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PERIODIC BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Applying monotone iterative technique, increasing operator fixed point theorem in Banach space and Poincaré operator, we obtain two theorems on existence of extreme solutions of periodic boundary value problems for Integro-differential equations in Banach space.

Key Words : Integro-Differential Equations; Monotone Iterative Technique; Fixed Point Theorems; Poincaré Operator

1. INTRODUCTION

In this paper, using the monotone iterative method[1-3], we study the existence of maximal and minimal solutions of the periodic boundary value problems (PBVP) for nonlinear integro-differential equations of Volterra type, namely

$$u'(t) = H(t, u(t), Su), \qquad \text{... (1.1)}$$

$$u(0) = u(2\pi) \qquad \text{... (1.2)}$$

in a real Banach space E, where

$$(Su)(t) = \int_0^t h(t, s)\, u(s)\, ds, \qquad \text{... (1.3)}$$

$$h(\cdot,\cdot) \in C[I \times I, R^+],\ h_0 = \max\{h(t, s) : t \in I, s \in I\}\ I = [0, 2\pi],$$

$$H \in C[I \times E \times E, E].$$

The case where H, h are Carathéodory functions, has been studied by several authors, see[3-6] The purpose of this paper is to study PBVP (1.1)(1.2) in a Banach space, no compact condition is used.

2. Preliminaries and Some Lemmas

Let K be a cone of E, then the cone K induces a partial ordering on E defined by $u \leq \upsilon$ if $\upsilon - u \in K$. Let E^* denote the set of continuous linear functions on E. Given a cone K, we let

$$K^* = \left\{ \phi \in E^* : \phi(u) \geq 0 \text{ for all } u \in K \right\}.$$

A cone K is said to be regular if every sequence in E, that is nondecreasing and bounded in order, has a limit. We shall assume in this paper that K is a regular cone, θ denotes the zero element in the Banach space E.

Definition 2.1 — The function $\alpha(t) \in C^1[I, E] = \{u : I \to E \mid u'(t) \text{ is continuous on } I\}$ is called a lower solution of PBVP (1.1)-(1.2) if

$$\alpha'(t) \leq H(t, \alpha(t), (S\alpha)(t)), \qquad \alpha(0) \leq \alpha(2\pi). \qquad \text{... (2.1)}$$

Similarly, a function $\beta \in C^1[I, E]$ is called an upper solution of PBVP (1.1)-(1.2) if

$$\beta'(t) \geq H(t, \beta(t), (S\beta)(t)), \qquad \beta(0) \geq \beta(2\pi). \qquad \text{... (2.1')}$$

Let us list the following assumptions for convenience:

(A1) (*i*) α, β are lower and upper solutions for PBVP (1.1)-(1.2) respectively and satisfy $\alpha(t) \leq \beta(t)$ for $t \in I$,

(*ii*) α, β are lower and upper solutions for PBVP (1.1)-(1.2) respectively and satisfy $\alpha(t) \geq \beta(t)$ for $t \in I$;

(A2) (*i*) $H(t, u, Su) - H(t, \tilde{u}, S\tilde{u}) \geq -M(t)(u - \tilde{u}) - N(t)(Su - S\tilde{u})$, $t \in I$;

whenever

$$\alpha(t) \leq \tilde{u} \leq u \leq \beta(t), \text{ where } M(t), N(t) \in C[I, R],\ M(t) > 0, N(t) \geq 0,$$

and

$$\int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt < 1.$$

(*ii*) $H(t, u, Su) - H(t, \tilde{u}, S\tilde{u}) \leq M(t)(u - \tilde{u}) + N(t)(Su - S\tilde{u})$, $t \in I$;

whenever

$$\beta(t) \leq \tilde{u} \leq u \leq \alpha(t), \text{ where } M(t), N(t) \in C[I, R],\ M(t) > 0, N(t) \geq 0,$$

and

$$\int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt < \frac{1}{2}.$$

In this paper, we need the following lemmas.

Lemma 2.1[7-8,10] — Let E be a Banach space, partially ordered by the cone $K \subset E$. Let $[x_0, y_0]$ be an order interval and $F ; [x_0, y_0] \to [x_0, y_0]$ increasing. Then F has a fixed point if K is regular and F is continuous.

The following Lemma 2.2 and Lemma 2.2′ are taken from [4]. We list the proof in appendix so that the reader can read this paper with no difficulty.

Lemma 2.2[4] — Assume that $m(t) \in C^1[I, R]$ and satisfies

$$m'(t) \le -M(t)\, m(t) - N(t) \int_0^t h(t, s)\, m(s)\, ds, \qquad t \in I, \qquad \text{... (2.2)}$$

where $M(t), N(t) \in C[I,R], M(t) > 0, N(t) \ge 0$, suppose further that

$$\int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt < 1.$$

Then either $m(0) \le 0$ or $m(0) \le m(2\pi)$ implies that $m(t) \le 0$ on I.

*Lemma*2.2′[4] — Assume that $m(t) \in C^1[I, R]$ and satisfies

$$m'(t) \ge M(t)\, m(t) + N(t) \int_0^t h(t, s)\, m(s)\, ds, \quad t \in I; \qquad \text{... (2.2′)}$$

where $M(t), N(t) \in C[I, r], M(t) > 0, N(t) \ge 0$, suppose further that

$$\int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt < \frac{1}{2}.$$

Then $m(0) \ge m(2\pi)$ implies that $m(t) \le 0$ on I.

Lemma 2.3 — If K is a cone in E, and assumptions (A1) (i) and (A2)(i) hold. Then for each $\eta \in [\alpha, \beta]$, the following linear initial value problem (IVP).

$$u'(t) = H_\eta(t) - M(t)\, u(t) - N(t) \int_0^t h(t, s)\, u(s)\, ds, \quad t \in I;$$

$$u(0) = u_0, \quad u_0 \in [\alpha(0), \beta(0)]. \qquad \text{... (2.3)}$$

has a unique solution $u(t)$ such that $u(t) \in [\alpha, \beta] = \{u \in C[I, R] : \alpha(t) \le u(t) \le \beta(t)\}$, where

$$H_\eta(t) = H(t, \eta(t), (S\eta)(t)) + M(t)\,\eta(t) + N(t) \int_0^t h(t,s)\,\eta(s)\,ds.$$

PROOF : Problem (2.3) is equivalent to the following operator equation $u(t) = (Tu)(t)$, where

$$(Tu)(t) = u_0 + \int_0^t \left(H_\eta(r) - M(r)\,u(r) - N(r) \int_0^r h(r,s)\,u(s)\,ds \right) dr.$$

We show that T is a contraction in Banach space $C[I, R]$. Indeed, for any $u, \bar{u} \in C[I, R]$, we have

$$\| Tu - T\bar{u} \| = \max_I \left| (Tu(t) - (T\bar{u})(t) \right|$$

$$\leq \int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t,s)\,ds \right) dt \cdot \| u - \bar{u} \|.$$

Then, the above operator equation, or the equivalent IVP (2.3), has a unique solution $u(t; u_0)$.

Next, we show that $u(t; u_0) \in [\alpha, \beta]$.

For any $\phi \in K^*$, set $m(t) = \phi(\alpha(t) - u(t))$, so that $m(0) \leq 0$, and

$$\begin{aligned} m'(t) &= \phi(\alpha'(t) - u'(t)) \\ &\leq -M(t)\,\phi(\alpha(t) - u(t)) - N(t)\,\phi((S\alpha)(t) - (Su)(t)) \\ &\leq -M(t)\,m(t) - N(t) \int_0^t h(t,s)\,m(s)\,ds, \end{aligned}$$

in view of (A2) (*i*). By Lemma 2.2, we have $m(t) \leq 0, t \in I$. Since $\phi \in K^*$ is arbitrary, this implies that $\alpha(t) \leq u(t), t \in I$. Similar argument shows that $u(t) \leq \beta(t), t \in I$. The proof of Lemma 2.3 is completed..

Lemma 2.3′ — If K is a cone in E and assumptions (A1)(ii) and (A2)(ii) hold. Then for each $\eta \in [\beta, \alpha]$, the following linear initial value problem (IVP)

$$\begin{aligned} u'(t) &= \overline{H}_\eta(t) + M(t)\,u(t) + N(t) \int_0^t h(t,s)\,u(s)\,ds, \quad t \in I \\ u(2\pi) &= u_0, \quad u_0 \in [\beta(2\pi), \alpha(2\pi)], \end{aligned} \qquad \text{... (2.3′)}$$

has a unique solution $u(t) = u(t; 2\pi, u_0)$, $u(0; u_0) \in [\beta(0), \alpha(0)]$, where

$$\overline{H}_\eta(t) = H(t, \eta(t), (S\eta)(t)) - M(t)\eta(t) - N(t)\int_0^t h(t,s)\eta(s)\,ds.$$

PROOF : As in the proof of Lemma 2.3, using the Banach's contraction mapping theorem, it is easy to prove that (2.3′) has a unique solution $u(t) = u(t; 2\pi, u_0)$ on I.

Next, we show that $u(0; u_0) \in [\beta(0), \alpha(0)]$.

For any $\phi \in K^*$, set $m(t) = \phi(\beta(t) - u(t))$, so that $m(2\pi) \le 0$, and

$$\begin{aligned} m'(t) &= \phi(\beta'(t) - u'(t)) \\ &\ge M(t)\phi(\beta(t) - u(t)) + N(t)\phi((S\beta)(t) - (Su)(t)) \\ &= M(t)m(t) + N(t)\int_0^t h(t,s)m(s)\,ds, \end{aligned}$$

in view of (A2) (*ii*). If $m(0) \le 0$ is not true, then $m(0) > 0$, consequently $m(0) > m(2\pi)$. By Lemma 2.2′, we have $m(t) \le 0, t \in I$. In particular $m(0) \le 0$, which is a contradiction. Since $\phi \in K^*$ is arbitrary, this implies that $\beta(0) \le u(0)$. Similar argument shows that $u(0) \le \alpha(0)$. The proof of Lemma 2.3′ is complete.

Under the same conditions of Lemma 2.3, it is clear that $\alpha(0) \le \alpha(2\pi) \le \beta(2\pi) \le \beta(0)$, that is $[\alpha(2\pi), \beta(2\pi)] \subseteq [\alpha(0), \beta(0)]$. By Lemma 2.3, for any $u_0 \in [\alpha(2\pi), \beta(2\pi)]$, IVP (2.3) has a unique solution $u(t; u_0)$.

Lemma 2.4 — Under the same conditions of Lemma 2.3, the Poincaré operator

$$P_{2\pi}: \ u_0 \to u(2\pi; u_0), \ u_0 \in [\alpha(2\pi), \beta(2\pi)], \quad \dots (2.4)$$

is continuous, increasing on $[\alpha(2\pi), \beta(2\pi)]$.

PROOF : First, we show that $P_{2\pi}$ is an increasing operator. Let $u_1, u_2 \in [\alpha(2\pi), \beta(2\pi)]$ and $u_1 \le u_2$, suppose that $\overline{u}_i(t) = u(t; u_i)$ $(i = 1, 2)$ are the solutions of

$$\begin{cases} u'(t) = H_\eta(t) - M(t)u(t) - N(t)\int_0^t h(t,s)u(s)\,ds, & t \in I; \\ u(0) = u_n, & n = 0, 1, 2. \end{cases}$$

For any $\phi \in K^*$, set $m(t) = \phi(\bar{u}_1(t) - \bar{u}_2(t))$, so that $m(0) = 0$, and

$$\begin{aligned} m'(t) &= \phi(\bar{u}_1'(t) - \bar{u}_2'(t)) \\ &\le -M(t)\,\phi(\bar{u}_1(t) - \bar{u}_2(t)) - N(t)\,\phi((S\bar{u}_1)(t) - (S\bar{u}_2)(t)) \\ &\le -M(t)\,m(t) - N(t)\int_0^t h(t,s)\,m(s)\,ds. \end{aligned}$$

By Lemma 2.2, we have $m(t) \le 0, t \in I$. Since $\phi \in K^*$, is arbitrary, this implies that $\bar{u}_1(t) \le \bar{u}_2(t), t \in I$. Especially, $\bar{u}_1(2\pi) \le \bar{u}_2(2\pi)$, that is $P_{2\pi}u_1 \le P_{2\pi}u_2$.

Next, we show that $P_{2\pi}$ is a continuous operator. Let $u_n \in [\alpha(2\pi), \beta(2\pi)]$, $n = 0, 1, 2 \ldots$ and $\| u_n - u_0 \| \to 0$, as $n \to \infty$. Let $u(t; u_n)$ are the solutions of

$$\begin{cases} u'(t) = H_\eta(t) - M(t)\,u(t) - N(t)\displaystyle\int_0^t h(t,s)\,u(s)\,ds, \quad t \in I; \\ u(0) = u_n, \quad n = 0, 1, 2 \ldots . \end{cases}$$

Indeed, for any $t \in I$, we have

$$\begin{aligned} \| u(t; u_n) - u(t; u_0) \| &\le \| u_n - u_0 \| \\ &+ \int_0^{2\pi} [M(r)\,\| 4(r; u_n) - u(r; 0) \|]\,dr \\ &+ \int_0^{2\pi} N(r)\int_0^r h(r,s)\,\| u(s; u_n) - u(s; 0) \|\,ds\,dr \\ &\le \| u_n - u_0 \| + \int_0^{2\pi}\left(M(r) + N(r)\int_0^r h(r,s)\,ds\right)dr \cdot \| u(t; u_n) - u(t; u_0) \|_C, \end{aligned}$$

then from (A2) (i), it follows that

$$\| u(t; u_n) - u(t; u_0) \|_C \le \frac{\| u_n - u_0 \|}{1 - \displaystyle\int_0^{2\pi}\left(M(r) + N(r)\int_0^r h(r,s)\,ds)\,dr\right)}$$

Thus as $n \to \infty$, we have $\| u(t; u_n) - u(t; u_0) \|_C \to 0$, consequently $\| u(2\pi; u_n) -$

$u(2\pi; u_0) \|_C \to 0$. Therefore, operator $P_{2\pi}$ is continuous on $[\alpha(2\pi), \beta(2\pi)]$. Similarly, we can prove that

Lemma 2.4′ — Under the same conditions of Lemma 2.3′, the Poincaré operator

$$P_0 : u_0 \to u(0, 2\pi; u_0), \quad u_0 \in [\beta(0), \alpha(0)], \qquad \dots (2.4)$$

is continuous, increasing on $[\beta(0), \alpha(0)]$.

Lemma 2.5 — If the cone K is regular and assumptions (A1)(*i*) and (A2)(*i*) hold, then for each $\eta \in [\alpha, \beta]$, the following PBVP

$$u'(t) = H_\eta(t) - M(t)\,u(t) - N(t) \int_0^t h(t,s)\,u(s)\,ds, \quad t \in I; \qquad \dots (2.5)$$

$$u(0) = u(2\pi)$$

has a unique solution $u(t)$ such that $u(t) \in [\alpha, \beta]$.

PROOF : From Lemma 2.4, we see that $P_{2\pi} : u_0 \to u(2\pi; u_0)$ is a continuous, increasing operator on $[\alpha(2\pi), \beta(2\pi)]$. Further, we show that $P_{2\pi}\,\alpha(2\pi) \geq \alpha(2\pi)$. In fact, by Lemma 2.3, for $u_0 = \alpha(2\pi) \in [\alpha(0), \beta(0)]$, IVP (2.3) has a unique solution $u(t; \alpha(2\pi)) \in [\alpha, \beta]$, consequently, $u(2\pi; \alpha(2\pi)) \in [\alpha(2\pi), \beta(2\pi)]$, hence $P_{2\pi}\,\alpha(2\pi) \geq \alpha(2\pi)$.

Similarly, we can prove that $P_{2\pi}\,\beta(2\pi) \leq \beta(2\pi)$.

Therefore, the Poincaré operator $P_{2\pi} : u_0 \to u(2\pi; u_0)$, for $u_0 \in [\alpha(2\pi), \beta(2\pi)]$, maps the interval $[\alpha(2\pi), \beta(2\pi)]$ into itself. By Lemma 2.1, we can conclude that $P_{2\pi}$ has a fixed point $u_0^* \in [\alpha(2\pi), \beta(2\pi)]$. Therefore, IVP (2.3) has a solution $u(t, u_0^*) \in [\alpha, \beta]$ satisfying $u(0, u_0^*) = u_0^* = u(2\pi; u_0^*)$. Hence, $u(t; u_0^*)$ is a solution of PBVP (2.5), and the solution is unique in view of Lemma 2.2.

Lemma 2.5′ — If the cone K is regular and assumptions (A1)(*ii*) and (A2)(*ii*) hold, then for each $\eta \in [\beta, \alpha]$, the following PBVP

$$u'(t) = \overline{H}_\eta(t) + M(t)\,u(t) + N(t) \int_0^t h(t,s)\,u(s)\,ds, \quad t \in I; \qquad \dots (2.5')$$

$$u(0) = u(2\pi)$$

has a unique solutuion $u(t)$ such that $u(0) \in [\beta(0), \alpha(0)]$.

We omit the proof of this theorem because it is similar to that of Lemma 2.5.

3. Main Results

***Theorem* 3.1** — *Let the cone K be regular and assumptions (A1)(i), (A2)(i) hold. Then there exist monotone sequences* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *with* $\alpha_0(t) = \alpha(t)$, $\beta_0(t) = \beta(t)$ *such that* $\lim_{n \to \infty} \alpha_n(t) = \rho(t)$, $\lim_{n \to \infty} \beta_n(t) = \gamma(t)$ *uniformly on I and* $\rho(t), \gamma(t)$ *are respectively the minimal and maximal solutions of the Problem (1.1)-(1.2) in* $[\alpha, \beta]$.

Proof : We define a mapping $T_\eta = u$ for $\eta \in [\alpha, \beta]$, where u is the unique solution of PBVP (2.5). We shall show that

(a) $\alpha \le T\alpha$, $T\beta \le \beta$;

(b) T possesses a monotone nondecreasing property on the segment $[\alpha, \beta]$, that is, if

$$\eta_1, \eta_2 \in [\alpha, \beta] \text{ with } \eta_1(t) \le \eta_2(t) \text{ on } I, \text{ then } (T\eta_1)(t) \le (T\eta_2)(t) \text{ on } I.$$

To prove (a), from Lemma 2.5, if we take $\eta = \alpha$, then $\alpha \le T\alpha$; if we take $\eta = \beta$, we can get $\beta \ge T\beta$.

In order to prove (b), denote $u_1 = T\eta_1$, $u_2 = T\eta_2$, for any $\phi \in K^*$, let $m(t) = \phi(u_1 - u_2)$, then it follows from Lemma 2.2, that $m(t) \le 0$ on I, so we get $(T\eta_1)(t) \le (T\eta_2)(t)$ on I.

Now we can define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that

$$\alpha_{n+1} = T\alpha_n, \quad \beta_{n+1} = T\beta_n, \quad n = 0, 1, 2, \ldots .$$

From the properties of operator T, we have

$$\alpha_0(t) \le \alpha_1(t) \le \ldots \le \alpha_n(t) \le \ldots \le \beta_n(t) \le \ldots \le \beta_1(t) \le \beta_0(t), \quad \forall t \in I. \qquad \ldots (3.1)$$

Let $B = \{\alpha_n\} \subset [\alpha_0, \beta_0]$, $B(t) = \{\alpha_n(t)\} \subset E$, $t \in I$. By virtue of the regularity of K and (3.1), we know that $B(t)$ is a relatively compact. Since K is a regular cone, K is also a normal cone. Thus, $\{\alpha_n\}$, $\{\beta_n\}$ are uniformly bounded. For any $t \in I$, $n = 0, 1, 2, \ldots$, we have $\alpha_n, \beta_n \in [\alpha, \beta]$, $S\alpha_n, S\beta_n \in [S\alpha, S\beta]$, therefore

$$H(t, \beta_0(t), (S\beta_0)(t)) - H(t, \alpha_n(t), (S\alpha_n)(t))$$

$$\ge -M(t)(\beta_0(t) - \alpha_n(t)) - N(t))(S\beta_0)(t) - (S\alpha_n)(t)),$$

$$H(t, \alpha_n(t), (S\alpha_n)(t)) \le H(t, \beta_0(t), (S\beta_0)(t))$$

$$+ M(t)(\beta_0(t) - \alpha_0(t)) + N(t)((S\beta_0)(t) - (S\alpha_0)(t)) = H^+(t).$$

Similar argument shows that

$$H(t, \alpha_n(t), (S\alpha_n)(t)) \geq H(t, \alpha_0(t), (S\alpha_0)(t))$$

$$- M(t)\beta_0(t) - \alpha_0(t)) - N(t)((S\beta_0)(t) - (S\alpha_0)(t)) = H^-(t).$$

Hence

$$\theta \leq H(t, \alpha_n(t), (S\alpha_n)(t)) - H^-(t) \leq H^+(t) - H^-(t),$$

Since K is a normal cone, there exist constant L_1 such that

$$\| H(t, \alpha_n(t), (S\alpha_n)(t)) \|_C \leq \| H^-(t) \|_C + L_1 \| H^+(t) - H^-(t) \|_C.$$

Then $H(t, \alpha_n(t), (S\alpha_n)(t))$ is uniformly bounded. In view of the facts that both $\{\alpha_n\}$, $\{S\alpha_n\}$ are uniformly bounded and

$$\alpha_n' = H(t, \alpha_{n-1}, S\alpha_{n-1}) - M(t)(\alpha_n - \alpha_{n-1}) - N(t)(S\alpha_n - S\alpha_{n-1}),$$

it implies the equicontinuity of the sequence $\{\alpha_n\}$. By the Ascoli-Arzela theorem, $\{\alpha_n\}$ is a relatively compact set in $C[I, E]$. In view of the normality of K and (3.1), we can get $\{\alpha_n(t)\}$ converges to $\rho(t)$ uniformly on I. It is easy to prove that $\rho(t)$ is a solution of PBVP (1.1)-(1.2). Similarly, we can show that $\{\beta_n\}$ in $C[I, E]$ converges to $\gamma(t)$, and that $\gamma(t)$ is a solution of PBVP (1.1)-(1.2). It follows by using standard arguments that $\rho(t), \gamma(t)$ are minimal and maximal solutions of PBVP (1.1)-(1.2)

Similarly, we have

Theorem 3.1′ — *Let the cone K be regular and assumptions (A1)(ii), (A2)(ii) hold. Then there exist monotone sequences $\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0(t) = \alpha(t)$, $\beta_0(t) = \beta(t)$ such that* $\lim_{n \to \infty} \beta_n(t) = \rho(t)$, $\lim_{n \to \infty} \alpha_n(t) = \gamma(t)$ *uniformly on I, and ρ, γ are respectively the minimal and maximal solutions of the Problem (1.1)-(1.2) in* $[\beta, \alpha]$.

We omit the proof of this theorem because it is similar to that of Theorem 3.1, except for Lemma 2′, 3′, 4′, 5′ taking the place of Lemma 2, 3, 4, 5 respectively.

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APPENDIX

Let us list the following proof to Lemma 2.2 and Lemma 2.2′ for convenience:

The proof of Lemma 2.2:

PROOF : First consider the case $m(0) \le 0$. If the conclusion is not true, then there exists $t_1 \in (0, 2\pi]$ such that $m(t_1) > 0$. If $m(t) \ge 0, t \in (0, t_1]$, then from (2.2) it follows that $m'(t) \le 0, t \in (0, t_1]$. We get $m(t_1) \le m(0) \le 0$, which is a contradiction. Hence, there exists $t_2 \in [0, t_1)$ such that $m(t_2) = \min_{t \in [0, t_1)} m(t) = -\lambda, \lambda > 0$. By integrating from t_2 to t_1 on both sides of (2.2), we have

$$\lambda < m(t_1) + \lambda = \int_{t_2}^{t_1} m'(t)\, dt$$

$$\le \int_{t_2}^{t_1} \left(-M(t)\, m(t) - N(t) \int_0^t h(t, s)\, m(s)\, ds \right) dt$$

$$\le \lambda \int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt.$$

Thus

$$\int_0^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\, ds \right) dt > 1,$$

which is a contradiction.

Next consider the proof of the case $m(0) \le m(2\pi)$. If the conclusion is not true, then we have the cases : (a) $m(t) \ge 0$ for $t \in I$ and $m(t) \equiv 0$ is not true; (b) there exist $\bar{t}, \underline{t} \in I$ such that $m(\bar{t}) > 0, m(\underline{t}) < 0$.

For case (a), from (2.2) we have $m'(t) \le 0$ on I and $\int_0^{2\pi} m'(t)\,dt < 0$. We get $m(0) > m(2\pi)$ which is a contradiction.

For case (b), there are two situations: (i) $m(2\pi) \ge 0$ and (ii) $m(2\pi) < 0$.

When $m(2\pi) \ge 0$, there exists $t^* \in [0, 2\pi)$ such that $m(t^*) = \min_I m(t) = -\varepsilon < 0$. By integration from t^* to 2π on both sides of (2.2), we get

$$\varepsilon \le m(2\pi) - m(t^*) = \int_{t^*}^{2\pi} m'(t)\,dt$$

$$\le \int_{t^*}^{2\pi} \left(-M(t)\,m(t) - N(t) \int_0^t h(t,s)\,m(s)\,ds \right) dt$$

$$< \varepsilon L \le \varepsilon$$

which deduces a contradiction. When $m(2\pi) < 0$, we also have $m(0) < 0$. Then, there exists a $t_1 \in (0, 2\pi)$ such that $m(t_1) = 0$. It is clear that $m(t^*) = \min_{[0,t_1]} m(t) = -\varepsilon < 0$ with $t^* \in [0, t_1)$. From (2.2), we have

$$\varepsilon = m(t_1) - m(t^*) = \int_{t^*}^{t_1} m'(t)\,dt$$

$$\le \int_{t^*}^{t_1} \left(-M(t)\,m(t) - N(t) \int_0^t h(t,s)\,m(s)\,ds \right) dt$$

$$\le \varepsilon \int_0^{2\pi} (M(t) + N(t) \int_0^t h(t,s)\,ds)\,dt < \varepsilon$$

which leads to a contradiction again.

The proof of Lemma 2.2′:

PROOF : If the conclusion is not true, then we have the following two situations:

(*a*) $m(t) \geq 0$ for $t \in I$ and $m(t) \equiv 0$ is not true.

(*b*) there exist $\bar{t}, \underline{t} \in I$ such that $m(\bar{t}) > 0, m(\underline{t}) < 0$.

If (*a*) holds, from (2.2′), we have $m'(t) \geq 0$ on I and $\int_0^{2\pi} m'(t)\,dt > 0$. Since $M(t) > 0$, we get $m(2\pi) - m(0) = \int_0^{2\pi} m'(t)\,dt > 0$, which is a contradiction.

If (*b*) holds, then there exists $t_1 \in (0, 2\pi]$ such that $m(t_1) = \min_I m(t) = -\lambda, \lambda > 0$. Denote

$$\delta(t) = \int_0^t \left(M(\xi) + N(\xi) \int_0^{\xi} h(\xi, s)\,ds \right) d\xi$$

and notice that $\delta(t)$ is increasing on I. By integration from 0 to t_1 on both sides of (2.2′), we get

$$\begin{aligned} -\lambda - m(0) &= \int_0^{t_1} m'(t)\,dt \\ &\geq \int_0^{t_1} \left(M(t)\,m(t) + N(t) \int_0^t h(t, s)\,m(s)\,ds \right) dt \\ &\geq -\lambda\,\delta(t_1) \geq -\lambda\,\delta(2\pi) \geq -\frac{1}{2}\lambda. \end{aligned}$$

It follows that $m(0) < 0$ and $\lambda + m(0) \leq \lambda\,\delta(2\pi)$. Hence

$$\lambda \leq \frac{-m(0)}{1 - \delta(2\pi)}.$$

On the other hand, since $0 > m(0) \geq m(2\pi)$, there exists $t_2 \in (0, 2\pi)$ such that $m(t_2) = 0$, and then we get

$$m(2\pi) = \int_{t_2}^{2\pi} m'(t)\,dt \geq -\lambda \int_{t_2}^{2\pi} \left(M(t) + N(t) \int_0^t h(t, s)\,ds \right) dt > -\lambda\,\delta(2\pi).$$

Hence,

$$m(0) \geq m(2\pi) > -\lambda\,\delta(2\pi) \geq \frac{m(0)\,\delta(2\pi)}{1 - \delta(2\pi)}$$

Therefore, $1 - \delta(2\pi) < \delta(2\pi)$ or $\delta(2\pi) > \frac{1}{2}$, which is a contradiction to assumption. The proof of the Lemma is complete.