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CONNECTED DOMSATURATION NUMBER OF A GRAPH

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The connected domsaturation number ds_c of a graph $G = (V, E)$ is the least positive integer k such that every vertex of G lies in a connected dominating set of cardinality k. In this paper we initiate a study of this parameter.

Key Words: Domination; Domination Number; Connected Domination Number; Domsaturation Number; Connected Domsaturation Number

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. Terms not defined here are used in the sense of Harary[3].

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex $V \setminus S$ is adjacent to at least one vertex in S. The domination number γ in G is the minimum cardinality of a dominating set in G. Fundamentals of domination and several advanced topics are given in Haynes *et al.*[4,5].

Sampathkumar and Walikar[8] introduced the concept of connected domination in graphs. A dominating set S of a connected graph G is called a connected dominating set of G if the induced subgraph $\langle S \rangle$ is connected. The connected domination number γ_c of G is the minimum cardinality of a connected dominating set of G.

Acharya[1] introduced the concept of domsaturation number ds of a graph. The least positive integer k such that every vertex of G lies in a dominating set of cardinality k is called the domsaturation number of G and is denoted by $ds(G)$. Several results concerning $ds(G)$ are given in Arumugam *et al.*,[2]. In this paper we extend the concept of domsaturation number of a graph to connected domsaturation number ds_c and initiate a study of this parameter. We denote by C_n and P_n the cycle and the path on n vertices.

We use the following theorems.

Theorem 1.1[5] — *For any graph G such that G and* $\overline{G}$ *are connected,* $\gamma_c + \overline{\gamma}_c \leq p + 1$ *and equality holds if and only if* $G \cong C_5$.

Theorem 1.2[5] — *For any graph G such that G and* $\overline{G}$ *are connected,* $\gamma_c + \overline{\gamma}_c = p$ *if and only if* $G \cong C_p$ $(p \geq 6)$, P_p $(p \geq 4)$ *or the graph* G_1 *given in Fig. 1.*

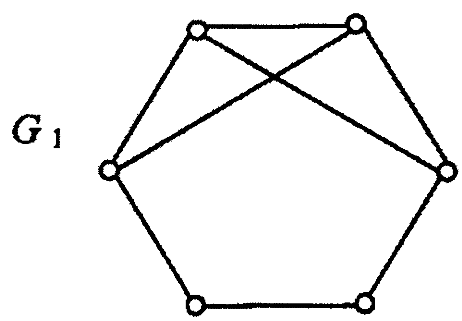

FIG. 1

Theorem 1.3[8] — *For any connected (p, q)-graph G with maximum degree* Δ, $\left(\frac{p}{\Delta+1}\right) \leq \gamma_c(G) \leq 2q - p$. *The lower bound is attained if any only if G has a vertex of degree p – 1 and the upper bound is attained if and only if G is a path.*

Theorem 1.4[8] — *Let G be a connected graph of order* $p \geq 4$ *such that both G and* $\overline{G}$ *are connected. Then* $\gamma_c + \overline{\gamma}_c \leq p(p-3)$. *The bound is attained if and only if G is* P_4.

2. MAIN RESULTS

Definition 2.1 — Let G be a connected graph. The connected domsaturation number ds_c of G is the least positive integer k such that every vertex of G lies in a connected dominating set of cardinality k.

If S is a γ_c-set of G, then for any vertex $u \in V \setminus S$, $S \cup \{u\}$ is a connected dominating set and hence it follows that $ds_c = \gamma_c$ or $\gamma_c + 1$.

The following definition is basically due to Sampathkumar and Neeralagi[9].

Definition 2.2 — Let $G=(V,E)$ be a connected graph and let $v\in V$. The vertex v is said to be γ_c-totally-free if v does not lie in any γ_c-set of G. The vertex v is said to be γ_c-free if there exist γ_c-sets A and B such that $v\in A$ and $v\notin B$. The vertex v is said to be γ_c-fixed if v lies in every γ_c-set of G.

For example, in a tree any pendant vertex is γ_c-totally-free and any non-pendant vertex is γ_c-fixed. In a complete graph, every vertex is γ_c-free.

Clearly $ds_c=\gamma_c+1$ if and only if there exists at least one vertex v which is γ_c-totally free. Hence, $ds_c=\gamma_c+1$ for any connected graph with $\delta=1$. Also for any connected graph G with $\Delta=p-1$ and $G\neq K_p$, we have $ds_c=\gamma_c+1$. If G is any connected graph with $p\geq 3$, having two non-adjacent vertices of degree $p-2$, then every vertex of G is γ_c-free, so that $ds_c=\gamma_c=2$. Also for a complete bipartite graph $K_{m,n}$ with $m,n\geq 2$ we have $ds_c=\gamma_c=2$.

The following theorem gives bounds for ds_c.

Theorem 2.3 — *For any connected graph G with maximum degree* Δ, $\left(\frac{p}{\Delta+1}\right)\leq ds_c\leq 2q-p+1$. *Further* $ds_c=\frac{p}{\Delta+1}$ *if and only if* $G\cong K_p$ *and* $ds_c=2q-p+1$ *if and only if G is a path.*

PROOF : Since $ds_c=\gamma_c$ or γ_c+1, the bounds follow from Theorem 1.3. Now, $ds_c=\frac{p}{\Delta+1}$ if and only if $ds_c=\gamma_c$ and $\gamma_c=\frac{p}{\Delta+1}$. Hence it follows from Theorem 1.3 that $ds_c=\frac{p}{\Delta+1}$ if and only if G is a complete graph.

Also $ds_c=2q-p+1$ if and only if $ds_c=\gamma_c+1$ and $\gamma_c=2q-p$. Hence it follows from Theorem 1.3 $=2q-p+1$ if and only if G is a path. ■

The following theorem gives a Nordhaus-Gaddum type result for ds_c.

Theorem 2.4 — *Let G be a graph such that G and* $\overline{G}$ *are connected then*

(i) $ds_c+\overline{ds}_c\leq p(p-3)+2$ *and equality holds if and only if G is* P_4.

(ii) $ds_c+\overline{ds}_c\leq p+2$ *and equality holds if and only if G is* P_4 *or* P_5.

PROOF : (*i*) Follows from Theorem 1.4

(*ii*) It follows from Theorem 1.1 that $ds_c + \overline{ds}_c \le p+3$. Further C_5 is the only graph for which $\gamma_c + \overline{\gamma}_c = p+1$ and for this graph $ds_c + \overline{ds}_c = \gamma_c + \overline{\gamma}_c = p+1$, so that there is no graph for which $d_c + \overline{ds}_c = p+3$. Hence $ds_c + \overline{ds}_c = p+2$. Moreover $ds_c + \overline{ds}_c = p+2$ if and only if $\gamma_c + \overline{\gamma}_c = p$ and $ds_c + \gamma_c + 1$, $\overline{ds}_c + \overline{\gamma}_c = +1$. By Theorem 1.2, $\gamma_c + \overline{\gamma}_c = p$ if and only if $G \cong C_p$ $(p \ge 6)$, P_p $(p \ge 4)$ or the graph G_1, given in Fig. 1. Among these P_4 and P_5 are the only graphs for which $ds_c = \gamma_c + 1$ and $\overline{ds}_c = \overline{\gamma}_c = +1$ and hence the result follows. The converse is obvious. ■

***Theorem* 2.5** — *Let G be a connected graph with p vertices, q edges and maximum degree* Δ. *Then* $ds_c \le p - \Delta + 1$ *and equality holds for a tree T if and only if T is a spider.*

PROOF : Since $\gamma_c \le p - \Delta$, we have $ds_c \le p - \Delta + 1$. Also, for any tree T, $ds_c = \gamma_c + 1$ and hence $ds_c = p - \Delta + 1$ if and only if $\gamma_c = p - \Delta$. But for a tree $\gamma_c = p - \Delta$ if and only if T is a spider[6] and hence the result follows. ■

***Theorem* 2.6** — *Let G be graph such that both G and* $\overline{G}$ *are connected. Then the following are equivalent :*

(*i*) $\overline{ds}_c = 2$

(*ii*) *For every* $u \in V$, *there exists* $v \in V(G)$ *such that* $d_G(u, v) \ge 3$.

(*iii*) *diam* $(G) \ge 5$.

PROOF : (*i*) ⇒ (*ii*); Since $\overline{ds}_c = 2$, for every $u \in V$, there exists a vertex $v \in V$ such that $\{u, v\}$ is a connected dominating set of $\overline{G}$. Hence u and v are non-adjacent in G and $N(u) \cap N(v) = \phi$, so that $d_G(u, v) \ge 3$.

(*ii*) ⇒ (*i*) is obvious.

(*ii*) ⇒ (*iii*) : It follows from (*ii*) that *diam* $(G) \ge 3$. Suppose *diam* $(G) = 3$ or 4. Let u and v be two vertices of G such that $d(u, v) =$ *diam* (G). Let (u, u_1, u_2, v) or $(u, u_1, u_2, u_3\ v)$ be a shortest $u - v$ path in G. Clearly there is no vertex w such that $d(u_2, w) \ge 3$ which is a contradiction. Hence *diam* $(G) \ge 5$.

(*iii*) ⇒ (*ii*) Let u and v be two vertices of G such that $d(u, v) =$ *diam* (G). Then for any vertex $w \in V(G)$, either $d(w, u) \ge 3$ or $d(w, v) \ge 3$ and hence (*ii*) holds. ■

In the following theorems we obtain a characterization of trees and unicyclic graphs with $ds_c + \overline{ds}_c = p + 1$.

***Theorem* 2.7** — *For a tree T such that $\overline{T}$ is connected $ds_c + \overline{ds}_c = p+1$ if and only if $T \cong P_p$ with $p \geq 6$ or one of the graphs given in Fig. 2.*

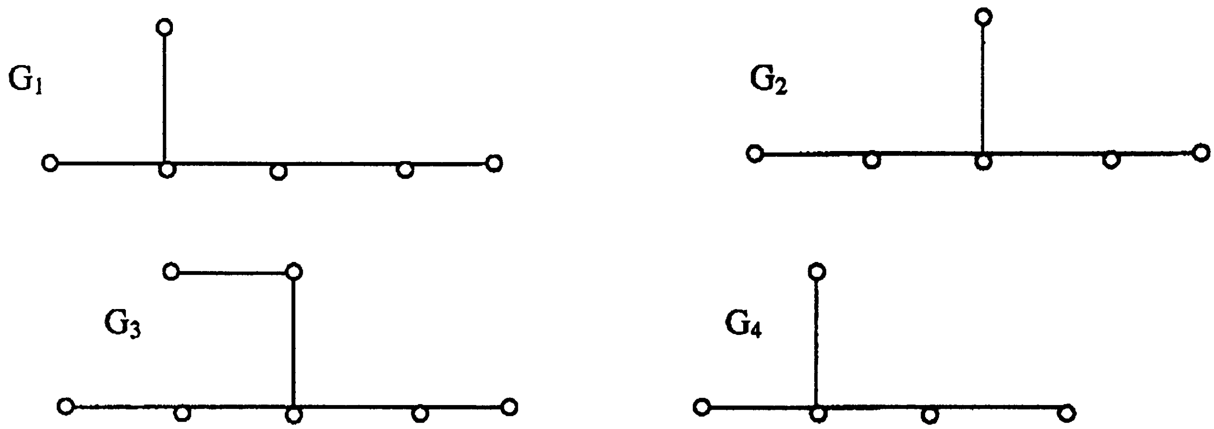

FIG. 2

PROOF : Let $ds_c + \overline{ds}_c = p + 1$. Then one of the following is true.

(*a*) $\gamma_c + \overline{\gamma}_c = p - 1$, $ds_c = \gamma_c + 1$ and $\overline{ds}_c = \overline{\gamma}_c + 1$

(*b*) $\gamma_c + \overline{\gamma}_c = p$ and either $ds_c = \gamma_c + 1$, $\overline{ds}_c = \overline{\gamma}_c$ or $ds_c = \gamma_c$, $\overline{ds}_c = \overline{\gamma}_c + 1$.

If (*b*) holds, then by Theorem 1.2, T is isomorphic to P_p with $p \geq 6$. Suppose (*a*) holds. Since $\overline{\gamma}_c = 2$, we have $\gamma_c = p - 3$ so that T has exactly 3 pendant vertices. Since $\overline{ds}_c$ =3, it follows that $diam\,(T)$ = 3 or 4, so that T is isomorphic to one of the graphs G_i, $1 \leq i \leq 4$, given in Fig. 2. The converse is obvious. ■

***Theorem* 2.8** — *Let G be a unicyclic graph with cycle C_n such that $\overline{G}$ is connected. Then $ds_c + \overline{ds}_c = p + 1$ if and only if G is isomorphic to one of the twelve graphs given in Fig. 3.*

PROOF : For C_5, $ds_c = \overline{ds}_c$ = 3 and for all other graphs given in Fig. 3, $\gamma_c = p - 3$, ds_c = $p - 2$, $\overline{\gamma}_c$ = 2 and $\overline{ds}_c$ = 3 so that $ds_c + \overline{ds}_c = p + 1$.

Conversely, let G be any unicyclic graph with $ds_c + \overline{ds}_c = p+1$. If G is a cycle, then G is isomorphic to C_5. Suppose G is not a cycle. Then $\overline{\gamma}_c = 2$, so that $\overline{ds}_c \le 3$ and $ds_c \ge p-2$. Further $\gamma_c \le p - \Delta \le p-3$ and so it follows that $ds_c = p-2$, $\gamma_c = p-3$, $\Delta = 3$ and $\overline{ds}_c = 3$. Thus, G has at most three pendant vertices. Theorem 2.6 implies that $n \le 5$ and $diam\,(G) \in \{3, 4\}$. We claim that every vertex not on C has degree 1 or 2. Suppose to contrary that some vertex in $V - C$ has degree three. Then the cycle C has two adjacent vertices of degree two in G, and G has at least two pendant vertices. Hence, $\gamma_c < p-3$, a contradiction. Thus every vertex not on C has degree 1 or 2. It is a simple exercise to show that if $diam\,(G) = 3$, then G is isomorphic to one of the graphs G_1, G_3, G_5, G_7 or G_{10} and if $diam\,(G) = 4$, G is isomorphic to one of the graphs G_2, G_4, G_6, G_8, G_9 or G_{11} given in Fig. 3. ■

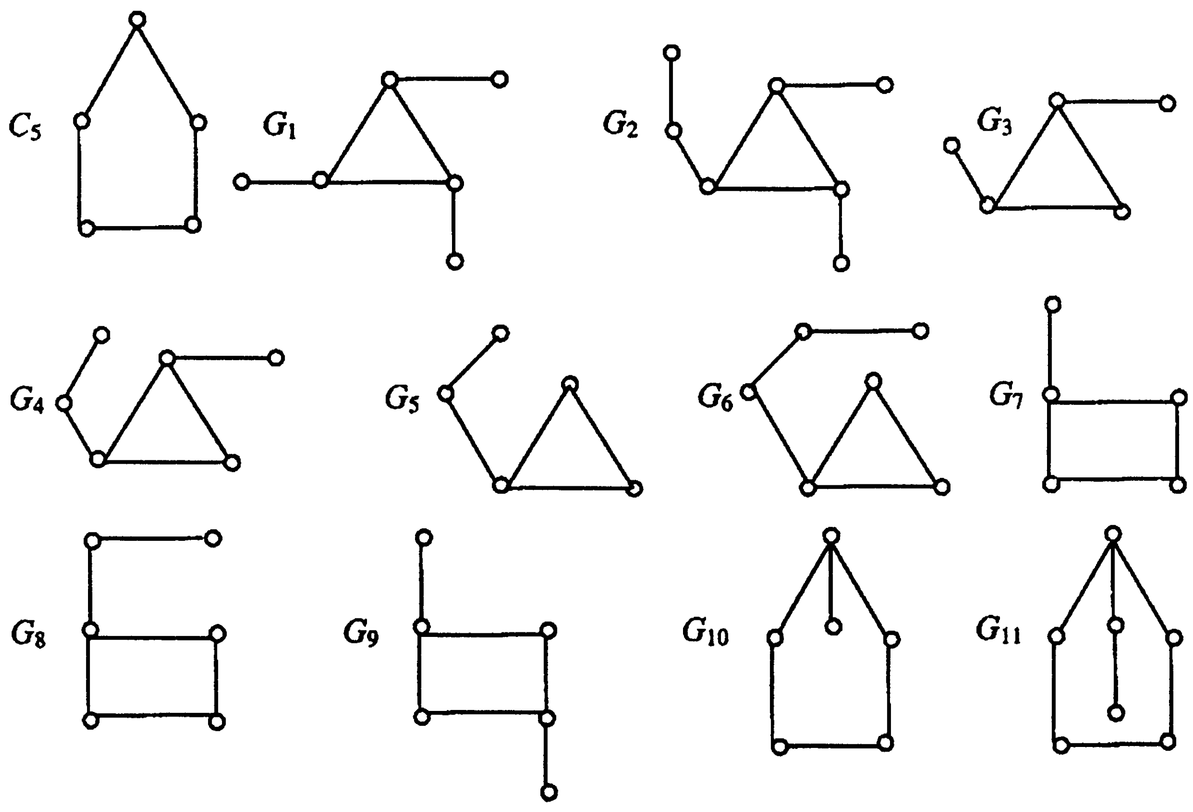

FIG 3

***Theorem* 2.9** — *Let G be an r-regular graph with $diam\,(G) \ne 2$ such that both G and $\overline{G}$ are connected. Then $ds_c + \overline{ds}_c \le p$.*

PROOF : Suppose G is an r-regular graph with $diam\,(G) \ne 2$ such that $ds_c + \overline{ds}_c = p+1$. If $\overline{ds}_c = 2$, $ds_c = p-1$ so that $\gamma_c = p-2$ and hence $r = 2$. But for a cycle $ds_c + \overline{ds}_c = p+1$ if and only if $G \cong C_5$ and $diam(C_5) = 2$. Hence $r \geq 3$. Further $\overline{ds}_c \geq 3$ and $3 \leq diam(G) \leq 4$. Hence $\overline{\gamma}_c = 2$ so that $\overline{ds}_c = 3, ds_c = p-2$ and $\gamma_c = p-3$. Hence $r \leq 3$ so that G is a cubic graph with $p \geq 8$ vertices.

Let T be a spanning tree of G with $\Delta(T) = 3$. Since $\gamma_c = p-3$, G has exactly three pendant vertices, say u_1, u_2, u_3. Let v_i be the vertex which is adjacent to u_i in T. If a pendant vertex u_i is adjacent to an internal vertex $\omega \neq v_i$, then $T^1 = T - u_i v_i + u_i\omega$ is a spanning tree of G with four pendant vertices. Otherwise $\{u_1, u_2, u_3\}$ forms a complete graph in G and $T^1 = T - u_2v_2 - u_3v_3 + u_1u_3 + u_1u_2$ is a spanning tree of G with 4 pendant vertices. Hence $\gamma_c \leq p-4$ which is a contradiction. Hence $ds_c + \overline{ds}_c \leq p$. ■

The following are some problems for further investigation.

1. Characterize γ_c-free, γ_c-totally-free and γ_c-fixed vertices of a connected graph.
2. Characterize graphs for which $ds_c = \gamma_c$ or $ds_c = \gamma_c + 1$.

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