MEASURES WITH $\tau$-SMOOTH MARGINALS

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An example is given of two $\tau$-smooth positive measures $\mu_1, \mu_2$ on completely regular Hausdorff spaces $X_1, X_2$, and a net $(\nu_\alpha)$ of $\tau$-smooth positive measures on $X_1 \times X_2$ such that the marginals of $(\nu_\alpha)$ converge to $\mu_1, \mu_2$ but $(\nu_\alpha)$ converges to a non-$\tau$-smooth measure. It is proved that if $\mu_1$ is tight then $\nu_\alpha$ converges to a $\tau$-smooth measure. As a consequence, a Strassen type theorem is proved for $\tau$-smooth measures.

Keywords: Measure; Finitely Additive Measures; $\tau$-Smooth Measures; Marginal of Measures; Lifting of Measures; Lifting Topology

In this note, all vector spaces are over the field of real numbers $R$ (we will call them scalars). For a completely regular Hausdorff space $X$, the following notations will be used:

$C(X)(C_b(X))$ will denote the space of all (all bounded) $R$-valued continuous functions on $X$,

elements of $\{ f^{-1}(0) \mid Lf \in C_b(X) \}$ will be called zero-sets of $X$ and their complements the positive-sets of $X$,

$\mathcal{F}(X)$ will be algebra generated by zero-sets,

$M(X) = (C_b(X), \| \cdot \|)'$ is the lattice of bounded finitely additive measures on $\mathcal{F}(X)$, inner regular by zero-sets and outer regular by positive-sets of $X$; $M^+(X)$ denotes the positive elements of $M(X)$,

$M_\tau(X)$ are scalar-valued, countably additive Borel measures $\mu$ on $X$, which are inner regular by closed sets and outer regular by open sets of $X$, and for any increasing net of open sets $\{ V_\alpha \}$ $\subset X$ with $\bigcup V_\alpha = V$, we have $\lim \mu(V_\alpha) = \mu(V)$; $M^+_\tau(X)$ denotes the positive elements of these measures. These measures are called $\tau$-smooth measures.
\( M_t(X) \) are scalar-valued, countably additive tight Borel measures \( \mu \) on \( X \); they are inner regular by compact sets and outer regular by open sets of \( X \); \( M_t^+(X) \) denotes the positive elements of these measures (7,9,10,1).

If \( X \) is compact \( M(X) = M_t(X) = M_t(X) \).

The topologies on these measures will be the one induced by \( \sigma(M(X), C_b(X)) \).

For a completely regular Hausdorff space \( X, \overline{X} \) will denote its Stone-Cech compactification; a \( \mu \in M(X) \) gives a \( \overline{\mu} \in M(\overline{X}) \) with \( \overline{\mu}(f) = \mu(f|_X) \), for all \( f \in C(\overline{X}) \); this \( \mu \) will be in \( M_t(X) \) iff \( |\overline{\mu}|(\overline{X} \setminus C) = 0 \) for all compact \( C \subset \overline{X} \setminus X \) (9,7).

For \( i = (1, 2) \), let \( \mathcal{A}_i \) be algebras of subsets of the sets \( X_i \) and \( \mathcal{A}_1 \times \mathcal{A}_2 \) the algebra generated by \( \{A_1 \times A_2 : A_i \in \mathcal{A}_i \ (i = 1, 2)\} \). Suppose \( \nu : \mathcal{A}_1 \times \mathcal{A}_2 \to [0, 1] \) is a finitely additive measure. The finitely additive measures \( \nu^{(1)} : \mathcal{A}_1 \to [0, 1] \), \( \nu^{(1)}(A_1) = \nu(A_1 \times A_2) \) and \( \nu^{(2)} : \mathcal{A}_2 \to [0, 1] \), \( \nu^{(2)}(A_2) = \nu(X_1 \times A_2) \) are called the marginals of \( \nu \); they are projections of \( \nu \) on \( X_1 \) and \( X_2 \). If \( X_1 \) and \( X_2 \) are completely regular Hausdorff spaces and \( \nu \in M_t^+(X_1 \times X_2) \) then its marginals \( \nu^{(i)} \), \( i = 1, 2 \), will be in \( M_t^+(X_i) \) with \( \nu(f_1 \otimes 1) = \nu^{(1)}(f_1) \) and \( \nu(1 \otimes f_2) = \nu^{(2)}(f_2) \), with \( f_i \in C_b(X_i) \), \( i = 1, 2 \), (here \( f_1 \otimes 1(x_1, x_2) = f_1(x_1) \) and \( 1 \otimes f_2(x_1, x_2) = f_2(x_2) \)); also it is simple verification that if \( \nu \in M_t^+(X_1 \times X_2) \) and \( \mu_i \in M^+(X_i) \) such that \( \nu(f_1 \otimes 1) = \mu_1(f_1) \) and \( \nu(1 \otimes f_2) = \mu_2(f_2) \) with \( f_i \in C_b(X_i) \), \( i = 1, 2 \), then \( \mu_1 \) and \( \mu_2 \) are the marginals of \( \nu \).

In the celebrated Strassen's paper (8), conditions are obtained for the existence of measures with given marginals. In (3, Theorem 1, p. 160), a result is proved about the limit of a net of \( \tau \)-smooth measures, having \( \tau \)-smooth marginals; unfortunately there is a gap in the proof. The following example proves that.

**Example 1** — For \( i = 1, 2 \), there exist completely regular Hausdorff spaces \( X_i \), \( \mu_i \in M_t^+(X_i) \) and a net \( \{\nu_\alpha\} \subset M_t^+(X_1 \times X_2) \) having the following properties:

(i) \( \nu^{(i)}_\alpha \to \mu_i \) in \( \sigma(M_t(X_i), C_b(X_i)) \)

(ii) \( \nu_\alpha \to \nu \in M^+(X_1 \times X_2) \setminus M_t^+(X_1 \times X_2) \).
PROOF: The following construction is done in (4, p. 167):

\( l = [0,1], \) \( m \) the Lebesgue measure on \( L, B \) all Borel subsets of \( I, Z_1, Z_2 \) a decomposition of \( I \) such that the Lebesgue outer measures of \( Z_1, Z_2 \) are 1 each;

for \( i = 1, 2, \mu_i : Z_i \cap B \to I, \mu_i (Z_i \cap B) = m (B) \), for all \( B \in B \) are countably additive probability measures;

there is a finitely additive measure \( \nu_0 \) on the algebra generated, in \( (Z_1 \times Z_2) \), by \( \{(Z_1 \times Z_2) \cap (B_1 \times B_2) : B_1 \text{ and } B_2 \text{ Borel subsets of } I\}, \nu_0 ((Z_1 \times Z_2) \cap (B_1 \times B_2)) = m (\{(x,x) \in B_1 \times B_2\}) \); this \( \nu_0 \) has total mass 1, is not countably additive and has \( \mu_1, \mu_2 \) as its marginals.

The completions of the measures \( \mu_i \) (\( i = 1, 2 \)) are also denoted by \( \mu_i \). We fix some liftings \( \rho_i \) for these finite measures \( \mu_i \) (2), (5). Putting \( \mathcal{A}_i = \{\rho_i (B \cap Z_i) : B \in B\} \) (\( i = 1, 2 \), \( \mathcal{A}_1, \mathcal{A}_2 \) are algebras of subsets of \( Z_1 \) and \( Z_2 \) and forms a base for the lifting topology \( T_{\rho_i} \) on \( Z_i \) (these are clopen sets; note these topologies are extremely disconnected). We denote by \( \left( X_p, T_{\rho_i} \right) \) the associated completely regular Hausdorff space. The canonical mapping \( Z_i \to X_i \) is denoted by \( \phi_i \) (note for an \( A \in \mathcal{A}_i, \phi_i^{-1} (\phi_i (A)) = A \). The space \( X_i \) is extremely disconnected and the algebra of clopen sets, \( \mathcal{S}_i = \{\phi_i (A) : A \in \mathcal{A}_i\} \), forms a base of the topology of \( X_i \) for \( i = 1, 2 \). As is well-known (2), the measures \( \mu_i \) can be considered as elements of \( M^+ (X_p) \).

Let \( \mathcal{S} \) be the algebra generated by \( \{S_1 \times S_2 : S_1 \in S_1, S_2 \in S_2\} \). Take \( S_i \in \mathcal{S}_i, i = 1, 2 \) and put \( \phi_i^{-1} (S_i) = S_i^z \). There are \( B_i \in B \) such that \( S_i^z = B_i \cap Z_i \), a.e. \([\mu_i]\). Define \( \nu (S_1 \times S_2) = \nu_0 ((B_1 \times B_2) \cap (Z_1 \times Z_2)); \nu \) can be extended to a finitely additive measure on \( \mathcal{S} \). Since \( \nu_0 \) is not countably additive, it is a routine verification that this measure in not countably additive. This means there is sequence \( \{W_n\} \subset \mathcal{S} \) such that \( W_n \downarrow 0 \) but \( \nu (W_n) \) does not converge to 0.
For any \( S_i \in \mathcal{S}_i, i = 1, 2, \chi_{S_1} \chi_{S_2} \) can be considered an element of \( C_b(X_1 \times X_2) \). Let \( F \) be the subspace of \( C_b(X_1 \times X_2) \), generated by \( \{ \chi_{S_1} \chi_{S_2} : S_i \in \mathcal{S}_i, i = 1, 2 \} \); \( F \) contains constant functions and \( \nu : F \to \mathbb{R}, \nu(f) = \int f \, dv \), is linear, positive and \( \| \nu(f) \| \leq \| f \|, \forall f \in F \) (\( \| f \| \) is the sup-norm). By Hahn-Banach theorem, \( \nu \) extends to \( \nu : C_b(X_1 \times X_2) \to \mathbb{R} \) which is also linear, continuous (in sup-norm topology), and \( \| \nu(f) \| \leq \| f \|, \forall f \in C_b(X_1 \times X_2) \). We claim this is also positive: to prove this, take an \( f \in C_b(X_1 \times X_2), \) \( 0 \leq f \leq 1 \) and suppose \( \nu(f) = -c \) (\( c > 0 \)); this means \( \nu(1-f) = 1+c > 1 \); but \( \| 1-f \| \leq 1 \) and so we have a contradiction. Thus \( \nu \in M^+(X_1 \times X_2) \). For any \( S_i \)-simple function \( f \in C_b(X_1) \), we have \( \nu(f \otimes 1) = \mu_1(f) \). Now the elements of \( C_b(X_1) \) are, individually, bounded \( \mu_1 \)-measurable functions on \( X_1 \); this implies that for an \( f \in C_b(X_1) \), there is a sequence \( \{ f_n \} \), of \( \mu_1 \) measurable simple functions, such that \( f_n \to f \) uniformly on \( X_1 \). Taking liftings, it immediately follows that \( f \) is the uniform limit of a sequence of \( S_i \)-simple functions. Since \( \nu \) is continuous in sup-norm, we get \( \nu(f \otimes 1) = \mu_1(f) \) for every \( f \in C_b(X_1) \). Similarly for every \( f \in C_b(X_2) \), we have \( \nu(f \otimes 1) = \mu_2(f) \).

Take a net \( \{ \nu_\alpha \} \subset M^+_\tau(X_1 \times X_2) \), of discrete probability measures, such that \( \nu_\alpha \to \nu \). This implies that, for an \( f \in C_b(X_1), \) \( \nu_\alpha(f \otimes 1) \to \nu(f \otimes 1) = \mu_1(f) \). Similar result holds for an \( f \in C_b(X_2) \). From this it follows that \( \lim \nu_\alpha^{(i)} = \mu_i \) (\( i = 1, 2 \)). Were (3, Theorem 1, p. 160) true, \( \nu \) must be in \( M^+_\tau(X_1 \times X_2) \), which is not the case.

However, the following theorem holds for \( \tau \)-smooth measures:

**Theorem 2** — For \( i = 1, 2 \), let \( X_1, X_2 \) be completely regular Hausdorff spaces, \( \mu_1 \in M^+_\tau(X_1), \mu_2 \in M^+_\tau(X_2) \), and a \( \nu \in M^+(X_1, X_2) \) such that for every \( f \in C_b(X_1) \), \( \nu(f \otimes 1) = \mu_1(f) \) and for every \( f \in C_b(X_2), \nu(1 \otimes f) = \mu_2(f) \). Then \( \gamma \in M^+_\tau(X_1 \times X_2) \).

**Proof**: Let \( \bar{X}_1 \) and \( \bar{X}_2 \) be the Stone-Cech compactifications of \( X_1 \) and \( X_2 \). For any \( f \in C(\bar{X}_1 \times \bar{X}_2) \), define \( \tilde{\nu}(f) = \nu(f_{|X_1 \times X_2}) \). We get \( \tilde{\nu} \in M^+(\bar{X}_1 \times \bar{X}_2) \). For any \( \tilde{f}_i \in C(\bar{X}_1), i = 1, \)
2, put $f_i = f_{i_X}$; we have $\bar{\nu}(f_1 \otimes 1) = \mu_1(f_1)$ and $\bar{\nu}(1 \otimes f_2) = \mu_2(f_2)$. This means for any Borel $B_i$ in $\bar{X}$, $i = 1, 2$, \( \bar{\nu}(B_1 \times B_2) = \mu_1(B_1 \cap X_1) \) and \( \bar{\nu}((\bar{X}_1 \times B_2) = \mu_2(B_2 \cap X_2) \) (note $\mu_1$ and $\mu_2$ are $\tau$-smooth). Take an increasing sequence \( \{C_n\} \) of compact subsets of $X_1$ such that $\bar{\nu}((\bar{X}_1 \setminus C_n) \times X_2) \leq \frac{1}{n}$, for all $n$ (here we are very much using that $\mu_1 \in M^+(X_1)$). First we prove that for any Borel $K$ of $(\bar{X}_1 \times \bar{X}_2)$, $K \subset (\bar{X}_1 \times (\bar{X}_2 \setminus X_2))$, $\bar{\nu}(K) = 0$. Because of the regularity of $\bar{\nu}$.

It is enough to prove the result when $K$ is compact. Let $\psi_1 : \bar{X}_1 \times \bar{X}_2 \to \bar{X}_1$ and $\psi_2 : \bar{X}_1 \times \bar{X}_2 \to \bar{X}_2$ be the canonical mappings; they are continuous. $K_i = \psi_i(K)$ are compact subsets of $\bar{X}$, $i = 1, 2$, $K \subset K_1 \times K_2$ and $K_2 \subset (\bar{X}_2 \setminus X_2)$. Since $\mu_2$ is $\tau$-smooth, $\bar{\nu}(\bar{X}_1 \times K_2) = 0$. This means $\bar{\nu}(K_1 \times K_2) = 0$ and so $\bar{\nu}(K) = 0$, proving the result. Also if a Borel $B$ in $(\bar{X}_1 \times \bar{X}_2)$ has the property that, for some $n$, $B \cap (C_n \times X_2) = \phi$, then $\bar{\nu}(B \cap (C_n \times X_2) = 0$; to prove this, one has only to note that the Borel set $B \cap (C_n \times \bar{X}_2)$ is a subset of $(\bar{X}_1 \times (\bar{X}_2 \setminus X_2))$. Now take a Borel $B$ of $(\bar{X}_1 \times \bar{X}_2)$ such that $B \subset (\bar{X}_1 \times \bar{X}_2) \setminus (X_1 \times X_2)$. This means $B \cap (C_n \times X_2) = \phi$, for all $n$ and so $\bar{\nu}(B \cap (C_n \times \bar{X}_2) = 0$, for all $n$. This implies that $\bar{\nu}(B) = 0$.

Let $X$ be the Stone-Cech compactification of $(X_1 \times X_2)$. This means $\bar{\nu} \in M^+(X)$. Let $\phi : X \to (\bar{X}_1 \times \bar{X}_2)$ be the unique continuous extension of the identity mapping $(X_1 \times X_2) \to (\bar{X}_1 \times \bar{X}_2)$; $\phi$ maps $X \setminus (X_1 \times X_2)$ onto $(\bar{X}_1 \times \bar{X}_2) \setminus (X_1 \times X_2)$. It is easily verified that for any $f \in C(\bar{X}_1 \times \bar{X}_2)$, $\bar{\nu}(f) = \bar{\nu}(f \circ \phi)$. By regularity, we get $\bar{\nu}(K) = \bar{\nu}(\phi^{-1}(K))$, for any compact $K \subset (\bar{X}_1 \times \bar{X}_2)$. Take a compact $C \subset X \setminus (X_1 \times X_2)$. Then $C_1 = \phi^{-1}(\phi(C))$ is compact and contains $C$, and $\phi(C)$ is disjoint from $(X_1 \times X_2)$. Now $\bar{\nu}(C) \leq \bar{\nu}(C_1) = \bar{\nu}(\phi(C)) = 0$. This means $\nu \in M^+(X_1 \times X_2)$. This proves the result.

**Remark 3**: ([3], Theorem 1, p. 160) holds in the following form:

**Corollary 4**. --- Let $X_1, X_2$ be completely regular Hausdorff spaces, $\mu_1 \in M^+(X_1)$, $\mu_2 \in M^+(X_2)$ and a net $\{\nu_\alpha\} \subset M^+(X_1 \times X_2)$ such that $\nu_\alpha^{(i)} \to \mu_i$, for $i = 1, 2$. Then there exist a subnet of $\{\nu_\alpha\}$ which converges to some $\nu \in M^+(X_1 \times X_2)$ with $\nu^{(i)} = \mu_i$, for $i = 1, 2$. 


PROOF: A subnet of \( \{ \nu_{\alpha} \} \) converges to some \( \nu \in M^{+}(X_1 \times X_2) \) such that for every \( f \in C_b(X_1) \), \( \nu(f \otimes 1) = \mu_1(f) \) and for every \( f \in C_b(X_2) \), \( \nu(1 \otimes f) = \mu_2(f) \). By Theorem 2, the result follows.

Now the Strassen’s theorem for \( \tau \)-smooth measures can be put in the form:

**Theorem 5** — Let \( X_1, X_2 \) be completely regular Hausdorff spaces, \( \mu_1 \in M^{+}_\tau(X_1) \), \( \mu_2 \in M^{+}_\tau(X_2) \) and \( Q \) a uniformly bounded, convex and closed subset of \( M^{+}_\tau(X_1 \times X_2) \). Then there exists a \( \lambda \in Q \) such that \( \lambda^{(i)} = \mu_i \) \( (i = 1, 2) \), iff for any \( \{ f_i \} \subset C_b(X_i) \) \( (i = 1, 2) \), \( \mu_1(f_1) + \mu_2(f_2) \leq \sup \{ \nu(f_1 \otimes 1 + \otimes f_2) : \nu \in Q \} \).

PROOF: The condition is trivially necessary. We take the topology \( \sigma(M(X_1 \times X_2), C_b(X_1 \times X_2)) \) on \( M(X_1 \times X_2) \) and the topology \( \sigma(M(X_i, C_b(X_i))) \) on \( M(X_i) \) \( (i = 1, 2) \). \( \overline{Q} \), the closure of \( Q \), is a compact and convex subset of \( M^{+}_\tau(X_1 \times X_2) \). For a \( \nu \in \overline{Q} \), define \( \nu^{(1)} \in M^{+}(X_1) \), \( \nu^{(1)}(f) = \nu(f \otimes 1) \) for every \( f \in C_b(X_1) \); \( \nu^{(2)} \in M^{+}(X_2) \) is defined in a similar way. Now \( Q_0 = \{ (\nu^{(1)}, \nu^{(2)}) : \nu \in \overline{Q} \} \) is a compact convex subset of \( M(X_1) \times M(X_2) \), with product topology. We claim that \( (\mu_1, \mu_2) \in Q_0 \); if not, by separation theorem ([6], 9.2, p. 65), there are, for \( i = 1, 2 \), \( f_i \in C_b(X_i) \) such that

\[
\mu_1(f_1) + \mu_2(f_2) > \sup \left\{ \nu^{(1)}(f_1) + \nu^{(2)}(f_2) : (\nu^{(1)}, \nu^{(2)}) \in Q_0 \right\}.
\]

Now the right hand side of this inequality is \( \geq \sup \{ \nu(f_1 \otimes 1 + \otimes f_2) : \nu \in Q \} \). This contradicts the given hypothesis. Thus \( (\mu_1, \mu_2) \in Q_0 \). So there is a \( \lambda \in \overline{Q} \) such that \( (\mu_1, \mu_2) = (\lambda^{(1)}, \lambda^{(2)}) \). By Theorem 2, \( \lambda \in M^{+}_\tau(X_1 \times X_2) \). Since \( Q \) is closed in \( M^{+}_\tau(X_1 \times X_2) \), \( \lambda \in Q \).

**Remark 6**: In ([3], Theorem 2, p. 164), one of the marginals should be tight.

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