TRANSIENT ANALYSIS OF LARGE SYSTEMS WITH COMPLEX PARAMETERS

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An improved waveform relaxation (WR) is proposed for solving initial value problem for ordinary differential equations with complex parameters. Firstly, the convergence conditions are given. Secondly, the initial waveforms are produced by using the Chebyshev collocation method instead of constant waveforms. Thirdly, model-reduction is performed for iterative systems to reduce computational cost. An illustrative example is also provided.

Key Words: Waveform Relaxation; Initial Waveform Guess; Collocation Method; Model-reduction

1. INTRODUCTION

Waveform relaxation (WR) is a dynamic iteration technique for differential equations. It was first introduced as a practically numerical method by Lelarasmee et al. for the time-domain analysis of large-scale integrated circuits. It differs from classical static iterative methods in that it iterates with functions in a function space, instead of with finite sets of discrete unknown vectors. Sometimes the method can be regarded as a natural extension to systems of differential equations on classical relaxation methods for solving algebraic equations.

WR has been used and discussed for a wide variety of problems (2-8). Many problems arising from electrical network modelling can be solved extremely efficiently, but in general the actual rate of convergence of WR may be very slow (2). We know that the choice of constant initial waveforms is one of factors which result in the slow convergence rate on WR. In order to improve the convergence rate of WR, in (2), a method was presented which used a technique called dynamic fitting to produce initial waveforms. In this paper, we apply WR to ordinary differential equations with complex parameters and choose initial waveforms by applying the Chebyshev collocation method (9-11). Model-reduction was adopted for each iterative system so that computational expense could be reduced significantly (12,13).

The paper is organized as follows. In Section 2, we recall some basic results and give the convergence conditions of WR. In Section 3, we give the initial waveforms by using the Chebyshev collocation method in small intervals and large intervals respectively. In Section 4, we perform reduction-model for approximate systems. In Section 5, an illustrative example is also provided in the paper.

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2. WAVEFORM RELAXATION

We consider a nonlinear initial-value problem with a parameter of the form

\[
\begin{align*}
M(\omega) \frac{dy(x, \omega)}{dx} + A(\omega) y(x, \omega) + g(t(x, \omega)) &= f(x, \omega) \\
y(x_0, \omega) &= y_0(\omega)
\end{align*}
\]  

... (1)

where \( x \in [0, l], \omega \in C \) is a complex parameter, \( y(x, \omega) \in \mathbb{R}^n, M(\omega) \) and \( A(\omega) \) are \( n \times n \) complex matrices, \( f(x, \omega) \in \mathbb{R}^n \) is a known function, and \( g(y(x, \omega)) \) is a nonlinear function. This system comes from nonlinear transient simulation of frequency-dependent nonuniform coupled lossy transmission lines. If \( M(\omega) \) is nonsingular, then (1) is an ordinary differential equation (ODE) system. If \( M(\omega) \) is singular, we then have a differential-algebraic equation (DAE) system, whose behaviour is more complicated than ODE system. In the paper, we consider only the case that \( M(\omega) \) is nonsingular for \( \omega \).

The general form of WR for (1) is given as

\[
\begin{align*}
M_1(\omega) \frac{dy^{(k+1)}(x, \omega)}{dx} &= M_2(\omega) \frac{dy^{(k)}(x, \omega)}{dx} - A_1(\omega) y^{(k+1)}(x, \omega) + A_2(\omega) y^{(k)}(x, \omega) \\
& \quad - \tilde{u}(y^{(k+1)}(x, \omega), y^{(k)}(x, \omega)) + f(x, \omega) \\
y^{(k+1)}(0, \omega) &= y_0(\omega), \quad k = 0, 1, 2, \ldots
\end{align*}
\]

... (2)

where \( M(\omega) = M_1(\omega) - M_2(\omega), A(\omega) = A_1(\omega) - A_2(\omega), y^0(-, \omega) \) is a given initial iteration. Usually one can take \( y^0(x, \omega) = y_0(\omega) \), if nothing better is available. The nonlinear splitting function \( \tilde{u}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies \( g(y) = \tilde{u}(y, y) \). The choice of the splitting matrices depends on classical relaxation methods. For example, in the waveform Jacobi (WJAC) method, \( M_1(\omega) \) and \( A_1(\omega) \) are the block diagonal parts of \( M(\omega) \) and \( A(\omega) \) respectively; in the waveform Gauss-Seidel (WGS) method, \( M_1(\omega) \) and \( A_1(\omega) \) are the block lower triangular parts of \( M(\omega) \) and \( A(\omega) \) respectively. We now consider convergence conditions of WR in small interval and large interval respectively.

2.1. SMALL INTERVAL

We first give a assumption on the splitting function \( \tilde{u} \) appearing in (2).

\textbf{Assumption 1} — There are constants \( \alpha_i (i = 1, 2) \) such that

\[
\| \tilde{u}(\nu_1, \nu_2) - \tilde{u}(\hat{\nu}_1, \hat{\nu}_2) \| \leq \sum_{i=1}^{2} \alpha_i \| \nu_i - \hat{\nu}_i \|. 
\]


Let \( M_1 (\omega) \) be invertible, then we can write (2) in another form as follows

\[
\begin{align*}
\frac{d y^{(k+1)}(x, \omega)}{dx} = & \quad M_1^{-1} (\omega) M_2 (\omega) \frac{d y^{(k)}(x, \omega)}{dx} - M_1^{-1} (\omega) A_1 (\omega) y^{(k+1)}(x, \omega) \\
& \quad + M_1^{-1} (\omega) A_2 (\omega) y^{(k)}(x, \omega) \\
& \quad - M_1^{-1} (\omega) \tilde{u} (y^{(k+1)}(x, \omega), y^{(k)}(x, \omega)) + M_1^{-1} (\omega) f(x, \omega),
\end{align*}
\]

\( y^{(k+1)}(0, \omega) = y_0(\omega), \quad k = 0, 1, 2, \ldots, \) \hspace{1cm} ... (3)

Now, we let

\[
h(x, v_1, v_2, v_3) = M_1^{-1} (\omega) M_2 (\omega) v_3 - M_1^{-1} (\omega) A_1 (\omega) v_1 \\
+ M_1^{-1} (\omega) A_2 (\omega) v_2 - (\omega) \tilde{u}(v_1, v_2) + M_1^{-1} (\omega) f(x, \omega).
\]

Thus, (3) may be described as

\[
\dot{y} = h(x, y, \dot{y}).
\]

Denoting \( z(x, \omega) = \dot{y}(x, \omega) \) for \( x \in [0, l] \), we have \( y(x, \omega) = (Jz)(x, \omega) \) where

\[
(Jz)(x, \omega) = y_0(\omega) + \int_0^x z(s, \omega) \, ds.
\]

In other words, (3) can be rewritten as

\[
z(x, \omega) = h(x, (Jz)(x, \omega), (Jz)(x, \omega), z(x, \omega)).
\]

Therefore, the solution of (3) may be regarded as the solution of the fixed-point equation in \( C([0, l]; \mathbb{R}^n) \) as

\[
z = h(z, z)
\]

for any fixed \( \omega \). Based on Assumption 1, we can find that

\[
\| h(x, v_1, v_2, v_3) - h(x, \hat{v}_1, \hat{v}_2, \hat{v}_3) \| \leq \left( \| M_1^{-1} (\omega) A_1 (\omega) \| + \alpha_1 \right) \| v_1 - \hat{v}_1 \| \\
+ \left( \| M_1^{-1} (\omega) A_2 (\omega) \| + \alpha_2 \right) \| v_2 - \hat{v}_2 \| \\
+ \| M_1^{-1} (\omega) M_2 (\omega) \| \| v_3 - \hat{v}_3 \|.
\]
Recall that the idea and the main results in [5], we can easily conclude that
\[ \left\| M_1^{-1}(\omega) M_2(\omega) \right\| < 1 \] is a convergence condition of the WR algorithm (3).

If \( g(y(x, \omega)) = 0 \), then we may know from (6) that if \( M_1(\omega) \) is a nonsingular matrix, then the convergence condition of the iterative method is that the spectral radius of \( M_1^{-1}(\omega) M_2(\omega) \) is less than one. However, if \( M(\omega) \) is singular and \( M_1(\omega) \) is nonsingular, then its convergence cannot be ensured. This is because \( \rho\left(M_1^{-1}(\omega) M_2(\omega)\right) \geq 1 \) by \( \det\left(I - M_1^{-1}(\omega) M_2(\omega)\right) \geq 1 = \det M_1^{-1}(\omega) \det M(\omega) = 0. \)

**Lemma (14)** — For \( A \in \mathbb{C}^{n \times n} \) and any fixed positive number \( \varepsilon \), there exists a matrix norm \( \| \cdot \|_M \) such that
\[ \rho(A) \leq \| A \|_M \leq \rho(A) + \varepsilon. \]

Now, we also let \( M(\omega) = \{ m_{ij}(\omega) \}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n \). The following theorem can be proved.

**Theorem 1** — If \( M(\omega) \) is a normal matrix, \( | m_{ii}(\omega) | \geq 1 \), and \( \rho(M_2(\omega)) < 1 \), then (2) is convergent for the WJAC splitting.

**Proof:** Applying WJAC for (2) we know that
\[ M_1(\omega) = \text{diag} \{ m_{11}(\omega), \ldots, m_{nn}(\omega) \}, \quad M_2(\omega) = M(\omega) - M_1(\omega). \]

Since \( M(\omega) \) is normal, namely \( M^T(\omega) M(\omega) = M(\omega) M^T(\omega) \), then \( M_2(\omega) \) is normal too. Obviously, \( M_1(\omega) \) is also normal. Further, for any normal matrix \( N \), we have \( \| N \|_2 = \rho(N) \) where \( \| \cdot \|_2 \) is the 2-norm (see (15)). Therefore, we know
\[
\left\| M_1^{-1}(\omega) M_2(\omega) \right\|_2 \leq \left\| M_1^{-1}(\omega) \right\|_2 \left\| M_2(\omega) \right\|_2
\]
\[
= \max_{1 \leq i \leq n} \frac{1}{| m_{ii}(\omega) |} \rho(M_2(\omega)) < 1.
\]

This completes the proof of Theorem 1.

**Theorem 2** — If \( \rho\left(M_1^{-1}(\omega)\right) < 1 \), then (2) is convergent for the WGS splitting.
PROOF: Adopting WGS for (2), we obtain

\[
M_1(\omega) = \begin{pmatrix}
  m_{11}(\omega) \\
  m_{21}(\omega) & m_{22}(\omega) \\
  \vdots & \ddots & \ddots \\
  \vdots & \ddots & \ddots \\
  m_{n1}(\omega) & \cdots & m_{n(n-1)}(\omega) & m_{nn}(\omega)
\end{pmatrix},
\]

\[
M_2(\omega) = \begin{pmatrix}
  0 & -m_{12}(\omega) & \cdots & -m_{1n}(\omega) \\
  0 & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & -m_{2n}(\omega)
\end{pmatrix}.
\]

By virtue of \( \rho\left(M_1^{-1}(\omega)\right) < 1 \) and \( \rho(M_2(\omega)) = 0 \), we can find a fixed positive number \( \varepsilon \) such that \( \rho\left(M_1^{-1}(\omega)\right) + \varepsilon < 1 \) and \( \rho(M_2(\omega)) + \varepsilon < 1 \). From Lemma 1, there exists an equivalent norm \( || \cdot ||_M \) in \( \mathbb{C}^{n \times n} \) such that \( \left\| M_1^{-1}(\omega) \right\|_M \leq \rho\left(M_1^{-1}(\omega)\right) + \varepsilon \) and \( ||M_2(\omega)||_M \leq \rho(M_2(\omega)) + \varepsilon \). Hence, it follows

\[
\left\| M_1^{-1}(\omega) M_2(\omega) \right\|_M \leq \left\| M_1^{-1}(\omega) \right\|_M \left\| M_2(\omega) \right\|_M \leq \left( \rho\left(M_1^{-1}(\omega)\right) + \varepsilon \right) \varepsilon < 1.
\]

This completes the proof of Theorem 2.

2.2. LARGE INTERVAL

In practice we would not usually iterate on \([0, \infty)\), we are more interesting in iterating on \([0, T]\), but the infinitely long window is a natural setup for a stiff problem and for a DAE system: for fast transients a long interval \([0, l]\) should be regarded as infinitely long. So, we regard \([0, l]\), in which \( l(>0) \) is large, as \([0, \infty]\). In this section, we consider only the linear case, that is, \( g(y(x, \omega)) = 0 \) in (1). Namely,
\[
\begin{aligned}
M(\omega) \frac{dy(x, \omega)}{dx} + A(\omega) y(x, \omega) &= f(x, \omega), \\
y(x_0, \omega) &= y_0(\omega).
\end{aligned}
\] ...
(4)

The general form of WR for (4) is
\[
\begin{aligned}
M_1(\omega) \frac{dy^{(k+1)}(x, \omega)}{dx} + A_1(\omega) y^{(k+1)}(x, \omega) &= M_2(\omega) \frac{dy^{(k)}(x, \omega)}{dx} \\
&+ A_2(\omega) y^{(k)}(x, \omega) + f(x, \omega)
\end{aligned}
\]
...
(5)
\[
y^{(k)}(x_0, \omega) = y_0(\omega),
\]

where \( M(\omega) = M_1(\omega) - M_2(\omega), A(\omega) = A_1(\omega) - A_2(\omega), \) and \( M_1(\omega) \) is nonsingular. For the purpose at hand, the iteration process (5) will be written as an operator iteration scheme. For \( \nu \in L^2([0, l], C^n) \), we define an operator \( \mathbf{K}_1 \) as follows,

\[
(\mathbf{K}_1 \nu)(x, \omega) = \int_0^x \left( M_1^{-1}(\omega) A_1(\omega) - I \right) e^{-M_1^{-1}(\omega) A_1(\omega)(x-s)} M_1^{-1}(\omega) A_2(\omega) \nu(x, \omega) ds, x \in [0, l].
\]

We also let

\[
\varphi(x, \omega) = e^{-M_1^{-1}(\omega) A_1(\omega)x} \left( I - M_1^{-1}(\omega) M_2(\omega) \right)x_0
\]

\[
+ \int_0^x e^{-M_1^{-1}(\omega) A_1(\omega)(x-s)} M_1^{-1}(\omega) f(s, \omega) ds.
\]

Thus, we can write (5) as

\[
y^{(k)}(x, \omega) = \mathbf{K} y^{(k-1)}(x, \omega) + \varphi, \quad k = 1, 2, ...
\]

where \( \mathbf{K} = M_1^{-1}(\omega) M_2(\omega) + \mathbf{K}_1 \). This operator \( \mathbf{K} : L^2([0, l], C^n) \to L^2([0, l], C^n) \) is bounded and is called a waveform relaxation operator. We can easily prove that \( \rho(\mathbf{K}_1) = 0 \) using a similar method to (6). So, \( \rho(\mathbf{K}) = \rho\left(M_1^{-1}(\omega) M_2(\omega)\right) \). Further, we conclude that the iterative process (5) is convergent if \( \rho\left(M_1^{-1}(\omega) M_2(\omega)\right) < 1 \). Of course, from the point of view of scientific computation,
the pseudospectra of operators are more reliable than the spectra of non-normal operators (16). Waveform relaxation operator on a finite interval is often non-normal.

In this section, we define an infinite interval waveform relaxation operator \( \mathcal{R}_\infty : L^2 ([0, \infty)), C^0 \rightarrow L^2 ([0, \infty), C^0) \). The expressions of the operators \( \mathcal{R}_\infty \) and \( \mathcal{R}_L \), where \( \mathcal{R}_L : L^2 ([0, l]), C^0 \rightarrow L^2 ([0, l), C^0) \), are the same as the above operator \( \mathcal{R} \). Taking the Laplace transform on (5), we obtain

\[
\hat{y}^{(k)} (x, \omega) = (M_1 (\omega) z + A_1 (\omega))^{-1} (M_2 (\omega) z + A_2 (\omega)) \hat{y}^{(k-1)} (x, \omega)
+ (M_1 (\omega) z + A_1 (\omega))^{-1} \hat{f}(\omega).
\]

Here, \( (M_1 (\omega) z + A_1 (\omega))^{-1} (M_2 (\omega) z + A_2 (\omega)) \) is the symbol of \( \mathcal{R}_L \) and \( \mathcal{R}_\infty \), denoted by \( K(z) \). In order to seek the spectral radius of the operator \( \mathcal{R}_\infty \), we should first give out the spectrum of \( \mathcal{R}_\infty \) if it is bounded. Without loss of generality, we assume in the section that \( M_1^{-1} (\omega) A_1 (\omega) \) has eigenvalues with positive real parts, thus \( \mathcal{R}_\infty \) is a bounded operator in \( L^2 ([0, \infty), C^0) \). We can obtain the following conclusions by applying a similar approach in (3,6):

a) If \( M_1^{-1} (\omega) A_1 (\omega) \) has eigenvalues with positive real parts, then \( \mathcal{R}_\infty \) has spectrum

\[
\sigma(\mathcal{R}_\infty) = \bigcup_{R(\xi) \geq 0} \sigma(K(i\xi)), \xi \in \mathcal{R}.
\]

b) \( \rho(\mathcal{R}_\infty) = \max \{ \rho(K(i\xi)), \xi \in \mathcal{R} \} \).

The conclusions of (a) and (b) are strictly valid only if the infinite time interval is considered. In the same way, the iterative method (5) for \( x \in [0, \infty) \) is convergent if \( \rho(\mathcal{R}_\infty) < 1 \).

3. INITIAL WAVEFORM GUESS BASED ON COLLOCATION

We can know from (2) that the convergence rate of Jacobi waveforms may be far from linear in the first many iterations. Hence, the usual choice of constant initial waveforms should be improved. We adopt the Chebyshev collocation method to get these initial waveforms.

3.1. SMALL INTERVAL

In a small interval, we choose \( T_k (x) = \cos(k \rho \text{ccos}(x)) \) (\( k = 0, 1, 2, \ldots \)) as base functions. So, the exact solution \( y(x, \omega) \) of (1) is approximated by \( p(x, \omega) \) which is obtained by
\[ p(x, \omega) = \sum_{k=0}^{N} b_k(\omega) T_k(x) \quad \ldots \quad (6) \]

We perform a variable transform on (1), \( x = l(\xi + 1)/2 \), then (1) may be rewritten as

\[ \frac{2/l M(\omega)}{d\xi} \left( l \left( \frac{\xi + 1}{2} \right) \right)^{2} + A(\omega) p(l(\xi + 1)/2, \omega) \]

\[ + g(p(x, \omega)) = f(l(\xi + 1)/2, \omega), \quad \xi \in [-1, 1]. \quad \ldots \quad (7) \]

Substituting \( p(l(\xi + 1)/2, \omega) \) with \( \sum_{k=0}^{N} b_k(\omega) T_k(\xi) \), the coefficient vectors \( b_k(\omega) \in \mathbb{R}^n \) for all \( k \) are determined in the following way. We enforce

\[ \frac{2/l M(\omega)}{d\xi} \sum_{k=0}^{N} b_k(\omega) \dot{T}_k(\xi_j) + A(\omega) \sum_{k=0}^{N} b_k(\omega) T_k(\xi_j) \]

\[ + g\left( \sum_{k=0}^{N} b_k(\omega) T_k(\xi_j) \right) = f(l(\xi_j + 1)/2, \omega), \quad \ldots \quad (8) \]

where \( \xi_j = \cos(2j\pi/(2N+1)), j = 1, 2, \ldots N \). We also impose the initial condition

\[ y(0, \omega) = \sum_{k=0}^{N} b_k(\omega) T_k(-1) \quad \ldots \quad (9) \]

thus, (8) and (9) become nonlinear algebraic equations on the unknown coefficient vectors \( b_k(\omega) \).

We can solve \( b_k \) by applying a conventional method, and get an initial waveform guess for (2). By virtue of \( M(\omega) \) is nonsingular, furthermore, (8) may be written as follows

\[ \sum_{k=0}^{N} b_k \dot{T}_k(\xi_j) + l/2 M^{-1}(\omega) A(\omega) \sum_{k=0}^{N} b_k(\omega) T_k(\xi_j) \]

\[ + l/2 M^{-1}(\omega) g\left( \sum_{k=0}^{N} b_k \dot{T}_k(\xi_j) \right) \]

\[ = l/2 M^{-1}(\omega) f(l(\xi_j + 1)/2, \omega), \quad j = 1, 2, \ldots, N. \quad \ldots \quad (10) \]

Recall that Kronecker product of two matrices \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{l \times k} \), denoted by \( A \otimes B \), is a large matrix with the size of \((ml) \times (nk)\) formed by
$$A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B 
\end{pmatrix}.$$  

where $a_{ij}$ is the $(i, j)$ element of $A$. Furthermore, by invoking of the following fact as

$$\sum_{k=0}^{N} b_{k}(\omega) \hat{T}_{k}(\xi_j) = \sum_{k=0}^{N} \hat{T}_{k}(\xi_j) b_{k}(\omega)$$

$$= \begin{pmatrix} b_{0}(\omega) \\
b_{1}(\omega) \\
\vdots \\
b_{N}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} T_{0}(\xi_j), T_{1}(\xi_j), \ldots, T_{N}(\xi_j) \end{pmatrix}$$

$$\sum_{k=0}^{N} b_{k}(\omega) T_{k}(\xi_j) = \sum_{k=0}^{N} T_{k}(\xi_j) b_{k}(\omega)$$

$$= \begin{pmatrix} b_{0}(\omega) \\
b_{1}(\omega) \\
\vdots \\
b_{N}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} T_{0}(\xi_j), T_{1}(\xi_j), \ldots, T_{N}(\xi_j) \end{pmatrix}.$$  

we can also write (9) and (10) as follows
\[
(E \otimes I_n + F \otimes (1/2 M^{-1}(\omega) A(\omega)) \begin{pmatrix}
 b_0(\omega) \\
 b_1(\omega) \\
 \vdots \\
 b_{N-1}(\omega) \\
 b_N(\omega)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
 0 \\
 1/2 M^{-1}(\omega) \left( \sum_{k=0}^{N} b_k \dot{T}_k(\xi_1) \right) \\
 1/2 M^{-1}(\omega) \left( \sum_{k=0}^{N} b_k \dot{T}_k(\xi_{N-1}) \right) \\
 1/2 M^{-1}(\omega) \left( \sum_{k=0}^{N} b_k \dot{T}_k(\xi_N) \right)
\end{pmatrix}
\]

\[
y(0, \omega)
\]

\[
= \begin{pmatrix}
 y(0, \omega) \\
 1/2 M^{-1}(\omega) f(\xi_1, \omega) \\
 \vdots \\
 1/2 M^{-1}(\omega) f(\xi_{N-1}, \omega) \\
 1/2 M^{-1}(\omega) f(\xi_N, \omega)
\end{pmatrix}, \quad \ldots \quad (11)
\]

where
$$E = \begin{pmatrix}
T_0 (-1) & T_1 (-1) & \cdots & T_N (-1) \\
\dot{T}_0 (\xi_1) & \dot{T}_1 (\xi_1) & \cdots & \dot{T}_N (\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
\dot{T}_0 (\xi_{N-1}) & \dot{T}_1 (\xi_{N-1}) & \cdots & \dot{T}_N (\xi_{N-1}) \\
\dot{T}_0 (\xi_N) & \dot{T}_1 (\xi_N) & \cdots & \dot{T}_N (\xi_N)
\end{pmatrix}$$

and

$$F = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
T_0 (\xi_1) & T_1 (\xi_1) & \cdots & T_N (\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
T_0 (\xi_{N-1}) & T_1 (\xi_{N-1}) & \cdots & T_N (\xi_{N-1}) \\
T_0 (\xi_N) & T_1 (\xi_N) & \cdots & T_N (\xi_N)
\end{pmatrix},$$

in which $E$ and $F$ are $(N+1) \times (N+1)$ matrices, $I_n \in \mathbb{R}^{n \times n}$ is an identity matrix. By the definition of $T_k (x) = \cos (\kappa r c o s x)$, we know $T_k (-1) = (-1)^k, T_k (1) = 1$. Recall that, see also\textsuperscript{17,18},

$$T_0 (x) = 1, \quad T_1 (x) = x,$$

$$T_{k+1} (x) = 2x T_k (x) - T_{k-1} (x), \quad k = 1, 2, \ldots,$$

\begin{equation}
T_{2k} (x) = 4k \sum_{m=0}^{k-1} T_{2m+1} (x), \quad \ldots \quad (12)
\end{equation}

and

$$T_{2k+1} (x) = (2k+1) T_0 (x) + 2(2k+1) \sum_{m=1}^{k} T_{2m} (x), \quad \ldots \quad (13)$$

By solving the nonlinear algebraic eqs. (11), we obtain $b_k (\omega)$ for all $k$. Furthermore, we can get the initial waveform guess of (2).

Let $y^* (x, \omega)$ be the exact solution of (1), from\textsuperscript{17}, we may know that for any $y^* (l (\xi + 1)/2, \omega) \in H^\sigma_\chi [-1, 1], \sigma \geq 1$ it holds
\[ \left\| y^* \left( l \frac{(\xi + 1)}{2}, \omega \right) \right\|_{l, \chi} - \left\| y^0 \left( l \frac{(\xi + 1)}{2}, \omega \right) \right\|_{l, \chi} \leq cN^{2l - \sigma} \left\| y^* (\omega) \right\|_{\sigma, \chi}, \quad 0 \leq l \leq \sigma, \]

where \( N \) is the number of collect points, \( \chi (x) = (1 - x^2)^{-1/2} \).

### 3.2. LARGE INTERVAL

In this case, we adopt the same way in the previous section to produce the initial waveforms. Now, \( T_k (x) \) \((k = 1, 2, \ldots)\) are not suitable as basis functions for \([0, \infty)\). In function space, it is well known that we often need to use orthogonal functions as a basis to approximate exact solution of a complex system, see also (19). Now, we will induce the Chebyshev rational functions \( R_l (x) \) \((l = 1, 2, \ldots)\) as basis functions. In (9), these functions \( R_l (x) \) are defined by

\[
R_l (x) = T_l \left( \frac{x - 1}{x + 1} \right), \quad l = 1, 2, \ldots.
\]

We know that the Chebyshev rational functions \( R_l (x) \) construct a set of orthogonal basis for \( L^2_w [0, \infty) \) where \( w (x) = \frac{1}{(x + 1) \sqrt{x}} \). From the recurrence relation (12), the functions \( R_l (x) \) for all satisfied the following recurrence formula,

\[
R_0 (x) = 1, \quad R_1 (x) = \frac{x - 1}{x + 1},
\]

\[
R_{l+1} (x) = 2 \frac{x - 1}{x + 1} R_1 (x) - R_{l-1} (x), \quad l \geq 1.
\]

Further, due to (13), we also get

\[
\hat{R}_{2k} (x) = 4k \frac{2}{(x + 1)^2} \sum_{m=0}^{k-1} R_{2m+1} (x),
\]

\[
\hat{R}_{2k+1} (x) = (2k + 1) \frac{2}{(x + 1)^2} \left( R_0 (x) + 2 \sum_{m=1}^{k} R_{2m} (x) \right).
\]

Since the interval \([0, \infty)\) is considered in this section, as for the solution \( u (x, \omega) \) of (1), so we will approximate it by the rational Chebyshev collocation method. That is,

\[
p (x, \omega) = \sum_{k=0}^{N} c_k (\omega) R_k (x), \quad \ldots (15)
\]
where \( c_k(\omega) \in \mathbb{R}^n \). For the sake of determining coefficient vectors, we substitute (15) into (1) and impose

\[
\sum_{k=0}^{N} c_k(\omega) \hat{R}_k(\delta_i) + M^{-1}(\omega) A(\omega) \sum_{k=0}^{N} c_k(\omega) R_k(\delta_i)
\]

\[+ M^{-1}(\omega) g(c_k(\omega) R_k(\delta_i)) \]

\[= M^{-1}(\omega) f(\delta_i, \omega), \quad i = 1, 2, ..., N, \quad \ldots \quad (16)\]

and

\[y(0, \omega) = \sum_{k=0}^{N} c_k(\omega) R_k(0), \quad \ldots \quad (17)\]

where \( \delta_i = (1 + \xi_i) (1 + \xi_i)^{-1} \), in which \( \xi_i = \cos \frac{2j\pi}{2N+1} \). Similar to the previous section, we now write (16) and (17) as follows

\[
(G \otimes I_n + H \otimes M^{-1} A) \begin{pmatrix} c_0(\omega) \\ c_1(\omega) \\ \vdots \\ c_N(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ M^{-1} g(c_k(\omega) R_k(\delta_1)) \\ \vdots \\ M^{-1} g(c_k(\omega) R_k(\delta_N)) \end{pmatrix} y(0, \omega) \]

\[M^{-1}(\omega) f(\delta_1, \omega) \]

\[M^{-1}(\omega) f(\delta_N, \omega) \]

, \ldots \quad (18)

where
\[
G = \begin{pmatrix}
R_0(0) & R_1(0) & \ldots & R_N(0) \\
\dot{R}_0(\delta_1) & \dot{R}_1(\delta_1) & \ldots & \dot{R}_N(\delta_1) \\
& \ddots & \ddots & \ddots \\
& & \dot{R}_0(\delta_N) & \dot{R}_1(\delta_N) & \ldots & \dot{R}_N(\delta_N)
\end{pmatrix}
\]

and

\[
H = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
R_0(\delta_1) & R_1(\delta_1) & \ldots & R_N(\delta_1) \\
& \ddots & \ddots & \ddots \\
& & R_0(\delta_N) & R_1(\delta_N) & \ldots & R_N(\delta_N)
\end{pmatrix},
\]

in which \(G\) and \(H\) are \((N+1) \times (N+1)\) matrices. Accordingly, the coefficient vectors \(c_k\) are achieved by (18). The initial waveform of (2), \(y^0(x, \omega) = \sum_{k=0}^{N} c_k(\omega) R_k(x)\), is produced for \(x \in [0, \infty)\).

Let \(y^*(x, \omega)\) be the exact solution of (1), from\(^9\) we also know that for any \(y^*(x, \omega) \in H_{\chi, A}^r[0, \infty)\) and \(0 \leq \mu \leq 1 \leq r\), it holds

\[
\| y^0(x, \omega) - y^*(x, \omega) \|_{\mu, \chi} \leq c N^{2 \mu - r} \| y^0(\omega) \|_{r, \chi, A},
\]

where \(H_{\chi, A}^r[0, \infty) = \{ v \mid v \text{ is measurable and } \| v \|_{r, \chi, A} < \infty \}\), \(N\) is the number of collect points, \(\chi(x) = 1/(x + 1 \sqrt{x})\), and

\[
\| v \|_{r, \chi, A} = \left( \sum_{k=0}^{r} \left( (x+1)^{\gamma/2 + k} \partial_x^k v \right)^2 \chi \right)^{1/2}.
\]

4. MODEL-REDUCTION OF ITERATIVE EQUATIONS

Computational expense significantly increases as the scale of systems increases, so a great deal of attention has been paid to reduction-model. WR can simplify original systems, and it is suitable for parallel processing. But it does not reduce the dimension of systems. Therefore, in order to reduce the computational expense, we should also reduce the order of iterative systems. In (12), an algorithm based on Krylov subspace process is proposed for model-reduction of large nonlinear circuits. Making
use of this algorithm, we can find that the first ‘q’ derivatives of the time-response of the original
system are identical with those of the response obtained from the reduced system.

In the kth iterative system of (2), we let

\[ v(x, \omega) = y^{(k+1)}(x, \omega), \]

\[ \varphi(x, \omega) = -M_2(\omega) \frac{\partial y^{(k)}(x, \omega)}{\partial x} - A_2(\omega) y^{(k)}(x, \omega) - f(x, \omega). \]

Thus, (2) can be rewritten as

\[ M_1(\omega) \dot{v}(x, \omega) A_1(\omega) v(x, \omega) + \tilde{u}(v(x, \omega)) + \varphi(x, \omega) = 0. \quad \text{... (19)} \]

Let us expand \( v(x, \omega) \) of (19) in Taylor series' form as

\[ v(x, \omega) = \sum_{k=0} a_k(\omega) (x-x_0)^k, \]

where \( a_0(\omega) = y_0(\omega), \) and \( a_k(\omega) = v^{(k)}(x_0, \omega)/k! \) \( (k = 1, 2, \ldots) \) are the normalized spatial domain
derivatives of \( v, \) and they are computed by

\[ (k+1) M_1(\omega) a_{k+1}(\omega) + A_1(\omega) a_k(\omega) + \tilde{u}_k(\omega) = \varphi_k(\omega)/k!, \quad \text{... (20)} \]

where \( \tilde{u}_i(\omega) \) and \( \varphi_i(\omega) \) denote the \( i \)th derivatives of \( \tilde{u}(v(x, \omega)) \) and \( \varphi(x, \omega) \) evaluated at \( x=x_0, \)
and

\[ \tilde{u}_k = \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-j-1)!} (j+1) \zeta^{k-j-1} a_{k+1}, \quad k \geq 1, \]

in which \( \zeta = \tilde{u}(v(x, \omega)) \bigg|_{v(x_0)}. \)

The iterative system (2) is now reduced to a smaller set of unknown vectors through a
congruent transformation obtained from the Krylov subspace \( K. \) The subspace \( K \) is formed by
the derivatives computed in (20), and it is defined as

\[ K = [a_0(\omega), a_1(\omega), a_2(\omega), \ldots, a_q(\omega)], \]

where \( q \) is the order of reduction and \( q \ll n. \) Thus, carrying out an orthogonal decomposition on
\( K \) we have

\[ K = Q(\omega) R(\omega), \]

where \( Q(\omega) \) is an \( n \times q \) matrix, and \( Q(\omega)^T Q(\omega) = I_q, \) in which \( I_q \in \mathbb{R}^{q \times q} \) is an identity matrix.
For a given \( k, \) we do a congruent transformation on (2), that is

\[ y^{(k+1)}(x, \omega) = Q(\omega) \rho(x, \omega) \quad \text{... (21)} \]
where \( \rho(x, \omega) \in \mathbb{R}^q \). This variables' transform reduces the iterative system to a system with a lower dimension as follows

\[
\begin{align*}
\bar{M}_1(\omega) \frac{d\rho(x, \omega)}{dx} + \bar{A}_1(\omega) \rho(x, \omega) + \sigma(x, \omega) &= \bar{\Phi}(x, \omega), \quad \cdots \ (22)
\end{align*}
\]

where

\[
\begin{align*}
\bar{M}_1(\omega) &= Q^T(\omega) M_1(\omega) Q(\omega), \\
\bar{A}_1(\omega) &= Q^T(\omega) A_1(\omega) Q(\omega), \\
\sigma(x, \omega) &= \tilde{u}(Q \rho(x, \omega)), \quad \bar{\Phi}(x, \omega) = Q^T(\omega) \varphi(x, \omega),
\end{align*}
\]

in which \( \bar{M}_1(\omega) \) and \( \bar{A}_1(\omega) \) are \( q \times q \) matrices and depend on \( \omega \), and \( \bar{\Phi}(x, \omega), \sigma(x, \omega) \in \mathbb{R}^q \).

Because of (21) and \( Q(\omega)^T Q(\omega) = I_q \), we also have \( \rho(0, \omega) = Q^T(\omega) y_0(\omega) \). Thus, we can conveniently solve the small system (22) instead of solving the large system (2).

5. An Illustrative Example

We consider an initial problem of multiconductor transmission lines as follows

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial x} &= -Ri(x, t) - L \frac{\partial i(x, t)}{\partial t}, \\
\frac{\partial i(x, t)}{\partial x} &= -Gi(x, t) - C \frac{\partial u(x, t)}{\partial t}, \\
u(0, t) &= g_1(t), \quad i(0, t) = g_2(t), \\
u(x, 0) &= r_1(x), \quad i(x, 0) = r_2(x),
\end{align*}
\]

where \( i, u \in \mathbb{R}^2 \). Further, \( R, G, L, C \) are resistance, conductance, inductance, and capacitance. They are four matrices of \( 2 \times 2 \). Here, \( x \in [0, 1], t \in [0, 3], \) and \( g_i(t) (i = 1, 2) \) are known input functions.

Taking the Laplace transforms with respect to \( t \) on (23), we write

\[
\begin{align*}
\frac{\partial V(x, s)}{\partial x} &= -(R + sL) I(x, s), \quad \cdots \ (24) \\
\frac{\partial I(x, s)}{\partial x} &= -(G + sC) V(x, s), \quad \cdots \ (25)
\end{align*}
\]

where
\[ R = \begin{pmatrix} 7.7 & 1.1 \\ 1.1 & 7.7 \end{pmatrix}, \quad L = \begin{pmatrix} 4.7 & 2.6 \\ 2.6 & 4.7 \end{pmatrix}, \]
\[ C = \begin{pmatrix} 6.2 & -3.3 \\ -3.3 & 6.2 \end{pmatrix}, \quad G = \begin{pmatrix} 9.7 & -1.5 \\ -1.5 & 9.7 \end{pmatrix}. \]

Let \( g_1(t) = (0.8, 0.8)^T \), \( g_2(t) = (0.5, 0.5)^T \), \( r_1(x) = (1.0, 1.0)^T \), and \( r_2(x) = (0.6, 0.6)^T \).

Thus, (24) and (25) can be further written

\[
\frac{d}{dx} \begin{pmatrix} V_1(x, s) \\ V_2(x, s) \\ I_1(x, s) \\ I_2(x, s) \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 7.7 + 4.7s & 1.1 + 2.6s \\ 0 & 0 & 1.1 + 2.6s & 7.7 + 4.7s \\ 6.2 + 9.7s & -3.3 - 1.5s & 0 & 0 \\ -3.3 - 1.5s & 6.2 + 9.7s & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1(x, s) \\ V_2(x, s) \\ I_1(x, s) \\ I_2(x, s) \end{pmatrix}, \quad \ldots \text{(26)}
\]

\( V(0, \omega) = (0.8/\omega, 0.8/\omega)^T, \quad I(0, \omega) = (0.5/\omega, 0.5/\omega)^T, \quad V(x, 0) = (1.0, 1.0)^T, \quad I(x, 0) = (0.6, 0.6)^T. \)

Comparing (26) with (1), we have

\[ M = I, A(\omega) = \begin{pmatrix} 0 & 0 & 7.7 + 4.7\omega & 1.1 + 2.6\omega \\ 0 & 0 & 1.1 + 2.6\omega & 7.7 + 4.7\omega \\ 6.2 + 9.7\omega & -3.3 - 1.5\omega & 0 & 0 \\ -3.3 - 1.5\omega & 6.2 + 9.7\omega & 0 & 0 \end{pmatrix}, \]

\[ g(y(x, \omega)) = 0, \quad f(x, \omega) = 0. \]

Now, no matter adopting any classic splitting, for example, WJAC or WGS for (26), it is
easy to know that \( \rho(M_1^{-1}M_2) < 1 \) where \( M = M_1 - M_2 \). That is, the corresponding WR algorithm is convergent for the above model problem. Let \( N = 2 \). We use the backward Euler method as the basic solver for ODEs and the space step is 0.01. We define "error" as sum of the squared differences of successive waveform taken over all time points. The error tolerance value is \( 10^{-6} \). The experimental results for the two cases (\( \omega = 0 \) and \( \omega = -2 \)) of the WR Gauss-Seidel method with different initial value are drawn in Fig. 1 and Fig. 2, where, solid line: choose constant initial value;

dashed line: choose the initial guess obtained by applying the proposed method. Two figures clearly indicate that the WR algorithm is convergent and the proposed method to get initial waveform guess is efficient.
6. CONCLUSION

In this paper we establish some convergence criteria for waveform relaxation (WR) method for nonlinear initial value problem (1) on small windows; By rewriting the WR method as an operator iteration scheme, we discuss the convergence of WR method for linear initial value problem (4) on infinite interval \([0, \infty)\) and obtain more general results than previous results in (3&6). In order to increase the convergence rate of the overall WR method, we address the question of how to choose the initial guess: using the Chebyshev collocation approach to get it on the small interval and using the Chebyshev rational functions approach (see (9)). The question is of great practical importance. By reason of the dimension of multiconductor transmission line system is often very large, we adopt model-reduction for iterative systems to reduce computational cost.

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REFERENCES


