STABILIZATION OF THERMOELASTIC PLATES WITH VARIABLE COEFFICIENTS AND DYNAMICAL BOUNDARY CONTROL

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We consider the stabilization of thermoelastic plates with variable coefficients and dynamical boundary under some suitable feedbacks. A rational energy decay rate for the strong solutions is established by Riemannian geometry method. Finally, the uniform energy decay rate for a simplified system is obtained.

Key Words: Thermoelastic Plate; Rational Energy Decay Rate; Riemannian Geometry Method

1. INTRODUCTION

We consider the stabilization problem of thermoelastic plates with variable coefficients and dynamical boundary where, for convenience, our problems start out on a Riemannian manifold $M$ of dimension 2 with metric $g = \langle \ldots \rangle$. When there is no thermal effect, the problem mentioned above has been analyzed by Guo et al., (2). For the classical case where $M = R^2$ and $g$ is the dot product, elastic plates with dynamical boundary has been studied by Rao (10), thermoelastic plates by Lagnese (5), Eller et al., (1). Here we use the Riemannian geometry method to obtain the stabilization results for the thermoelastic plate with variable coefficients and dynamical boundary control. This method is first introduced into the boundary control problem by Yao (15) for the wave equation.

Our paper is organized as follows: Firstly, we introduce some notations and the model we are working on. In Section 2, we prove the existence and regularity of solutions to the model, and establish the rational energy decay rate for the strong solutions. In Section 3, we consider a simplified thermoelastic plate model, and obtain the uniform energy decay rate of the system.

1.1 Some notations — We introduce some notations in preparation for our systems of the thermoelastic plates with dynamical boundary control. It is mentioned that all definitions and notations in this subsection are standard and classical in the literature.

Let $(M, g)$ be a Riemannian manifold with metric $g = \langle \ldots \rangle$. For each $x \in M, M_x$ is the

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tangential space of $M$ at $x$. Denote the set of all vector fields on $M$ by $\mathcal{X}(M)$. Denote the set of all $k$-order tensor fields and all $k$-forms on $M$ by $T^k(M)$ and $\Lambda^k(M)$ respectively, where $k$ is a nonnegative integer. Then

$$\Lambda^k(M) \subset T^k(M).$$ \hfill (1.1)

In particular, $\Lambda^0(M) = T^0(M) = C^\infty(M)$ is the set of all $C^\infty$ functions on $M$ and

$$T^1(M) = T(M) = \Lambda(M) = \mathcal{X}(M)$$ \hfill (1.2)

where $\Lambda(M) = \mathcal{X}(M)$ has to be interpreted as the following isomorphism: for $X \in \mathcal{X}(M)$ given, equation

$$U(Y) = \langle Y, X \rangle \quad \forall Y \in \mathcal{X}(M)$$ \hfill (1.3)

determines a unique $U \in \Lambda(M)$.

It is well known that, for each $x \in M$, $k$-order tensor space $T^k_x$ on $M_x$ is an inner product space defined as follows: Let $e_1, e_2$ be an orthonormal basis of $M_x$ for any $\alpha, \beta \in T^k_x$, $x \in M$. The inner product is given by

$$\langle \alpha, \beta \rangle_{T^k_x} = \sum_{i_1, i_2, \ldots, i_k = 1}^2 \alpha(e_{i_1}, \ldots, e_{i_k}) \beta(e_{i_1}, \ldots, e_{i_k}), \text{ at } x.$$ \hfill (1.4)

Let $\Omega$ be a bounded region of $M$ with a regular boundary $\Gamma$. Then $T^k(\Omega)$ is an inner product space in the following meaning

$$(T^1, T^2)_{T^k(\Omega)} = \int_{\Omega} \langle T^1, T^2 \rangle_{T^k_x} dx, \quad T^1, T^2 \in T^k(\Omega),$$ \hfill (1.5)

where $dx$ is the volume element of $M$ in its Riemannian metric $g$.

The completion of $T^k(\Omega)$ in the inner product (1.5) is denoted by $L^2(\Omega, T^k)$. In particular, $L^2(\Omega, \Lambda) = L^2(\Omega, T)$. The completion of $C^\infty(\Omega)$ in the following inner product is defined by $L^2(\Omega)$

$$(f, h)_{L^2(\Omega)} = \int_{\Omega} f(x) h(x) dx, \quad f, h \in C^\infty(\Omega).$$ \hfill (1.6)

Let $D$ be the Levi-Civita connection on $M$ in the metric $g$. For $U \in \mathcal{X}(M)$, $DU$ is the covariant differential of $U$ which is a 2-order covariant tensor field in the following sense:
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\[ DU(X,Y) = D_y U(X) = \langle D_y U, X \rangle, \quad \forall X, Y \in M, \quad x \in M. \quad \ldots \quad (1.7) \]

For any \( T \in T^2(M) \), the trace of \( T \) at \( x \) is defined by

\[ tr \, T = \sum_{i=1}^{2} T(e_i, e_i), \quad \ldots \quad (1.8) \]

where \( e_1, e_2 \) is an orthonormal basis of \( M_x \). It is obvious that \( tr \, T \in C^\infty(M) \) if \( T \in T^2(M) \). The exterior derivative \( d : \Lambda^k(M) \rightarrow \Lambda^{k+1}(M) \) satisfies \( d^2 = 0 \). There is a first-order differential operator \( \delta : \Lambda^{k+1}(M) \rightarrow \Lambda^k(M) \), which is the formal adjoint of \( d \) and characterized by \( (d\alpha, \beta)_{L^2(\Omega, \Lambda^k)} = (\alpha, \delta \beta)_{L^2(\Omega, \Lambda^{k+1})} \) for \( \alpha \in \Lambda^k(\Omega) \) and \( \beta \in \Lambda^{k+1}(\Omega) \) with compact support.

The Sobolev space \( H^k(\Omega) \) is the completion of \( C^\infty(\Omega) \) with respect to the norm

\[ \| f \|_{H^k(\Omega)}^2 = \sum_{i=1}^{k} \| D^i f \|_{L^2(\Omega, T^i\Omega)}^2 + \| f \|_{L^2(\Omega)}^2, \quad f \in C^\infty(\Omega), \]

where \( D^i f \) is the \( i \)-th covariant differential of \( f \) in metric \( g \), and \( \| \cdot \|_{L^2(\Omega, T^i\Omega)}, \| \cdot \|_{L^2(\Omega)} \) are the induced norms in the inner products (1.5), (1.6) respectively. For details on the Sobolev spaces on the Riemannian manifolds, we refer to Hebey [3] or Taylor [11].

We will use many times the following divergence formulae

\[ \int_{\Omega} \text{div} \, X \, dx = \int_{\Gamma} \langle X, v \rangle \, d\Gamma, \quad \ldots \quad (1.9) \]

where \( \text{div} \, X \) is the divergence of vector field \( X \) in Riemannian metric \( g \), \( v \) is the normal of \( \Gamma \) pointing towards the exterior of \( \Gamma \), see Wu et al., (12).

1.2. Model — Let \( \Omega \) denote a bounded domain in Riemannian manifold \((M, g)\) with boundary \( \Gamma \) consisting of two disjoint parts \( \Gamma_0 \cup \Gamma_1 = \Gamma \). In the following we assume that boundary \( \Gamma \) is smooth. Here we consider a curvy plate with the dynamical boundary whose middle surface \( \Omega \) is part of a surface \( M \) and where the extension effects along the tangential direction are neglected. We assume that the material undergoing obeys Hooke's law. Let us assume that, in addition to mechanical loads, a homogeneous plate is subject to an unknown temperature distribution. Thus the part of the strain energy involving the displacement \( y \) of the plate along the normal is to be defined by
\[
P_{\Omega}(y) = \int_{\Omega} \left[ (1 - \mu) \left| D^2 y \right|^2 + \mu (\text{tr} \, D^2 y)^2 + (1 + \mu) \theta \text{tr} \, D^2 y \right] \, dx. \quad \text{... (1.10)}
\]

where \( \theta \) is the temperature field of the plate, and \( 0 < \mu < \frac{1}{2} \) the Poisson coefficient. The energy produced by the dynamic boundary is to be defined by

\[
P_{\Gamma}(y) = J \int_{\Gamma_1} \left\| \frac{\partial y'}{\partial n} \right\|^2 \, d\Gamma + \rho \int_{\Gamma_1} \left| y' \right|^2 \, d\Gamma, \quad \text{... (1.11)}
\]

where \( \rho > 0 \) is the linear boundary density and \( J > 0 \) is the bending moment of inertia of the boundary.

When there is no external force, the equation of motion for \( y \) is obtained by setting to zero the first variation of the Lagrangian

\[
\int_0^T \left[ \int_{\Omega} \left| y' \right|^2 \, dx - P_{\Omega}(y) - P_{\Gamma}(y) \right] \, dt \quad \text{... (1.12)}
\]

(the "Principle of the Virtual Work"). Then the variation of (1.12) is taken with respect to kinematically admissible displacements. When the plate is clamped on a portion \( \Gamma_0 \) of \( \Gamma \) and free on \( \Gamma_1 \), we obtain, as the result of calculation by the variation of (1.12), the following system:

\[
\begin{cases}
  y'' + A y + \alpha \Delta \theta = 0, & \text{in } \Omega \times [0, \infty) \\
  y = \frac{\partial y}{\partial n} = 0, & \text{on } \Gamma_0 \times [0, \infty) \\
  J \frac{\partial y''}{\partial n} + \Delta y + (1 - \mu) B_1 y + \alpha \theta = 0, & \text{on } \Gamma_1 \times [0, \infty) \\
  \rho y'' - \frac{\partial \Delta y}{\partial n} - (1 - \mu) B_2 y + \alpha \frac{\partial \theta}{\partial n} = 0, & \text{on } \Gamma_1 \times [0, \infty),
\end{cases} \quad \text{... (1.13)}
\]

where operator \( A \) is defined by

\[
A y = \Delta^2 y - (1 - \mu) \delta (kdy) \quad \text{... (1.14)}
\]

\( \nu \) the unit normal along \( \Gamma \) pointing towards the exterior of \( \Gamma \) and \( \Delta \) the Laplace operator in the Riemann metric \( g \). In the above equation \( k \) is the Gaussian curvature function on \( M : \alpha = (1 + \mu)/2; \)
\( d \) is the exterior derivative; \( \delta \) is the formal adjoint of \( d \); \( B_1, B_2 \) are the boundary operators defined by

\[
B_1 y = -D^2 y (\tau, \nu) \quad \text{... (1.15)}
\]
and

\[ B_2 y = \frac{\partial}{\partial \tau} \left( D^2 y(\tau, v) \right) + k \frac{\partial y}{\partial v}, \]  

... (1.16)

respectively, where \( D^2 y \) is the Hessian of \( y \), which is a 2-order tensor, and \( \tau \) is the tangential along curve \( \Gamma \).

**Remark 1.1:** The term \( (1 - \mu) \delta (dky) \) in the system (1.13) comes from the curvedness of the metric. For the flat case where \( M = R^2 \) and \( k = 0 \), the equation in system (1.13) is similar as that in Lagnese (5).

On the other hand, when there are no heat sources within the plate, and the heat flux through either face of the plate vanishes, we may derive the following equation relating to the dynamics of \( \theta \) and \( y \)

\[ \beta \theta - \gamma \Delta \theta + \sigma \theta - \alpha \Delta y' = 0, \quad \text{in} \ \Omega \times [0, \infty), \]  

... (1.17)

where \( \beta, \gamma, \sigma \) are positive constants, see Lagnese et al., (6).

### 2. RATIONAL ENERGY DECAY RATE

In this section, we will consider the control problem:

\[
\begin{align*}
\begin{cases}
y'' + Ay + \alpha \Delta \theta = 0, & \text{in } \Omega \times [0, \infty) \\
\beta \theta' - \gamma \Delta \theta + \sigma \theta - \alpha \Delta y' = 0 & \text{in } \Omega \times [0, \infty) \\
y = \frac{\partial y}{\partial v} = 0, & \text{on } \Gamma_0 \times [0, \infty) \\
y \frac{\partial y''}{\partial v} + \Delta y + (1 - \mu) B_1 y + \alpha \theta = m, & \text{on } \Gamma_1 \times [0, \infty) \\
\rho y'' - \frac{\partial \Delta y}{\partial v} - (1 - \mu) B_2 y + \alpha \frac{\partial \theta}{\partial v} = f, & \text{on } \Gamma_1 \times [0, \infty), \\
\frac{\partial \theta}{\partial v} = -\lambda \theta, & \text{on } \Gamma \times [0, \infty) \\
y(0) = y_0, \quad y'(0) = y_1, \quad \theta(0) = \theta_0, & \text{on } \Omega,
\end{cases}
\end{align*}
\]

... (2.1)

where \( \lambda > 0, m, f \) are the feedbacks denoted by

\[ m = -\frac{\partial y'}{\partial v}, \quad f = -y'. \]  

... (2.2)
2.1 Well-posedness — Set

\[ z = y', \quad \xi = \frac{\partial y'}{\partial v} \bigg|_{\Gamma_1}, \quad \eta = y' \bigg|_{\Gamma_1}. \quad \cdots (2.3) \]

Then we transform system (2.1) into an abstract evolutionary equation

\[ u' + Au = 0, \quad u(0) = u_0, \quad \cdots (2.4) \]

where

\[
 u = \begin{pmatrix}
 y \\
 z \\
 \theta \\
 \xi \\
 \eta 
\end{pmatrix}, \quad
 A u = \begin{pmatrix}
 -z \\
 \mathcal{A} y + \alpha \Delta \theta \\
 -\frac{1}{\beta} (\gamma \Delta \theta - \sigma \theta + \alpha \Delta z) \\
 \frac{1}{J} (\Delta y + (1 - \mu) B_1 y + \alpha \theta + \xi) \\
 -\frac{1}{\rho} \left( \frac{\partial \Delta y}{\partial v} + (1 - \mu) B_2 y - \alpha \frac{\partial \theta}{\partial v} - \eta \right)
\end{pmatrix}. \quad \cdots (2.5)
\]

According to the formulation (2.5), we are led to introduce the following energy space

\[ \mathcal{H} = H^2_{\Gamma_0} (\Omega) \times L^2 (\Omega) \times L^2 (\Omega) \times L^2 (\Gamma_1) \times L^2 (\Gamma_1). \quad \cdots (2.6) \]

For \( u = (y, z, \theta, \xi, \eta) \), \( \hat{u} = (\hat{y}, \hat{z}, \hat{\theta}, \hat{\xi}, \hat{\eta}) \in \mathcal{H} \), we define the inner product by

\[
 (u, \hat{u})_\mathcal{H} = \int_{\Omega} \left[ a(y, \hat{y}) + z \hat{\xi} + \beta \theta \hat{\theta} \right] dx + \int_{\Gamma_1} \left[ \xi \hat{\xi} + \eta \hat{\eta} \right] d\Gamma \quad \cdots (2.7)
\]

where

\[ a(y, \hat{y}) = (1 - \mu) \left( D^2 y, D^2 \hat{y} \right) + \mu (tr D^2 y) (tr D^2 \hat{y}). \quad \cdots (2.8) \]

The domain \( D(A) \) is denned by

\[
 D(A) = \left\{ u = (y, z, \theta, \xi, \eta) \in W \times H^2_{\Gamma_0} (\Omega) \times H^2 (\Omega) \times L^2 (\Gamma_1) \times L^2 (\Gamma_1) ; \right. \\
 \left. \text{such that } \xi = \frac{\partial z}{\partial v} \bigg|_{\Gamma_1}, \quad \eta = z \bigg|_{\Gamma_1}, \quad \frac{\partial \theta}{\partial v} + \lambda \theta = 0 \text{ on } \Gamma_1 \right\}. \quad \cdots (2.9)
\]

where \( W \) is defined by
\[
W = \begin{cases} 
\gamma \in H^2_{\Gamma_0} (\Omega), & \Delta^2 \gamma \in L^2 (\Omega) \\
\Delta \gamma + (1 - \mu) \cdot B_1 \cdot \gamma = \nu_1 \in L^2 (\Gamma_1) \\
\frac{\partial \Delta \gamma}{\partial \nu} + (1 - \mu) \cdot B_2 \cdot \gamma = \nu_2 \in L^2 (\Gamma_1)
\end{cases}
\] ... (2.10)

From Guo et al., (2), we have the following Green formulae

\[
\int_{\Omega} \mathcal{A} y \hat{y} \, dx = \int_{\Omega} a(y, \hat{y}) \, dx - \int_{\Gamma} \left[ \Delta \gamma + (1 - \mu) \cdot B_1 \cdot \gamma \right] \frac{\partial \hat{\gamma}}{\partial \nu} \, d\Gamma
\]

\[
+ \int_{\Gamma} \left[ \frac{\partial (\Delta \gamma)}{\partial \nu} + (1 - \mu) \cdot B_2 \cdot \gamma \right] \hat{\gamma} \, d\Gamma, \quad \ldots (2.11)
\]

if \( y \) and \( \hat{y} \) are smooth enough. So the traces of the function \( y \in W \) can be defined by means of (2.11) for any \( \hat{y} \in H^2_{\Gamma_0} (\Omega) \).

**Lemma 2.1** — Operator \( \mathcal{A} \) is maximal monotone.

**Proof:** Let \( u = (y, z, \theta, \xi, \eta) \in D (\mathcal{A}) \). Then we have

\[
(Au, u)_g = \int_{\Omega} \left[ a(-z, y) + (\mathcal{A} y + \alpha \Delta \theta) z + (-\gamma \Delta \theta + \sigma \theta - \alpha \Delta z) \theta \right] dx
\]

\[
+ \int_{\Gamma_1} \left[ (\Delta y + (1 - \mu) \cdot B_1 \gamma + \alpha \theta + \xi) \xi - \left( \frac{\partial \Delta y}{\partial \nu} + (1 - \mu) \cdot B_2 \gamma - \alpha \frac{\partial \theta}{\partial \nu} - \eta \right) \eta \right] d\Gamma
\]

\[
= \int_{\Gamma_1} \left[ z^2 + \left( \frac{\partial z}{\partial \nu} \right)^2 \right] d\Gamma + \int_{\Gamma} [\gamma D \theta^2 + \sigma \theta^2] \, dx + \lambda \gamma \int_{\Gamma} \theta^2 \, d\Gamma \geq 0
\]

... (2.12)

Now given \( u_0 \in \mathcal{H} \), we solve the equation \( u + Au = u_0 \). This means that
\[
\begin{cases}
y - z = y_0 & \text{in } \Omega \\
z + A\dot{y} + \alpha \Delta \theta = z_0 & \text{in } \Omega \\
\beta \theta - \gamma \Delta \theta + \sigma \theta - \alpha \Delta z = \beta \theta_0 & \text{in } \Omega \\
y = \frac{\partial y}{\partial n} = 0, & \text{on } \Gamma_0 \\
(J + 1) \xi + \Delta y + (1 - \mu) B_1 y + \alpha \theta = J \xi_0, & \text{on } \Gamma_1 \\
(\rho + 1) \eta - \frac{\partial \Delta y}{\partial n} - (1 - \mu) B_2 y + \alpha \frac{\partial \theta}{\partial n} = \rho \eta_0, & \text{on } \Gamma_1 \\
\frac{\partial \theta}{\partial n} = - \lambda \theta, & \text{on } \Gamma 
\end{cases}
\]

Then eliminating \(z\) and using \(\xi = \left| \frac{\partial z}{\partial n} \right|_{\Gamma_1} \), \(\eta = z \big|_{\Gamma_1}\), we find that \(y\) and \(\theta\) satisfy the following system

\[
\begin{cases}
y + A\dot{y} + \alpha \Delta \theta = z_0 + y_0, & \text{in } \Omega \\
\beta \theta - \gamma \Delta \theta + \sigma \theta - \alpha \Delta y = \beta \theta_0 - \alpha \Delta y_0 & \text{in } \Omega \\
y = \frac{\partial y}{\partial n} = 0, & \text{on } \Gamma_0 \\
(J + 1) \frac{\partial y}{\partial n} + \Delta y + (1 - \mu) B_1 y + \alpha \theta = J \xi_0 + (J + 1) \frac{\partial y_0}{\partial n}, & \text{on } \Gamma_1 \\
(\rho + 1) y - \frac{\partial \Delta y}{\partial n} - (1 - \mu) B_2 y + \alpha \frac{\partial \theta}{\partial n} = \rho \eta_0 + (\rho + 1) y_0, & \text{on } \Gamma_1 \\
\frac{\partial \theta}{\partial n} = - \lambda \theta, & \text{on } \Gamma 
\end{cases}
\]

The variational equation of system (2.14) is the following

\[
\int_{\Omega} (y\dot{\gamma} + a(y, \dot{y})) \, dx + \int_{\Omega} \alpha (\theta \Delta \dot{\gamma} - \dot{\theta} \Delta y) \, dx + (J + 1) \int_{\Gamma_1} \frac{\partial y}{\partial n} \frac{\partial \dot{\gamma}}{\partial n} \, d\Gamma \\
+ (\rho + 1) \int_{\Gamma_1} y \dot{\gamma} \, d\Gamma + \int_{\Omega} [(\beta + \sigma) \theta \dot{\gamma} + \gamma (D \theta, D \dot{\gamma})] \, dx + \lambda \int_{\Gamma} \theta \dot{\gamma} \, d\Gamma
\]
\[ = \int_{\Omega} (y_0 + z_0) \hat{\nabla} \hat{x} d\Gamma + \int_{\Gamma_1} [\rho \eta_0 + (\rho + 1) y_0] \hat{\nabla} d\Gamma \]

\[ + \int_{\Gamma_1} \left[ J \xi_0 + (J + 1) \frac{\partial y_0}{\partial v} \right] \frac{\partial \hat{\nabla}}{\partial v} d\Gamma \]

\[ + \int_{\Omega} (\beta \theta_0 - \alpha \Delta y_0) \hat{\theta} dx, \]

\[ \text{for all } \hat{y} \in H^2_{\Gamma_0} (\Omega), \hat{\theta} \in H^4 (\Omega). \text{ Thanks to the Lax-Milgram theorem, (2.15) admits a unique solution } y \in H^2_{\Gamma_0} (\Omega), \theta \in H^4 (\Omega). \]

On the other hand, \( \theta \) satisfies

\[ \begin{cases} 
- \gamma \Delta \theta &= \beta \theta_0 + \alpha \Delta y - \alpha \Delta y_0 - (\sigma + \beta) \theta \in L^2 (\Omega), \\
\frac{\partial \theta}{\partial v} &= -\lambda \theta \in H^2 (\Gamma). 
\end{cases} \]

... (2.16)

From the elliptic theory, see [8, Theorem 5.4 in Page 165], it follows that \( \theta \in H^2 (\Omega) \). Then \( y \) satisfies

\[ \begin{cases} 
y \in H^2_{\Gamma_0} (\Omega), \quad \mathcal{A}y = -y - \alpha \Delta \theta + z_0 + y_0 \in L^2 (\Omega), \\
\Delta y + (1 - \mu) B_1 y = -(J + 1) \frac{\partial \theta}{\partial v} - \alpha \theta + J \xi_0 + (J + 1) \frac{\partial y_0}{\partial v} \in L^2 (\Gamma_1), & \quad \text{... (2.17)} \\
- \frac{\partial \Delta y}{\partial v} - (1 - \mu) B_2 y = -(\rho + 1) y - \alpha \frac{\partial \theta}{\partial v} + \rho \eta_0 + (\rho + 1) y_0 \in L^2 (\Gamma_1). 
\end{cases} \]

that is, \( y \in W \). So \( R (I + A) = \mathcal{H} \). Furthermore using the elliptic theory, we have \( y \in H^2 (\Omega) \).

Thus we have the following result.

**Theorem 2.1** — 1) For any \( u_0 \in D (A) \), eq. (2.4) admits a unique strong solution

\[ u (t) = (y (t), z (t), \theta (t), \xi (t), \eta (t)) \]

such that

\[ y \in C (R^+ ; H^2 (\Omega)) \cap C^1 (R^+ ; H^2_{\Gamma_0} (\Omega)) \cap C^2 (R^+ ; L^2 (\Omega)), \]
\[ y \mid_{\Gamma_1} \in C^2 (R^+; L^2 (\Gamma_1)), \]  
\[ \frac{\partial y}{\partial v} \in C^2 (R^+; L^2 (\Gamma_1)), \quad \theta \in C (R^+; H^2 (\Omega)) \bigcap C^1 (R^+; L^2 (\Omega)). \]  
\[ \text{... (2.18)} \]  
\[ \text{... (2.19)} \]  

2) For any \( u_0 \in \mathcal{H} \), eq. (2.4) admits a unique weak solution 
\[ u (t) = (y (t), z (t), \theta (t), \xi (t), \eta (t)) \]

such that 
\[ y \in C (R^+; L^2 (\Omega)) \bigcap C^1 (R^+; L^2 (\Gamma_1)), \quad y \mid_{\Gamma_1} \in C^1 (R^+; L^2 (\Gamma_1)), \]  
\[ \frac{\partial y}{\partial v} \in C^1 (R^+; L^2 (\Gamma_1)), \quad \theta \in C (R^+; L^2 (\Omega)). \]  
\[ \text{... (2.20)} \]  
\[ \text{... (2.21)} \]  

2.2 Regularity of solutions — In order to obtain the regularity of solutions, we need more regularity of initial data. Let \( y_0 \in H^6 \mid_{\Gamma_0} (\Omega), y_1 \in H^6 \mid_{\Gamma_0} (\Omega) \) and \( \theta_0 \in H^4 (\Omega) \). Then system (2.1) has a solution \( y (t), \theta (t) \) satisfying (2.18), (2.19) and

\[ (y (0), z (0), \theta (0), \xi (0), \eta (0)) = \begin{pmatrix} y_0, y_1, \theta_0, \frac{\partial y_1}{\partial v} \mid_{\Gamma_1} \\ y_1 \mid_{\Gamma_1} \end{pmatrix}. \]  
\[ \text{... (2.22)} \]  

Set \( \Phi_1 = y, \Psi_1 = \theta' \). Then \( \Phi_1 \) and \( \Psi_1 \) satisfy the following system

\[
\begin{cases}
\Phi_1'' + A \Phi_1 + \alpha \Delta \Phi_1 = 0, & \text{in } \Omega \times [0, \infty) \\
\beta \Psi_1' - \gamma \Delta \Psi_1 + \sigma \Psi_1 - \alpha \Delta \Psi_1 = 0 & \text{in } \Omega \times [0, \infty) \\
\Phi_1 = \frac{\partial \Phi_1}{\partial v} = 0, & \text{on } \Gamma_0 \times [0, \infty) \\
J \frac{\partial \Phi_1'}{\partial v} + \Delta \Phi_1 + (1 - \mu) B_1 \Phi_1 + \alpha \Psi_1 = - \frac{\partial \Phi_1'}{\partial v}, & \text{on } \Gamma_1 \times [0, \infty) \\
\rho \Phi_1'' - \frac{\partial \Delta \Phi_1}{\partial v} - (1 - \mu) B_1 \Phi_1 + \alpha \frac{\partial \Psi_1}{\partial v} = - \Phi_1', & \text{on } \Gamma_1 \times [0, \infty), \\
\frac{\partial \Psi_1}{\partial v} = - \lambda \Psi_1, & \text{on } \Gamma \times [0, \infty). 
\end{cases}
\]  
\[ \text{... (2.23)} \]
It is easy to check that
\[
\begin{aligned}
\Phi_1 (0), \Phi_1' (0), \Psi_1 (0), \frac{\partial \Phi_1'}{\partial v} (0) \bigg|_{\Gamma_1}, \Phi_1' (0) \bigg|_{\Gamma_1} \in D (A),
\end{aligned}
\]  
so \( \Phi_1 \) and \( \Psi_1 \) satisfy
\[
\begin{aligned}
\Phi_1 \in C (R^+ ; H^2 (\Omega)) \cap C^1 (R^+ ; H^2_{\text{loc}} (\Omega)) \cap C^2 (R^+ ; L^2 (\Omega)),
\Phi_1' \bigg|_{\Gamma_1} \in C^2 (R^+ ; L^2 (\Gamma_1)),
\end{aligned}
\]  
\( \frac{\partial \Phi_1}{\partial v} \in C^2 (R^+ ; L^2 (\Gamma_1)), \Psi_1 \in C (R^+ ; H^2 (\Omega)) \cap C^1 (R^+ ; L^2 (\Omega)). \)  
\( \ldots \)  \( (2.25) \)
\[
\begin{aligned}
\frac{\partial \Phi_1}{\partial v} \in C^2 (R^+ ; L^2 (\Gamma_1)), \Psi_1 \in C (R^+ ; H^2 (\Omega)) \cap C^1 (R^+ ; L^2 (\Omega)). \)  
\end{aligned}
\]  
\( \ldots \)  \( (2.26) \)

Since \( y''' = \Phi_1' \in H^2 (\Omega) \), \( y \) satisfies the following system
\[
\begin{aligned}
\mathcal{A} y = - y'' - \alpha \Delta \theta \in L^2 (\Omega),
\Delta y + (1 - \mu) B_1 y = - \alpha \theta - \frac{\partial y'}{\partial v} - \frac{1}{2} \frac{\partial \psi'}{\partial v} \in H^2 (\Gamma_1),
\frac{- \partial \Delta y}{\partial v} - (1 - \mu) B_2 y = - \alpha \frac{\partial \theta}{\partial v} - y' - \rho y'' \in L^2 (\Gamma_1).
\end{aligned}
\]  
\( \ldots \)  \( (2.27) \)

From the elliptic theory, it follows that
\[
\begin{aligned}
y \in C (R^+ ; H^3 (\Omega)) \cap C^1 (R^+ ; H^2 (\Omega)),
\end{aligned}
\]  
\( \ldots \)  \( (2.28) \)

If initial data have more regularity, then we can obtain more regularity of solutions to system (2.1) by repeating the above step. Then the solution of system (2.1) is smooth if the initial is so.

2.3 Boundary stabilization — Define the energy of system (2.1) by
\[
E (t) = \frac{1}{2} \left\{ \int\limits_{\Omega} \left[ |y'|^2 + a (y, y) + \beta \theta^2 \right] dx + \int\limits_{\Gamma_1} \frac{1}{2} \left( |y'|^2 + J \left| \frac{\partial y'}{\partial v} \right|^2 \right) d \Gamma \right\} \ldots \)  \( (2.29) \)

First, we make some assumptions.

H.1 — There exists a vector field \( H \) on Riemannian manifold \( (M, g) \) such that
\[
\begin{aligned}
DH (X, X) = b (x) |X|^2, \forall X \in M_x, x \in \text{\overline{O}},
\end{aligned}
\]  
\( \ldots \)  \( (2.30) \)
where $b(x)$ is a positive function on $\Omega$.

H.2 — $\Gamma_0$ and $\Gamma_1$ satisfy that

$$\Gamma_0 \neq 0, \quad \Gamma_1 \cap \Gamma_1 = 0, \quad \ldots \ (2.31)$$

where

$$\Gamma_0 = \{ x \in \Gamma \mid \langle H, \nu \rangle \leq 0 \} \quad \text{and} \quad \Gamma_1 = \{ x \in \Gamma \mid \langle H, \nu \rangle > 0 \}.$$

H.3

$$2q_1 (q_2 + q_3) < 1, \quad \ldots \ (2.32)$$

where

$$q_2 = \max_{x \in \Omega} |k|_{H^1}, \quad q_3 = \max_{x \in \Omega} |D^2 H|,$$

$k$ is the Gauss curvature function, and $q_1$ is the best constant such that the following inequality is true:

$$\int_{\Omega} |Dy|^2 \, dx \leq q_1^2 \int_{\Omega} a(y, y) \, dx, \quad \forall y \in H^2_{\Gamma_0} (\Omega).$$

**Remark 2.2 :** The geometric condition (2.30) is used for some observability inequalities of the Euler-Bernoulli equation with variable coefficients and the shallow shell, see Yao (13), Yao (14). H.1 and H.2 are enough to get the uniform stabilization of the simplified model, see Theorem 3.1 in Sec 3. However, since we encounter difficulties when we try to use traditional method of compact-uniqueness to eliminate the lower order term in the proof of Theorem 2.2 below, we make H.3 to overcome it. For the classical case where $H = x - x_0$ and $k = 0$, H.3 is true since $DH = 0$, and therefore we have $q_1 (q_2 + q_3) = 0$. One can also find some other nontrivial example in Yao (14).

Furthermore, set

$$T(G, F) = (1 - \mu) \langle G, F \rangle + \mu \text{tr} \, G \, \text{tr} \, F,$$

where $G, F$ are second-order tensor and

$$L(y) = R(Dy, \cdot, H, \cdot) + D^2 H(Dy, \cdot, \cdot),$$

where "\cdot\cdot" denotes the position of the variable.

Now we give the following result.
**Theorem 2.2** — Assume that H.1, H.2 and H.3 hold. For any initial data \( u_0 \in D(A) \), there exists a constant \( K > 0 \) depending only on \( u_0 \) such that the following rational energy decay rate holds:

\[
E(t) \leq E(0) \frac{2K}{K+t}, \quad \forall t > 0,
\]

for all smooth solutions of system (2.1).

**Remark 2.3** : For the flat case Lagnese (5) established that the thermoelastic plate is strongly stable in the absence of dissipative boundary mechanisms. By the idea of Lagnese (5) we can obtain the same result, that is the system (2.1) and (2.2) is strongly stable even in the absence of dissipative boundary mechanisms. Here we omit it and only consider the energy decay rate of the system (2.1) and (2.2).

In order to prove Theorem 2.2, we need some lemmas. It is mentioned that in this paper \( C \) is a positive constant which may be different from line to line.

**Lemma 2.2** — The energy of system (2.1) satisfies

\[
\frac{d}{dt} E(t) = - \int_{\Gamma_1} \left( |y'|^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) d\Gamma - \int_{\Omega} (\gamma |D \theta|^2 + \sigma \theta^2) dx - \lambda \gamma \int_{\Gamma} \theta^2 d\Gamma \leq 0. \quad \text{... (2.34)}
\]

From system (2.1), it is easy to check Lemma 2.2. From Guo et al., (2), we have the following lemma.

**Lemma 2.3** — Assume that H.1 and (H.2) hold. Let \( y \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) \) and

\[
\begin{align*}
\Delta y + (1 - \mu) B_1 y &= v_1, & \text{on } \Gamma_1, \\
\frac{\partial \Delta y}{\partial v} + (1 - \mu) B_2 y &= v_2, & \text{on } \Gamma_1.
\end{align*}
\]

Then we have

\[
- \int_{\Omega} A y H(y) dx \leq - \frac{1}{2} \int_{\Omega} ba(y, y) dx
\]

\[
C \int_{\Gamma_1} (|v_1|^2 + |v_2|^2) d\Gamma + \int_{\Omega} T(D^2 y, L(y)) dx,
\]

where \( C \) is a positive constant depending only on the domain.

Set

\[
Q = \Omega \times [T, S], \quad \Sigma = \Gamma \times [T, S], \quad \Sigma_i = \Gamma_i \times [T, S], \quad i = 0, 1.
\]
Proof of Theorem 2.2 — Let \( 0 \leq T < S < +\infty \). Multiplying the plate equation in (2.1) from both sides by \( E(t)H(y) \) and integrating over \( \Omega \times [T, S] \) by parts, we then obtain in one hand

\[
\int_\Omega E(t) H(y) y'' \, dQ = \int_\Omega E(t) H(y) y' \, dx \bigg|_T^S - \int_\Omega E'(t) H(y) y' \, dQ
\]

\[
+ \frac{1}{2} \int_\Omega E(t) \left| y' \right|^2 \text{div} (H) \, dQ - \frac{1}{2} \int_{\Sigma_1} E(t) \left| y' \right|^2 \langle H, \nu \rangle \, d\Sigma.
\]

... (2.37)

By Cauchy-Schwartz inequality, we have

\[
\left| \int_\Omega H(y) y' \, dx \right| \leq C_0 E(t).
\]

... (2.38)

Then it follows that

\[
\int_\Omega E(t) H(y) y' \, dx \bigg|_T^S - \int_\Omega E'(t) H(y) y' \, dQ
\]

\[
\geq - C_0 (E^2(T) + E^2(S)) + C_0 \int_T^S E'(t) E(t) \, dt
\]

\[
\geq - 2 C_0 E^2(T).
\]

... (2.39)

Inserting (2.38) into (2.36), we have

\[
\int_\Omega E(t) H(y) y'' \, dQ \geq - 2 C_0 E^2(T) + b_0 \int_\Omega E(t) \left| y' \right|^2 \, dQ - \frac{R}{2} \int_{\Sigma_1} E(t) \left| y' \right|^2 \, d\Sigma
\]

... (2.40)

where

\[
b_0 = \min_{x \in \Omega} b(x), \quad R = \max_{x \in \Gamma} |H|.
\]

On the other hand, by Lemma 2.3 and the boundary condition in the system (2.1), we obtain

\[
- \int_\Omega E(t) H(y) (A_y + \alpha \Delta \theta) \, dQ
\]
\[
\leq - \frac{1}{2} \int_{\Sigma} bE(t) a(y, y) \, dQ + \int_{\Sigma} E(t) \alpha \langle D \theta, D(H(y)) \rangle \, dQ
\]
\[
+ \alpha \int_{\Sigma} E(t) \frac{\partial \theta}{\partial v} H(y) \, d\Sigma + C \int_{\Sigma} E(t) \left( |\rho y'' + y'\dot{l}|^2 + \left| J \frac{\partial y''}{\partial v} + \frac{\partial y'}{\partial v} \right|^2 + \theta^2 \right) \, d\Sigma
\]
\[
+ \int_{\Sigma} E(t) T(D^2 y, L(y)) \, dQ.
\]

In the following we estimate some terms in (2.41).

Since
\[
\langle D \theta, D(H(y)) \rangle = D \theta \langle H, Dy \rangle = D H(D \theta, Dy) + D^2 y(H, D \theta),
\]
we have
\[
|\langle D \theta, D(H(y)) \rangle| \leq |DH||D \theta||Dy| + |D^2 y||H||D \theta|
\]
\[
\leq C_\epsilon |D \theta|^2 + \epsilon |D^2 y|^2 + \epsilon |Dy|^2, \quad \forall \epsilon > 0.
\]

Applying embedding theorem and (2.43), we obtain that
\[
\int_{\Sigma} E(t) \alpha \langle D \theta, D(H(y)) \rangle \, dQ \leq C_\epsilon \int_{\Sigma} E(t) |D \theta|^2 \, dQ
\]
\[
+ \epsilon \int_{\Sigma} E(t) |D^2 y|^2 \, dQ.
\]

By the Cauchy inequality, we have
\[
\left( \int_{\Omega} T(D^2 y, L(y)) \, dx \right)^2 \leq \int_{\Omega} T(D^2 y, D^2 y) \, dx \int_{\Omega} T(L(y), L(y)) \, dx
\]
\[
= \int_{\Omega} a(y, y) \, dx \int_{\Omega} T(L(y), L(y)) \, dx
\]

It is easy to obtain
\[
T(R Dy, \cdot, H, \cdot), R(Ry, \cdot, H, \cdot))
\]
\[
= (1 - \mu) |H|^2 k^2 |Dy|^2 + \mu (H(y))^2 k^2 \leq k^2 |H|^2 |Dy|^2
\]

and
\[ T \left( D^2 H \left( D_y \cdot \cdot \cdot , D^2 H \left( D_y \cdot \cdot \cdot \right) \right) \right) \leq 1 \left| D^2 H \right|^2 \left| D_y \right|^2. \]  ... (2.47)

From (2.46), (2.47) and the definition of \( L(y) \), by the Cauchy inequality again, we have

\[
\int_{\Omega} T \left( D^2 y, L(y) \right) \, dx \leq \left( q_2 + q_3 \right)^2 \int_{\Omega} \left| D_y \right|^2 \, dx
\]

\[
\leq q_1^2 (q_2 + q_3)^2 \int_{\Omega} a(y, y) \, dx
\]

\[
\leq 2 q_1^2 (q_2 + q_3)^2 E(t) \]  ... (2.48)

Combining inequalities (2.45) and (2.48) yields

\[
\int_Q E(t) T \left( D^2 y, L(y) \right) \, dQ \leq 2q_1 (q_2 + q_3) \int_T E^2(t) \, dt. \]  ... (2.49)

Inserting (2.44) and (2.49) into (2.41) yields

\[
- \int_Q E(t) H(y) \left( A \dot{y} + \alpha \Delta \theta \right) \, dQ
\]

\[
\leq -\frac{1}{2} b_0 \int_Q E(t) a(y, y) \, dQ + \varepsilon \int_Q E(t) a(y, y) \, dQ
\]

\[
+ C \varepsilon \int_{\Sigma} E(t) \left( D_y \right)^2 \, dQ + C \int_{\Sigma} E(t) \theta^2 \, d\Sigma
\]

\[
+ \int_{\Sigma} E(t) \left( 1y^{\cdot \cdot \cdot}^2 + \left| \frac{\partial y^{\cdot \cdot \cdot}}{\partial \nu} \right|^2 + 1y^{\cdot \cdot \cdot \cdot}^2 + \left| \frac{\partial y^{\cdot \cdot \cdot \cdot}}{\partial \nu} \right|^2 \right) \, d\Sigma
\]

\[
+ 2q_1 (q_2 + q_3) \int_T E^2(t) \, dt. \]  ... (2.50)

Therefore combining H.3, (2.40) and (2.50) with the plate equation, and choosing \( \varepsilon \) small enough, we have

\[
\int_T E^2(t) \, dt \leq CE^2(T) + CE(T)
\]

\[
\int_Q \left( \left| D \theta \right|^2 + \left| \theta \right|^2 \right) \, dQ + \int_{\Sigma} \theta^2 \, d\Sigma
\]
\[
\left\{ + \int_{\Sigma_1} \left( |y'|^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) d\Sigma \right\} \quad \text{(2.51)}
\]

Now define the energy of higher order \( E_1 (t) \) by
\[
E_1 (t) = \frac{1}{2} \left\{ \int_{\Omega} \left[ |y''|^2 + a (y', y') + \beta |\theta'|^2 \right] dx + \int_{\Gamma_1} \left( \rho |y''|^2 + J \left| \frac{\partial y''}{\partial v} \right|^2 \right) d\Gamma \right\} \quad \text{(2.51)}
\]

Then differentiating the equation (2.4) gives \( \frac{dE_1 (t)}{dt} \leq 0 \). Thus
\[
\int_{\Sigma_1} \left( \rho |y''|^2 + J \left| \frac{\partial y''}{\partial v} \right|^2 \right) d\Sigma \leq C_1 E_1 (T) \quad \text{(2.53)}
\]

where \( C_1 \) is a positive constant which may be different from those below. And by Lemma 2.2, we have
\[
\int_{Q} \left( (l D \theta|^2 + l \theta|^2 \right) dQ \int_{\Sigma} \theta^2 d\Sigma \\
+ \int_{\Sigma_1} \left( |y'|^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) d\Sigma \leq C_1 E (T) \quad \text{(2.54)}
\]

From (2.51), (2.53) and (2.54), it follows that
\[
\int_{T} E^2 (t) dt \leq E (T) (CE (0) + C_1 E (0) + C_1 (E_1 (0)) \leq KE (T) R (0) \quad \text{(2.55)}
\]

where we have put \( K = C + C_1 + C_1 \frac{E_1 (0)}{\|u_0\|^2} \).

Finally, we deduce the rational energy decay rate from (2.55) according to the following classical result (see Komornik (4) and Lagnese (5)).

**Lemma 2.4** — Let \( E : R^+ \to R^+ \) be a nonincreasing function. Assume that there exists a positive constant \( K \) such that
\[
\int_{T} E^2 (t) dt \leq KE (0) E (T), \quad \forall T > 0,
\]
then we have

\[ E(t) \leq E(0) \frac{2K}{K + t}, \quad \forall T > 0, \quad \ldots \tag{2.56} \]

3. **Uniform Stabilization of a Simplified Model**

In this section, we consider the following simplified model, in which the bending moment of inertia of the boundary \( J \) is neglected,

\[
\begin{align*}
\rho y'' + \beta \theta - \gamma \Delta \theta + \alpha \theta &= \frac{\partial y'}{\partial v}, & \text{in } \Omega \times [0, \infty) \\
\frac{\partial y}{\partial v} &= 0, & \text{on } \Gamma_0 \times [0, \infty) \\
\Delta y + (1 - \mu) B_1 y + \alpha \theta &= -\frac{\partial y'}{\partial v'}, & \text{on } \Gamma_1 \times [0, \infty) \\
\rho y'' - \frac{\partial \Delta y}{\partial v} - (1 - \mu) B_2 y + \alpha \frac{\partial \theta}{\partial v} &= -y', & \text{on } \Gamma_1 \times [0, \infty), \\
\frac{\partial \theta}{\partial v} &= -\lambda \theta, & \text{on } \Gamma \times [0, \infty) \\
y(0) &= y_0, \quad y'(0) = y_1, \quad \theta(0) = \theta_0, & \text{on } \Omega.
\end{align*}
\]

Introduce the energy space \( \mathcal{H} \),

\[ \mathcal{H} = H^2_{\Gamma_0} (\Omega) \times L^2 (\Omega) \times L^2 (\Omega) \times L^2 (\Gamma_1), \quad \ldots \tag{3.2} \]

and the linear unbounded operator \( A \),

\[ Au = \begin{pmatrix}
-z \\
\mathcal{A} y + \alpha \Delta \theta \\
-\frac{1}{\beta} (\gamma \Delta \theta - \sigma \theta + \alpha \Delta z) \\
-\frac{1}{\rho} \left( \frac{\partial \Delta y}{\partial v} + (1 - \mu) B_2 y - \alpha \frac{\partial \theta}{\partial v} - \eta \right)
\end{pmatrix}, \quad \ldots \tag{3.3} \]

where \( u = (y, z, \theta, \eta) \). The domain of the operator \( A \) is defined as
\[
D(A) = \left\{ u = (y, z, \theta, \eta) \in W \times H^2_{\Gamma_0} (\Omega) \times H^2 (\Omega) \times L^2 (\Gamma_1), \quad \eta = z \big|_{\Gamma_1} \right. \\
\Delta y + (1 - \mu) B_1 y + \alpha \theta + \frac{\partial z}{\partial \nu} = 0, \quad \text{on } \Gamma_1, \quad \frac{\partial \theta}{\partial \nu} + \lambda \theta = 0, \quad \text{on } \Gamma 
\]

where

\[
W = \left\{ y \in H^2_{\Gamma_0} (\Omega), \quad \Delta^2 y \in L^2 (\Omega) \quad \frac{1}{\Gamma_1} \right. \\
\Delta y + (1 - \mu) B_1 y = \nu_1 \in H^2 (\Gamma_1) \\
\partial_{\nu} \Delta y + (1 - \mu) B_2 y = \nu_2 \in L^2 (\Gamma_1) \n\]

Setting

\[
z = y', \quad \eta = y' \big|_{\Gamma_1} \n\]

we transform system (3.1) into an abstract evolutionary equation:

\[
u' + Au = 0, \quad u(0) = u_0. \n\]

By the same arguments as in Section 2, we can get the well-posedness and regularity of the solution to system (3.1), but we omit it.

Let \((y, \theta)\) be a solution of system (3.1). Then we define the associated energy by

\[
E(t) = \frac{1}{2} \left\{ \int_\Omega [(y')^2 + a (y, y) + \beta \theta^2] \, dx + \int_{\Gamma_1} \rho \mid y' \mid^2 \, d\Gamma \right\}. \n\]

Set

\[
Q = \Omega \times [0, T], \quad \Sigma = \Gamma \times [0, T], \quad \Sigma_i = \Gamma_i \times [0, T], \quad i = 0, 1. \n\]

Now we give the following result.

**Theorem 3.1** — Assume that H.1 and (H.2) hold. For any solution of system (3.1), there exist two positive constants \(K\) and \(\omega\) such that

\[
E(t) \leq KE(0) e^{-\omega t}, \quad \forall t > 0. \n\]

Similar to Section 2, we have the following Lemmas.

**Lemma 3.1** — The energy of system (3.1) satisfies
\[
\frac{d}{dt} E(t) = - \int_{\Gamma_1} \left( |y'|^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) d\Gamma \\
- \int_{\Gamma} \left( \gamma |D \theta|^2 + \sigma \theta^2 \right) dx - \lambda \gamma \int_{\Gamma} \theta^2 d\Gamma \leq 0. \quad \text{... (3.10)}
\]

**Lemma 3.2**  — Assume that H.1 and H.2 hold. Let \( y \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) \) and

\[
\begin{align*}
\Delta y + (1 - \mu) B_1 y &= \nu_1, \quad \text{on } \Gamma_1, \\
\frac{\partial \Delta y}{\partial v} + (1 - \mu) B_2 y &= \nu_2 + \nu_3 \quad \text{on } \Gamma_1
\end{align*}
\quad \text{... (3.11)}
\]

with \( \nu_i \in L^2(\Gamma_1), \ i = 1, 2, 3 \). Then we have

\[
- \int_{\Omega} \mathcal{A} y H(y) dx \leq - \frac{1}{2} \int_{\Omega} b_1 (y, y) dx - \int_{\Gamma_1} \nu_3 H(y) d\Gamma \\
+ C \int_{\Gamma_1} (1 |v_1|^2 + 1 |v_2|^2) d\Gamma + l(y), \quad \text{... (3.12)}
\]

where \( l(y) \) is the lower order term with respect to the energy which may also be different from line to line.

**Proof of Theorem 3.1**  — Because of density of \( D(A) \), it is sufficient to consider the smooth initial data \( \nu_0 \in D(A) \). Multiplying the plate equation in system (3.1) from both sides by \( H(y) \) and integrating over \( \Omega \times [0, T] \) by parts, we then obtain in one hand

\[
\int_{\Omega} H(y) y'' dQ = - \int_{\Omega} H(y) y' dx \Bigg|_0^T \\
+ \int_{\Omega} b_1 y'^2 dQ - \frac{1}{2} \int_{\Sigma_1} |y|^2 \langle H, v \rangle d\Sigma. \quad \text{... (3.13)}
\]

By the Cauchy-Schwartz inequality, we have

\[
\left| \int_{\Omega} H(y) y' dx \right| \leq CE(t). \quad \text{... (3.14)}
\]

It follows that from Lemma 3.1 that
\[ \int_{\Sigma_1} \left( 1 y' \right)^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \, d\Sigma \leq E(0). \] ... (3.15)

Combining (3.14), (3.15) with (3.13) gives
\[ \int_Q y'' H(y) \, dQ \geq b_0 \int_Q 1 y' \, dQ - CE(0). \] ... (3.16)

On the other hand, using Lemma 3.2 and (2.43), we obtain
\[- \int_Q H(y) (A y + \alpha \Delta \theta) \, dQ \leq - \frac{1}{2} \int_Q ba(y, y) \, dQ + \int_Q \alpha \left\langle D \theta, D (H(y)) \right\rangle \, dQ + \alpha \int_{\Sigma} \frac{\partial \theta}{\partial v} H(y) \, d\Sigma \]
\[- \int_{\Sigma_1} \rho y'' H(y) \, d\Sigma + C \int_{\Sigma_1} \left( (1 y')^2 + \left| \frac{\partial y'}{\partial v} \right|^2 + \theta^2 \right) \, d\Sigma + \text{lot}(y) \]
\[\leq - \frac{1}{2} b_0 \int_Q a(y, y) \, dQ + C \int_{\Sigma_1} \left( (1 y')^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) \, d\Sigma + \int_{\Sigma_1} \rho y'' H(y) \, d\Sigma \]
\[+ \varepsilon \int_Q a(y, y) \, dQ + C \varepsilon \int_Q (1 D \theta)^2 \, dQ + C \int_{\Sigma} \theta^2 \, d\Sigma + \text{lot}(y), \] ... (3.17)

where \text{lot}(y) = \int_0^T l(y) \, dt. Integrating by parts, we have
\[- \int_{\Sigma_1} y'' H(y) \, d\Sigma = - \int_{\Sigma_1} y' H(y) \, d\Gamma_1 \bigg|_0^T + \int_{\Sigma_1} H(y') y' \, d\Sigma. \] ... (3.18)

Again by Cauchy-Schwartz inequality, we have
\[\left| \int_{\Gamma_1} H(y) y' \, d\Gamma \right|_0^T \leq CE(0). \] ... (3.19)

In addition, a direct computation gives
\[\int_{\Gamma_1} H(y') y' \, d\Gamma \leq \frac{R}{2} \int_{\Gamma_1} \left( (1 y')^2 + \left| \frac{\partial y'}{\partial v} \right|^2 \right) \, d\Gamma, \] ... (3.20)
where

\[ R = \sup_{x \in \Gamma} |H|, \]

Inserting (3.18)-(3.20) into (3.7), we obtain

\[- \int_Q (\mathcal{A} \psi + \alpha \Delta \theta) H(y) \, dQ \leq \left( -\frac{1}{2} b_0 + \varepsilon \right) \int_Q a(y, y) \, dQ + CE(0) + \]

\[ + C \int_{\Sigma} \left( l y' l^2 + \left| \frac{\partial y'}{\partial n} \right|^2 \right) d\Sigma + C \int_Q (1 D \theta l^2 + \theta^2) \, dQ + \int_{\Sigma} \theta^2 d\Sigma + \text{lot}(y), \]

\[ \text{... (3.21)} \]

Combining (3.16), (3.21) with the plate equation in system (3.1), and choosing \( \varepsilon \) small enough, we get

\[ \int_0^T E(t) \, dt \leq CE(0) + \]

\[ + C \left\{ \int_{\Sigma} \left( l y' l^2 + \left| \frac{\partial y'}{\partial n} \right|^2 \right) d\Sigma + \int_Q (1 D \theta l^2 + \theta^2) \, dQ + \int_{\Sigma} \theta^2 d\Sigma \right\} + \text{lot}(y) \]

\[ \text{... (3.22)} \]

From Lemma 3.1, it follows that

\[ E(t) \leq C_T \left\{ \int_{\Sigma} \left( l y' l^2 + \left| \frac{\partial y'}{\partial n} \right|^2 \right) d\Sigma + \int_Q (1 D \theta l^2 + \theta^2) \, dQ + \int_{\Sigma} \theta^2 d\Sigma \right\} + \text{lot}(y) \]

\[ \text{... (3.23)} \]

By a compactness/uniqueness argument which is similar to that in Lasiecka (7), the lower order terms can be absorbed. So

\[ E(t) \leq C_T \left\{ \int_{\Sigma} \left( l y' l^2 + \left| \frac{\partial y'}{\partial n} \right|^2 \right) d\Sigma + \int_Q (1 D \theta l^2 + \theta^2) \, dQ + \int_{\Sigma} \theta^2 d\Sigma \right\} \]

\[ \leq C_T (E(0) - E(T)), \]

\[ \text{... (3.24)} \]

for \( T \) big enough. Thus
\[ E(T) \leq \frac{C_T}{1 + C_T} E(0). \] \hfill \ldots (3.25)

Finally, by the known result of Pazy (9), we obtain the uniform energy decay rate (3.9). The theorem is thus proved.

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