delta E} = id$ for all t. By homotopy invariance and normalization of the degree (see for instance Deimling[10] [Theorem 1.3.1]), we have

$$deg\,(\pi \circ \theta, B_E, 0) = deg\,(id, B_E, 0) = 1.$$

Hence $0 \in \pi(\theta(B_E))$. So each θ intersects the hyperplane V.

Then a simple application of the Intermediate Value Theorem, implies that $\theta(E) \cap D \neq \phi$. Therefore, E_1 and D link in X. We can apply Linking Theorem[9] [Theorem 8.4, pp. 127] and obtain $u \in X$ such that $\varphi_\lambda'(u) = 0$.

Remark 3.1 : We note that our results do not assume neither that f is bounded, nor that f satisfies a sign condition. Theorem 1.2 is related to a class of resonance problems.

Remark 3.2 : The ideas in this section extend to the problem

$$\begin{cases} -\left(|u'|^{p-2}u'(t)\right)' = \lambda_g(t)\,|u|^{p-2}u + f(t,u) + h(t) \quad \text{a.e } t \in [0,T] \\ u(0) = u(T),\ u'(0) = u'(T),\ 1 < p < \infty. \end{cases}$$

Minor adjustments are only needed, so we leave the details to the reader.

Example 3.1 — Let us consider the scalar problem

$$\begin{cases} -(|u'|\,u') = \lambda\,|u|\,u + a\,[\sin(u - bsgnu) + \sin(bsgnu)] + u + e \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(1) = 0 \end{cases} \qquad \dots (3.6)$$

where $p = 3$, $a > 0$, $0 < b < \pi$, $e \in L^1(0,T)$ and $\int_0^T e(t)\,dt = 0$. In this case

$$F(t,u) = a\,[(\sin b)\,|u| - \cos(|u| - b) + \cos b] + \frac{1}{2}u^2 + e(t)\,u.$$

and hence

$$\int_0^T F(t,x)\,dt = Ta \sin b\,|x| - Ta\,|\cos(|x| - b) - \cos b\,| + \frac{T}{2}x^2 \to +\infty$$

if $|x| \to \infty$. Moreover,

$$|f(t,u)| \leq |u| + 2a.$$

Then there exists $\bar{\lambda} > 0$ such that for all $\lambda \in [-\bar{\lambda}, 0)$ problem (3.6) has at least three distinct solutions.

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