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NONLINEAR FIRST-ORDER IMPLICIT IMPULSIVE DIFFERENTIAL EQUATIONS IN BANACH SPACES†

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In this paper, by using Banach's fixed point theorem and the concept of directional Lipschitian condition due to Chen[5], we prove some new existence and uniqueness theorems of solution for a class of nonlinear first-order initial value implicit impulsive differential equations in Banach spaces.

Key Words : Nonlinear Implicit Impulsive Differential Equation; Initial Value Problem; Banach Fixed Point Theorem; Existence and Uniqueness

1. INTRODUCTION

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years (see, for example, [1-4, 6-11] and the references therein). In 2000, Heikkilä, Kumpulainen and Seikkala[9] introduced and studied a class of implicit impulsive differential equations and proved some uniqueness and existence results for the implicit impulsive differential equations under some suitable conditions.

In this paper, we consider the following nonlinear first-order initial value implicit impulsive differential equation:

$$\begin{cases} x'(t) = f(t, x(t), x'(t)), \ t \neq t_k, \\ \Delta x\,|_{t=t_k} = I_k(x(t_k)), \ (k = 1, 2, ..., m), \\ x(t_0) = x_0, \end{cases} \quad \ldots (1.1)$$

where E is a Banach space, $R = (-\infty, +\infty)$, $t_0 \in R, x_0, u_0 \in E, a, b, c \in R^+ = (0, +\infty)$, $J =$

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$[t_0, +\infty)$, $t_0 < t_1 < \ldots < t_m < t_0 + a < \infty$, $D = J \times B_b(x_0) \times B_c(u_0)$, $f: D \to E$, for each $k = 1, 2, \ldots, m$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e. $\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right)$, $I_k \in C[E, E]$, $x\left(t_k^+\right)$ and $\left(t_k^-\right)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, and $B_r(\omega)$ denotes the ball which center at ω and radius is r.

By using Banach's fixed point theorem and the concept of directional Lipschitian condition due to Chen[5], we prove some new existence and uniqueness theorems of solution for nonlinear first-order initial value implicit impulsive differential eq. (1.1) in Banach spaces.

2. PRELIMINARIES

Throughout this paper, let E be a real Banach space with the norm $\|\cdot\|$. Let $J = [t_0, +\infty)$, $t_0 < t_1 < \ldots < t_m < t_0 + a < +\infty$, $J_0 = [t_0, t_1], J_1 = (t_1, t_2], \ldots, J_k = (t_k, t_{k+1}], \ldots, J_m = (t_m, t_0 + a]$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$ and

$$PC^1(J, E) = \left\{ \begin{array}{c} x : x \text{ is a map from } J \text{ into } E \text{ such that } x(\mathrm{t}) \text{ is continuously} \\ \text{differentiable at } t \neq t_k, \text{ left continuous at } t = t_k, \text{ and} \\ x\left(t_k^+\right), x'\left(t_k^-\right), x'\left(t_k^+\right) \text{exist}, k = 1, 2, \ldots, m \end{array} \right\}, \quad \ldots (2.1)$$

where $x'\left(t_k^-\right)$ and $x'\left(t_k^+\right)$ represent the left and right derivatives of $x(t)$ at $t = t_k$ for all $k = 1, 2, \ldots, m$, respectively. For $x \in PC^1(J, E)$, by virtue of the mean value theorem

$$x(t_k) - x(t_k - h) \in h\overline{co}\left\{x'(t) : t_k - h < t < t_k\right\} \quad (h > 0),$$

it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h \to 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'_-(t_k^-).$$

In the following, $x'(t_k)$ is understood as $x'_-(t_k)$. Evidently, $PC^1(J, E)$ is a Banach space with norm $\|x\|_{PC} = \sup_{t \in J} \{\|x(t)\| + \|x'(t)\|\}$. A map $x \in PC^1(J, E)$ is called a solution of problem (1.1) if it satisfies (1.1).

Let $P_M = \{(t, x) \in R \times E : \|x\| \leq Mt\}$, where M is a positive constant. Then P_M is a cone on $R \times E$. We denote by $\prec$ the order induced by the cone P_M on $R \times E$. If $(t' - t, x' - x) \in P_M$, then we have $(t, x) \prec (t', x')$.

Now we recall the conception of directional continuous and directional Lipschitzian conditions described by Chen[5]. We then give a lemma.

Definition 2.1[5] — Let $(t_0, x_0) \in R \times E$ be a fixed point. A mapping $f: R \times E \to E$ is said to be (1) directional continuous with respect to P_M at (t_0, x_0), if

$$\lim_{(t,x) \to (t_0, x_0)} f(t, x) = f(t_0, x_0), \quad (t, x) \in R \times E, \quad (t_0, x_0) \prec (t, x).$$

(2) satisfying the local directional L-Lipschitzian condition with respect to P_M at (t_0, x_0), if there exists $\delta > 0$ such that

$$t_i \in [t_0, t_0 + \delta), \quad x_i \in E, \quad (t_0, x_0) \prec (t_i, x_i), i = 1, 2, \quad (t_1, x_1) \prec (t_2, x_2)$$

imply

$$\| f(t_2, x_2) - f(t_1, x_1) \| \leq L \| x_2 - x_1 \|,$$

where $L > 0$ is a constant and $\| x \| = \sup_{t \in J} \| x(t) \|$.

It is easy to see that $f: R \times E \to E$ is directional continuous (resp. local Lipschitz continuous) at (t_0, x_0) if $f(t, x)$ satisfies continuous condition (resp. local Lipschitzian condition) and the following example shows that the inverse is not true.

Example 2.1[5] — Consider the mapping $f: R \times E \to E$ defined as follows:

$$f(t, x) = \begin{cases} x, & \text{for } t \geq 0, \ \| x \| \leq 2t \\ x_0 + tx, & \text{otherwise,} \end{cases}$$

where $x_0 \in E$ and $x_0 \neq 0$. Let $M = 2$. It is easy to check that $f(t, x)$ is directional continuous with respect to P_M at (0, 0). However, $f(t, x)$ is discontinuous at (0, 0).

Lemma 2.1 — Let $f: R \times E \times E \to E$ be a mapping such that $u \mapsto f(t_0, x_0, u)$ is Lipschitz continuous with a constant $\beta \in (0, 1)$, where $t_0 \in R, x_0 \in E$ are two given points. Then there exists a unique point $u_0 \in E$ such that $u_0 = f(t_0, x_0, u_0)$.

PROOF : For all $u \in E$, define $Q(u) = f(t_0, x_0, u)$. Since $u \mapsto f(t_0, x_0, u)$ is Lipschitz continuous with constant $\beta \in (0, 1)$, we have

$$\| Q(u_2) - Q(u_1) \| = \| f(t_0, x_0, u_2) - f(t_0, x_0, u_1 \| \leq \beta \| u_2 - u_1 \|, \quad \forall u_1, u_2 \in E.$$

Therefore, $Q : E \to E$ is a contractive mapping and so there exists a unique point $u_0 \in E$ such that $u_0 = f(t_0, x_0, u_0)$. This completes the proof.

3. Main Results

In order to prove our main results, we need the following result.

Lemma 3.1 — Let $x \in PC^1(J, E)$, $s \mapsto f(s, x(s), x'(s))$ be continuous, and

$$Ax(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x'(s))\, ds + \sum_{t_0 < t_k < t} I_k(x(t_k)). \qquad \text{... (3.1)}$$

Then

(1) $(Ax(t))' = f(t, x(t), x'(t))$;

(2) $Ax \in PC^1(J, E)$;

(3) x is a solution of problem (1.1) in $PC^1(J, E)$ if and only if x is a fixed point of A.

Proof : It is easy to see that (1) and (2) hold and so we only prove (3). From Lemma 1 of Guo[7], we have

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\, ds + \sum_{t_0 < t_k < t} [x\left(t_k^+\right) - x(t_k)], \quad \forall t \in J. \qquad \text{... (3.2)}$$

If $x \in PC^1(J, E)$ is a solution of problem (1.1), then by substituting (1.1) into (3.2) we obtain

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x'(s))\, ds + \sum_{t_0 < t_k < t} I_k(x(t_k)). \qquad \text{... (3.3)}$$

Hence, it follows from (3.1) and (3.3) that x is a fixed point of A.

Conversely, assume that $x \in PC^1(J, E)$ is a fixed point of A, i.e., $x \in PC^1(J, E)$ is a solution of eq. (3.3). It is clear that

$$x(t_0) = x_0, \ \ \Delta x|_{t = t_k} = I_k(x(t_k)), \ (k = 1, 2, \ldots, m).$$

By performing direct differentiation of (3.3), we find

$$x'(t) = f(t, (t), x'(t)), \quad t \neq t_k.$$

This completes the proof.

We are now in a position to prove our main results concerning the solutions of problem (1.1) in Banach spaces.

Theorem 3.1 — *Let u_0 be given by Lemma 2.1 and $f: D \to E$ be a mapping such that $s \mapsto f(s, x(s), x'(s))$ is continuous. Suppose that*

(*i*) for each $u \in B_c(u_0)$, $(t, x) \mapsto f(t, x, u)$ safisfies the local directional L_1-Lipschitzian condition with respect to P_M for x at (t_0, x_0);

(*ii*) $u \mapsto f(t_0, x_0, u)$ is Lipschitz continuous with constant $0 < L_2 < 1$;

(*iii*) for every $k = 1, 2, ..., m$, there exist nonnegative numbers b_k such that, for any $t \in J$ and any $x, y \in E$,

$$\| I_k(x) - I_k(y) \| \le b_k \| x - y \| ;$$

(*iv*) $\alpha = \max \{ b_k \mid 1 \le k \le m \} < 1$;

(*v*) $L = \max\{L_1, L_2\} < \min \left\{ \frac{1}{2}, M, \frac{M - \| f(t_0, x_0, u_0) \|}{c} \right\}$, $L + m\alpha < 1$.

Then there exists a $\delta_1 \in (0, a]$ such that the following equation

$$\begin{cases} x'(t) = f(t, x(t), x'(t)), \ t \neq t_k, \quad t \in [t_0, t_0 + \delta_0], \\ \Delta x \mid_{t = t_k} = I_k(x(t_k)), \ (k = 1, 2, ..., m), \\ x(t_0) = x_0, \end{cases}$$

has a unique solution.

PROOF : Let $\alpha > 0$,

$$\delta_0 \in \left(0, \min \left\{ a, \frac{M - \| f(t_0, x_0, u_0) \| - Lc}{LM}, \frac{1 - L - m\,\alpha}{L}, \frac{b}{M}, \frac{c}{M} \right\} \right),$$

$$K = \max\{ L, \| f(t_0, x_0, u_0) \| + LM\,\delta_0 + Lc \}$$

and

$$C^K[t_0, t_0+\delta_0] = \left\{ x : [t_0, t_0+\delta_0] \to E \;\middle|\; \begin{array}{l} x(t_0) = x_0, \quad x'(t_0) = u_0, \\ \| x(t_2) - x(t_1) \| \le K \mid t_2 - t_1 \mid, \\ \| x'(t_2) - x'(t_1) \| \le K \mid t_2 - t_1 \mid, \\ t_1, t_2 \in [t_0, t_0+\delta_0] \cap J' \end{array} \right\}$$

Then it is easy to know that $C^K[t_0, t_0+\delta_0]$ is a complete metric space with

$$d(x, y) = \max_{t \in [t_0, t_0+\delta_0]} \{ \| x(t) - y(t) \| + \| x'(t) - y'(t) \| \}.$$

Further, we can check that $C^K[t_0, t_0+\delta_0]$ is a nonempty closed convex subset of $PC^1[t_0, t_0+\delta_0], E)$, where $PC^1[t_0, t_0+\delta_0], E)$ is the same definition as (2.1) and it is a real Banach space with

$$\| x \| = \max_{t \in [t_0, t_0+\delta_0]} \{ \| x(t) \| + \| x'(t) - y'(t) \| \}.$$

Let

$$\begin{aligned} Ax(t) &= x_0 + \int_{t_0}^{t} f(s, x(s), x'(s))\, ds + \sum_{t_0 < t_k < t} I_k(x(t_k)) \\ &= Fx(t) + Tx(t), \end{aligned} \qquad \ldots (3.4)$$

where

$$Fx(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x'(s))\, ds,$$

$$Tx(t) = \sum_{t_0 < t_k < t} I_k(x(t_k)).$$

We first prove that $F : C^K[t_0, t_0+\delta_0] \to C^K[t_0, t_0+\delta_0]$. In fact, since

$$\delta_0 < \frac{M - \| f(t_0, x_0, u_0) \| - Lc}{LM},$$

we know that $\| f(t_0, x_0, u_0) \| + LM\, \delta_0 + Lc < M$. It follows from $L < M$ that $K < M$. For each given

$x \in C^K[t_0, t_0, \delta_0]$, $t_1, t_2 \in [t_0, t_0, \delta_0] \cap J'$, without loss of generality, suppose that $t_1 < t_2$. Taking $t_1 = t_0$ and $t_2 = t$, we have

$$\| x(t) - x_0 \| = \| x(t) - x(t_0) \| \leq K | t - t_0 | \leq M (t - t_0),$$

i.e., $(t_0, x_0) \prec (t, x(t))$ for all $t \in [t_0, t_0 + \delta_0]$, and

$$\| x(t) - x_0 \| \leq K | t - t_0 | \leq M (t - t_0) \leq b,$$

$$\| x'(t) - x_0 \| \leq K | t - t_0 | \leq M (t - t_0) \leq c.$$

Therefore, for all $t \in [t_0, t_0 + \delta_0]$, we have $x(t) \in B_b(x_0)$, $x'(t) \in B_c(u_0)$, and

$$\begin{aligned} &\| f(t, x(t), x'(t) \| \\ &\leq \| f(t_0, x_0, u_0) \| + \| f(t_0, x_0, u_0) - f(t, x(t), x'(t)) \| \\ &\leq \| f(t_0, x_0, u_0) \| + \| f(t_0, x_0, u_0) - f(t, x(t), u_0 \| + \| f(t, x(t), u_0) - f(t, x(t), x'(t) \| \\ &\leq \| f(t_0, x_0, u_0) \| + L_1 \| x(t) - x_0 \| + L_2 \| x'(t) - u_0 \| \\ &\leq \| f(t_0, x_0, u_0) \| + LM\, \delta_0 + Lc. \end{aligned} \quad \text{... (3.5)}$$

It follows from (3.5) and $L < \frac{1}{2}$ that

$$\begin{aligned} \| Fx(t_2) - Fx(t_1) \| &= \left\| \int_{t_0}^{t_2} f(s, x(s), x'(s))\, ds \right\| \leq \int_{t_1}^{t_2} \| f(s, x(s), x'(s) \|\, ds \\ &\leq (\| f(t_0, x_0, u_0) \| + LM\, \delta_0 + Lc) | t_2 - t_1 | \\ &\leq K | t_2 - t_1 | \end{aligned}$$

and

$$\begin{aligned} \| (Fx)'(t_2) - (FX)'(t_1) &\leq \| f(t_2, x(t_2), x'(t_2)) - f(t_1, x(t_1), x'(t_1)) \| \\ &\leq \| f(t_2, x(t_2), x'(t_2)) - f(t_1, x(t_1), x'(t_2)) \| \\ &\quad + \| f(t_1, x(t_1), x'(t_2)) - f(t_1, x(t_1), x'(t_1)) \| \\ &= L_1 \| x(t_2) - x(t_1) \| + L_2 \| x'(t_2) - x'(t_1) \| \\ &\leq 2LK | t_2 - t_1 | \end{aligned}$$

$$\leq K|t_2 - t_1|.$$

Hence $Fx \in C^k[t_0, t_0 + \delta_0]$ for all $x \in C^k[t_0, t_0 + \delta_0]$. Let $x, y \in C^k[t_0, t_0 + \delta_0]$ and

$$z(t) = \frac{x(t-\xi) + y(t-\xi)}{2}, \quad t \in [t_0 + \xi, t_0 + \delta_0 + \xi] \cap J',$$

where

$$\xi = \frac{\sigma}{2M}, \qquad \sigma = \max_t \{\|x(t) - y(t)\|\}.$$

Further, since $K < M$ and

$$\begin{aligned} \|x(t) - y(t) &\leq \|x(t) - x(t_0)\| + \|x(t_0) - y(t)\| \\ &= \|x(t) - x(t_0)\| + \|y(t) - y(t_0)\| \\ &\leq 2K|t - t_0| \leq 2K\delta_0, \end{aligned}$$

we have $\sigma \leq 2K\delta_0$ and

$$0 \leq \xi = \frac{\sigma}{2M} \leq \frac{K}{M}\delta_0 < \delta_0.$$

It is clear that, for any given $s \in [t_0, t]$, we have

$$(t_0, x_0) \prec (s, x(s)), \ (t_0, x_0) \prec (s, y(s)), \ z(s+\xi) = \frac{x(s) + y(s)}{2}$$

and

$$\begin{aligned} \|z(s+\xi) - x(s)\| &= \frac{1}{2}\|x(s) - y(s)\| \\ &\leq \frac{1}{2} \max_{t \in [t_0, t_0 + \delta_0]} \{\|x(s) - y(s)\|\} \\ &= \frac{1}{2}\sigma = M \cdot \frac{\sigma}{2M} = M\xi \\ &= M|(s+\xi) - s|, \end{aligned}$$

i.e.,

$$(s, x(s)) \prec (s+\xi, z(s+\xi)).$$

Similarly, we have

$$(s, y(s)) \prec (s+\xi, z(s+\xi)).$$

Hence

$$\| Fx(t) - Fy(t) \| \leq \int_{t_0}^{t} \| f(s, x(s), x'(s)) - f(s, y(s), y'(s)) \| \, ds$$

$$\leq \int_{t_0}^{t} \| f(s, x(s), x'(s)) - f(s+\xi, z(s+\xi), x'(s)) \| \, ds$$

$$+ \int_{t_0}^{t} \| f(s+\xi, z(s+\xi), x'(s)) - f(s+\xi, z(s+\xi), y'(s)) \| \, ds$$

$$+ \int_{t_0}^{t} \| f(s+\xi, z(s+\xi), y'(s)) - f(s\, y(s), y'(s)) \| \, ds$$

$$\leq \int_{t_0}^{t} \{ L_1 \| z(s+\xi) - x(s) \| + L_2 \| x'(s) - y'(s) \|$$

$$L_1 \| z(s+\xi) - y(s) \| \} \, ds$$

$$= \int_{t_0}^{t} (L_1 \| x(s) - y(s) \| + L_2 \| x'(s) - y'(s) \|) \, ds$$

$$\leq L \, \delta_0 \, d(x, y). \qquad \text{... (3.6)}$$

On the other hand, from above proof we have

$$\| (Fx)'(t) - (Fy)'(t) \| = \| f(t, x(t), x'(t) - f(t, y(t), y'(t)) \|$$

$$\leq L_1 \| x(t) - y(t) \| + L_2 \| x'(t) - y'(t) \|$$

$$\leq Ld(x, y). \qquad \text{... (3.7)}$$

Next, it follows from the definition of $d(x, y)$ that

$$\| x(t_k) - y(t_k) \| \leq d(x, y), \qquad \forall \, 1 \leq k \leq m,$$

and

$$\| Tx(t) - Ty(t) \| \leq d(x, y) \sum_{t_0 < t_k < t} b_k.$$

Hence,

$$\sup_{t \in J} \| Tx(t) - Ty(t) \| \leq d(x, y) \sup_{t \in J} \sum_{t_0 < t_k < t}$$

$$b_k = d(x, y) \max_{1 \leq k \leq m} \sup_{t \in J_k} \sum_{t_0 < t_j < t} b_j$$

$$= d(x, y) \max_{1 \leq k \leq m} \{C_k\}, \quad \text{... (3.8)}$$

where for each $1 \leq k \leq m$,

$$C_k = \sup_{t \in J} \sum_{t_0 < t_j < t} b_j = \sum_{j=1}^{k-1} b_j + b_k$$

$$\leq \sum_{j=1}^{k-1} b_j + b_k$$

$$\leq \alpha(m-1) + \alpha = m\alpha. \quad \text{... (3.9)}$$

From (3.8) and (3.9), we have

$$\sup_{t \in J} \| Tx(t) - Ty(t) \| \leq m\, \alpha\, d(x, y),$$

i.e.,

$$\| Tx(t) - Ty(t) \| \leq m\, \alpha\, d(x, y). \quad \text{... (3.10)}$$

By $(Tx)'(t) = 0$ for all $x \in C^k[t_0, t_0 + \delta_0]$, we now have

$$\| (Tx)'(t) - (Ty)'(t) \| = 0. \quad \text{... (3.11)}$$

It follows from (3.4), (3.6), (3.7), (3.10) and (3.11) that

$$\| Ax(t) - Ay(t) \| = \| (Fx(t) + Tx(t)) - (Fy(t) + Ty(t) \|$$

$$\leq \| Fx(t) - Fy(t) \| + \| Tx(t) - Ty(t) \|$$

$$\leq (L\, \delta_0 + m\alpha)\, d(x, y) \quad \text{... (3.12)}$$

and

$$\| (Ax)'(t) - Ay'(t) \| = \| (Fx)'(t) - (Fy)'(t) \| + \| (Tx)'(t) - (Ty)'(t) \|$$

$$\leq Ld(x, y). \quad \text{... (3.13)}$$

It follow from (3.12) and (3.13) that

$$d(Ax, Ay) \leq [L(1 + \delta_0) + m\alpha]\, d(x, y).$$

By the definition of δ_0, we know that $L(1+\delta_0)+m\alpha<1$. Therefore, A is a contraction mapping, which shows that A has a unique fixed point in $C^k[t_0, t_0+\delta_0]$.

Now the conclusion directly follows from Lemma 3.1. This completes the proof.

Remark 3.1 : Using the same method as in Theorem 3.1, we can consider the following n-order initial value implicit impulsive differential equation in Banach space:

$$\begin{cases} x^n(t) = f(t, x(t), x'(t), ..., x^n(t)), \ t \neq t_k, \\ \Delta x \mid_{t=t_k} = I_k(x(t_k), x'(t_k), ..., x^{(n-1)}(t_k)), \\ \Delta x \mid_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k), ..., x^{(n-1)}(t_k)), \\ \ldots \\ \Delta x^{n-1} \mid_{t=t_k} = I_k^*(x(t_k), x'(t_k), ..., x^{(n-1)}(t_k)), \\ x(t_0) = x_0, x'(t_0) = x_0', ..., x^{(n-1)}(t_0) = x_0^{(n-1)} \end{cases}$$

for all $k = 1, 2, ..., m$.

From Theorem 3.1, we have the following results.

Theorem 3.2 — *Let $f: J \times B_b(x_0) \to E$ be a mapping such that $s \mapsto fs, x(s))$ is continuous. Suppose that*

(*i*) $f(t, x)$ satisfies the local directional L-Lipschitzian condition with respect to P_M at (t_0, x_0);

(*ii*) for every $k = 1, 2, ..., m$, there exist non-negative numbers b_k such that, for any $t \in J$ and any $x, y \in E$,

$$\| I_k(x) - I_k(y) \| \leq b_k \| x - y \| ;$$

(*iii*) $\alpha = \max\{b_k \mid 1 \leq k \leq m\} < 1$;

(*iv*) $\|f(t_0, x_0)\| < M, \ L < M, \ L + m\alpha < 1$.

Then there exists a $\delta_0(0, a]$ such that the following equation

$$\begin{cases} x'(t) = f(t, x(t)), \ t \neq t_k, \ t \in [t_0, t_0+\delta_0], \\ \Delta x \mid_{t=t_k} = I_k(x(t_k)), \ (k = 1, 2, ..., m), \\ x(t_0) = x_0, \end{cases}$$

has a unique solution.

PROOF : Let

$$PC(J,E) = \left\{ \begin{array}{l} x : J \to E, x(t) \text{ is continuous at } t \neq t_k, \text{ left continuous} \\ \\ \text{at } t = t_k, \text{ and } x\left(t_k^+\right) \text{exist}, k = 1, 2, \ldots, m \end{array} \right\}$$

and

$$\delta_0 \in \left(0, \min\left\{ a\frac{M - \| f(t_0, x_0, u_0) \|}{LM}, \frac{1 - L - m\alpha}{L}, \frac{b}{M} \right\}\right),$$

$$K = \max \; \{L, \| f(t_0, x_0, u_0) \| + LM\,\delta_0\},$$

where $\alpha > 0$. Let

$$C^K[t_0, t_0 + \delta_0] = \left\{ x : [t_0, t_0 + \delta_0] \to E \;\middle|\; \begin{array}{c} x(t_0) = x_0, \\ \| x(t_2) - x(t_1) \| \leq K \,|\, t_2 - t_1 \,|, \\ t_1, t_2 \in [t_0, t_0 + \delta_0] \cap J' \end{array} \right\}.$$

Then it is easy to know that $C^K[t_0, t_0 + \delta_0]$ is a complete metric space with

$$d(x, y) = \max_{t \in [t_0, t_0 + \delta_0]} \left\{ \| x(t) - y(t) \| \right\}.$$

Further, we can check that $C^K[t_0, t_0 + \delta_0]$ is a nonempty closed convex subset of $PC[t_0, t_0 + \delta_0], E) \cap C^1(J', E)$, and $PC[t_0, t_0 + \delta_0], E) \cap C^1(J', E)$ is a real Banach space with

$$\| x \| = \sup_{t \in [t_0, t_0 + \delta_0]} \left\{ \| x(t) \| \right\}.$$

Let

$$\begin{aligned} Ax(t) &= x_0 + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{t_0 < t_k < t} I_k(x(t_k)) \\ &= Fx(t) + Tx(t), \end{aligned}$$

where

$$Fx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

$$Tx(t) = \sum_{t_0 < t_k < t} I_k(x(t_k)).$$

The rest proof is similar to the proof of Theorem 3.1.

Theorem 3.3 — *Let u_0 be given by Lemma 2.1 and $f: D \to E$ be a mapping such that $s \mapsto f(s, x(s), x'(s))$ is continuous. Suppose that*

(i) for each $u \in B_c(u_0)$, $(t, x) \mapsto f(t, x, u)$ safisfies the local directional L_1-Lipschitzian condition with respect to P_M for x at (t_0, x_0);

(ii) $u \mapsto f(t_0, x_0, u)$ is Lipschitz continuous with constant $L_2 \in (0, 1)$;

(iii) $L = \max\{L_1, L_2\} < \min\left\{\frac{1}{2}, M, \frac{M - \| f(t_0, x_0, u_0) \|}{c}\right\}$.

Then there exists a $\delta_0 \in (0, a]$ such that the following equation

$$\begin{cases} x'(t) = f(t, x(t), x'(t)), & t \in [t_0, t_0 + \delta_0], \\ x(t_0) = x_0, & \end{cases}$$

has a unique solution.

PROOF : Let

$$\delta_0 \in \left(0, \min\left\{a, \frac{M - \| f(t_0, x_0, u_0) \| - Lc}{LM}, \frac{1-L}{L}, \frac{b}{M}, \frac{c}{M}\right\}\right),$$

$$K = \max\{L, \| f(t_0, x_0, u_0) \| + LM\,\delta_0 + Lc\},$$

and

$$C^K[t_0, t_0 + \delta_0] = \left\{ x : [t_0, t_0 + \delta_0] \to E \;\middle|\; \begin{array}{l} x(t_0) = x_0, \quad x'(t_0) = u_0, \\ \| x(t_2) - x(t_1) \| \le K | t_2 - t_1 |, \\ \| x'(t_2) - x'(t_1) \| \le K | t_2 - t_1 |, \\ \forall\, t_1, t_2 \in [t_0, t_0 + \delta_0]. \end{array} \right\}.$$

Set

$$Tx(t) \equiv 0, \quad \forall x \in C^K[t_0, t_0 + \delta_0], t \in [t_0, t_0 + \delta_0].$$

The rest proof is the same as in Theorem 3.1.

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