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ON A SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS INVOLVING THE OWA-SRIVASTAVA FRACTIONAL CALCULUS OPERATORS

JAE HO CHOI

Department of Mathematics Education, Daegu National University of Education, 1797-6 *Daemyong* 2 *Dong, Namgu, Daegu* 705-715, *Korea*
E-Mail : choijh@dnue.ac.kr

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We introduce a new subclass $\mathcal{T}(\alpha, \beta, \gamma)$ of analytic functions with negative coefficients which is defined by using the Owa-Srivastava fractional calculus operator Ω^{λ} in the unit disk $\mathbb{U}$. The main object of this paper is to investigate coefficient estimate, distortion theorem and various other interesting properties for analytic functions belonging to the class $\mathcal{T}(\alpha, \beta, \gamma)$.

Key Words : Analytic Functions; Starlike Functions; Convex Functions; Fractional Derivative; Fractional Integral; Owa-srivastava Fractional Calculus Operator

1. INTRODUCTION AND DEFINITIONS

Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad \text{... (1.1)}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}, \mathcal{S}^*(\rho)$ and $\mathcal{K}(\rho)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order ρ and convex of order ρ in $\mathbb{U}$ (see, e.g., [13]).

We note that

$$f(z) \in \mathcal{K}(\rho) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\rho), \qquad \text{... (1.2)}$$

and that $\mathcal{S}^*(\rho) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(\rho) \subset \mathcal{K}(0) \equiv \mathcal{K}(0 \leq \rho < 1)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the subclasses of $\mathcal{A}$ consisting of functions being starlike and convex in $\mathbb{U}$, respectively.

Several essentially equivalent definitions of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature (cf., e.g., (2), (7) and (9), p.28 et seq.). We state the following definitions due to Owa (3) which have been used rather frequently in the theory of analytic functions (see also (6) and (12)).

Definition 1 — The fractional integral of order $\lambda\,(\lambda > 0)$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta, \qquad \text{... (1.3)}$$

and the fractional derivative of order $\lambda\,(0 \leq \lambda < 1)$ by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\, d\zeta, \qquad \text{... (1.4)}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.3) (and that of $(z-\zeta)^{-\lambda}$ in (1.4)) is removed by requiring $\log\,(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2 — Under the hypotheses of Definition 1, the fractional derivative of order $n+\lambda$ $(0 \leq \lambda < 1;\, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z). \qquad \text{... (1.5)}$$

With the aid of the above definitions, Owa and Srivastava (6) defined the fractional calculus operator Ω^{λ} $(\lambda \in \mathbb{R};\, \lambda \neq 2, 3, 4, \ldots)$ by

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda)\, z^{\lambda}\, \mathcal{D}_z^{\lambda} f(z), \qquad \text{... (1.6)}$$

for functions (1.1) belonging to the class $\mathcal{A}$ (see also (10) and (11)).

Recently, Choi *et al.* (1) investigated the subclass $\mathcal{A}(\alpha, \beta, \gamma)$ of $\mathcal{A}$ for $\alpha < 2, \beta < 2$ and $\gamma < 1$, which was defined by using the Owa-Srivastava fractional calculus operator Ω^{λ} as follows:

$$\mathcal{A}(\alpha, \beta, \gamma) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)} \right) > \gamma \text{ in } \mathbb{U} \right\}. \qquad \text{... (1.7)}$$

We note that $\mathcal{A}(1, 0, \gamma) = \mathcal{S}^*(\gamma)$ and $\mathcal{A}(\alpha+1, 0, \gamma) = \mathcal{S}^*(\gamma, \alpha)$ which was studied by Owa and Shen (5).

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of all functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0; z \in \mathbb{U}). \qquad \text{... (1.8)}$$

We introduce the class of function $\mathcal{T}(\alpha, \beta, \gamma)$ as follows:

$$\mathcal{T}(\alpha, \beta, \gamma) = \mathcal{A}(\alpha, \beta, \gamma) \cap \mathcal{T}. \qquad \text{... (1.9)}$$

We also note that the class $\mathcal{T}(\alpha+1, 0, \gamma) = \mathcal{T}^*(\gamma, \alpha)$ was studied by Owa (4), and the special case $\mathcal{T}(1, 0, \gamma) = \mathcal{T}^*(\gamma, 0)$ was studied by Silverman (8).

In this paper, we investigate coefficient estimate, distortion theorem and radii of star-likeness and convexity for analytic functions belonging to the class $\mathcal{T}(\alpha, \beta, \gamma)$. Furthermore, some inclusion properties of this class are also considered.

2. Coefficient Bounds and Distortion Theorem

We begin by proving

***Theorem* 1** — *Let $\beta \leq \alpha < 2$ and $\gamma < 1$, and let the function $f(z)$ be defined by (1.8). Then $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} - \gamma \frac{\Gamma(2-\beta)}{\Gamma(n+1-\beta)} \right) n!\, a_n \leq 1 - \gamma. \qquad \text{... (2.1)}$$

The result (2.1) is sharp.

Proof : Using the definition of fractional calculus with (1.8), we have

$$\Omega^{\lambda} f(z) = z - \sum_{n=2}^{\infty} \phi(n, \lambda)\, a_n z^n \qquad (a_n \geq 0; \lambda < 2), \qquad \text{... (2.2)}$$

where

$$\phi(n, \lambda) = \frac{n!\, \Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \qquad (n \geq 2). \qquad \text{... (2.3)}$$

Assume that the function $f(z)$ defined by (1.8), is in the class $\mathcal{T}(\alpha, \beta, \gamma)$. Making use of (1.6) and (2.2), choosing values of z on the real axis, and then letting $z \to 1^-$ through real values, we obtain easily at the assertion (2.1) of Theorem 1.

Conversely, assume that the inequality (2.1) holds true and let $|z| = 1$. Then, from (2.2) we obtain

$$\left|\frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} (\phi(n, \alpha) - \phi(n, \beta))\, a_n z^n}{z - \sum_{n=2}^{\infty} \phi(n, \beta)\, a_n z^n}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} |\phi(n, \alpha) - \phi(n, \beta)|\, a_n}{1 - \sum_{n=2}^{\infty} |\phi(n, \beta)|\, a_n}.$$

This last expression is bounded above by $1-\gamma$ if

$$\sum_{n=2}^{\infty} \left\{ |\phi(n, \alpha) - \phi(n, \beta)| + (1-\gamma)\, |\phi(n, \beta)| \right\} a_n \leq 1 - \gamma. \qquad \ldots (2.4)$$

Since $\beta \leq \alpha < 2$, it is easily seen that $\phi(n, \beta) > 0$ and $\phi(n, \alpha) - \phi(n, \beta) \geq 0$. Therefore, we can rewrite the inequality (2.4) as

$$\sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta))\, a_n \leq 1 - \gamma. \qquad \ldots (2.5)$$

Hence, we conclude from (2.5) that $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$.

Finally, we note that the result (2.1) is sharp, the extremal function being

$$f(z) = z - \frac{(1-\gamma)\,\Gamma(n+1-\alpha)\,\Gamma(n+1-\beta)}{n!\left\{\Gamma(2-\alpha)\,\Gamma(n+1-\beta) - \gamma\Gamma(2-\beta)\,\Gamma(n+1-\alpha)\right\}}\, Z^n \quad (n \geq 2).$$

Remark : If $\alpha = \lambda + 1$ and $\beta = 0$ in Theorem 1, then it would immediately yield the result of Owa [4, Theorem 2.1].

Corollary 1 — Let $\beta \leq \alpha < 2$ and $\gamma < 1$. If $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, then

$$a_n \leq \frac{(1-\gamma)\,\Gamma(n+1-\alpha)\,\Gamma(n+1-\beta)}{n!\left\{\Gamma(2-\alpha)\,\Gamma(n+1-\beta) - \gamma\Gamma(2-\beta)\,\Gamma(n+1-\alpha)\right\}} \quad (n \geq 2). \qquad \ldots (2.6)$$

***Theorem* 2** — *Let* $0 \leq \alpha < 2$, $\beta \leq \alpha$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, *then*

***Theorem* 2** — *Let* $0 \leq \alpha < 2, \beta \leq \alpha$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, *then*

$$|z| - \frac{(1-\gamma)(2-\alpha)(2-\beta)}{(2\{2-\beta) - \gamma(2-\alpha)\}}|z|^2 \leq |f(z)| \leq |z|$$

$$+ \frac{(1-\gamma)(2-\alpha)(2-\beta)}{(2\{2-\beta) - \gamma(2-\alpha)\}}|z|^2$$

for $z \in \mathbb{U}$.

PROOF : Let $0 \leq \alpha < 2, \beta \leq \alpha$ and $0 \leq \gamma < 1$ and let $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$. Then, by virtue of Theorem 1, we obtain

$$(\phi(2, \alpha) - \gamma\phi(2, \beta)) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta)) a_n$$

$$\leq 1 - \gamma,$$

where $\phi(n, \alpha)$ and $\phi(n, \beta)$ are given by (2.3). This readily yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\gamma}{\phi(2, \alpha) - \gamma\phi(2, \beta)} = \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2\{(2-\beta) - \gamma(2-\alpha)\}}. \quad \text{... (2.7)}$$

Consequently, we get

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2\{(2-\beta) - \gamma(2-\alpha)\}}|z|^2$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{(1-\gamma)(2-\alpha)(2-\beta)}{(2\{2-\beta) - \gamma(2-\alpha)\}}|z|^2,$$

which completes the proof of Theorem 2.

***Theorem* 3** — *Let* $\beta \leq 0, \beta \leq \alpha < 2$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, *then*

$$\left| \mathcal{D}_z^{\alpha} f(z) \right| \leq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \left(1 + \frac{(1-\gamma)(2-\beta)}{2-\beta-\gamma(2-\alpha)}|z| \right) \quad \text{... (2.8)}$$

and

$$\left| \mathcal{D}_z^{\alpha} f(z) \right| \geq \frac{|z|^{1-\alpha}}{\Gamma(2-\alpha)} \left(1 - \frac{(1-\gamma)(2-\beta)}{2-\beta-\gamma(2-\alpha)}|z| \right) \quad \text{... (2.9)}$$

for $z \in \mathbb{U}$. *The results (2.8) and (2.9) are sharp.*

PROOF : Let $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$. Then, in view of Theorem 1, we obtain

$$\sum_{n=2}^{\infty} \phi(n, \alpha)\, a_n \leq 1-\gamma+\gamma \sum_{n=2}^{\infty} \phi(n, \beta)\, a_n, \qquad \text{... (2.10)}$$

where $\phi(n, \alpha)$ and $\phi(n, \beta)$ are given in (2.3). Since $\beta \leq 0$, it is easily seen that

$$0<\phi(n, \beta) \leq \phi(2, \beta) = \frac{2}{2-\beta}. \qquad \text{... (2.11)}$$

By applying (2.7), (2.10) and (2.11), we obtain

$$\sum_{n=2}^{\infty} \phi(n, \alpha)\, a_n \leq 1-\gamma+\gamma\phi(2, \beta) \sum_{n=2}^{\infty} a_n \qquad \text{... (2.12)}$$

$$\leq \frac{(1-\gamma)(2-\beta)}{2-\beta-\gamma(2-\alpha)}. \qquad \text{... (2.12)}$$

Hence, by virtue of (1.6) and (2.12), we observe that

$$\left| \Gamma(2-\alpha)\, z^{\alpha} \mathcal{D}_z^{\alpha} f(z) \right| \leq |z|+|z|^2 \sum_{n=2}^{\infty} \phi(n, \alpha)\, a_n$$

$$\leq |z|+|z|^2 \frac{(1-\gamma)(2-\beta)}{2-\beta-\gamma(2-\alpha)}$$

and

$$\left| \Gamma(2-\alpha)\, z^{\alpha} \mathcal{D}_z^{\alpha} f(z) \right| \geq |z|-|z|^2 \frac{(1-\gamma)(2-\beta)}{2-\beta-\gamma(2-\alpha)},$$

which prove the assertions (2.8) and (2.9) of Theorem 3.

3. FURTHER PROPERTIES OF THE CLASS $\mathcal{T}(\alpha, \beta, \gamma)$

Next, we prove

***Theorem* 4** — *Let* $\beta_1 \leq \beta_2 \leq \alpha < 2$ *and* $0 \leq \gamma < 1$. *Then*

$$\mathcal{T}(\alpha, \beta_1, \gamma) \subset \mathcal{T}(\alpha, \beta_2, \gamma).$$

PROOF : Let $f(z) \in \mathcal{T}(\alpha, \beta_1, \gamma)$. Since $\beta_1 \leq \beta_2 < 2$, we see that $\phi(n, \beta_j) > 0$ $(j = 1, 2)$ and

$$\frac{\phi(n, \beta_2)}{\phi(n, \beta_1)} \geq 1 \qquad (n \leq 2),$$

where $\phi(n, \beta_j)$ $(j = 1, 2)$ is given by (2.3). Therefore, by using Theorem 1, we have

$$\sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta_2))\, a_n \leq \sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta_1))\, a_n$$

$$\leq 1 - \gamma$$

which implies that $f(z) \in \mathcal{T}(\alpha, \beta_2, \gamma)$.

***Theorem* 5** — Let $\beta \leq \alpha_1 \leq \alpha_2 < 2$ and $\gamma < 1$. Then

$$\mathcal{T}(\alpha_2, \beta, \gamma) \subset \mathcal{T}(\alpha_1, \beta, \gamma).$$

The proof of Theorem 5 uses Theorem 1 in a straightforward manner. The details may be omitted.

Corollary 2 — Let $\beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 < 2$ and $0 \leq \gamma < 1$. Then

$$\mathcal{T}(\alpha_2, \beta_1, \gamma) \subset \mathcal{T}(\alpha_1, \beta_2, \gamma).$$

***Theorem* 6** — Let $\beta \leq \alpha < 2$ and $\gamma < 1$, and let $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1(\alpha, \beta, \gamma, \rho)$, where

$$r_1(\alpha, \beta, \gamma, \rho)$$

$$= \inf_n \left[\frac{(1-\rho)n!\,\{\Gamma(2-\alpha)\,\Gamma(n+1-\beta) - \gamma\Gamma(2-\beta)\,\Gamma(n+1-\alpha)\}}{(n-\rho)\,(1-\gamma)\,\Gamma(n+1-\alpha)\,\Gamma(n+1-\beta)} \right]^{1/(n-1)} \qquad (n \geq 2).$$

PROOF : Let $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$. Then it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$$

for $|z| < r_1(\alpha, \beta, \gamma, \rho)$. The rest of the details involved are fairly straightfoward and may be omitted.

Corollary 3 — Let $\beta \leq \alpha < 2$ and $\gamma < 1$, and let $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2(\alpha, \beta, \gamma, \rho)$, where

$$r_2(\alpha, \beta, \gamma, \rho)$$

$$= \inf_{n} \left[\frac{(1-\rho)n! \{ \Gamma(2-\alpha)\Gamma(n+1-\beta) - \gamma\Gamma(2-\beta)\Gamma(n+1-\alpha) \}}{n(n-\rho)(1-\gamma)\Gamma(n+1-\alpha)\Gamma(n+1-\beta)} \right]^{1/(n-1)}$$

$(n \geq 2)$.

***Theorem* 7** — *Let* $\beta \leq \alpha < 2$ *and* $\gamma < 1$. *Also let* $f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n$ $(a_{n,j} \geq 0;\ j = 1, \ldots, m)$ *be in the class* $\mathcal{T}(\alpha, \beta, \gamma)$. *Then the function* $h(z)$ *defined by*

$$h(z) = \frac{1}{m} \sum_{j=1}^{m} f_j(z) \qquad \ldots (3.1)$$

also belongs to the class $\mathcal{T}(\alpha, \beta, \gamma)$.

PROOF : From (3.1), we obtain the expansion

$$h(z) = z - \sum_{n=2}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) z^n.$$

Since $f_j(z) \in \mathcal{T}(\alpha, \beta, \gamma)$ $(j = 1, \ldots, m)$, by using Theorem 1, we have

$$\sum_{n=2}^{\infty} \{ \phi(n, \alpha) - \gamma \phi(n, \beta) \} \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) \leq 1 - \gamma,$$

where $\phi(n, \alpha)$ and $\phi(n, \beta)$ are given by (2.3). Hence, by Theorem 1, $h(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, which completes the proof of Theorem 5,

***Theorem* 8** — *Let* $\beta \leq \alpha < 2$, $\gamma < 1$ *and* $\sigma > -1$. *If* $f(z) \in \mathcal{T}(\alpha, \beta, \gamma)$, *then the function* $F(z)$ *given by*

$$F(z) = \frac{\sigma + 1}{z^{\sigma}} \int_0^z t^{\sigma - 1} f(t)\, dt \qquad (\sigma > -1;\ z \in \mathbb{U}) \qquad \ldots (3.2)$$

is also in the class $\mathcal{T}(\alpha, \beta, \gamma)$.

PROOF : Let the function $f(z)$ defined by (1.8) be in the class $\mathcal{T}(\alpha, \beta, \gamma)$. Then, from (3.2) we have

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n,$$

where $b_n = ((c+1)/(n+c))\, a_n$. Furthermore, by using Theorem 1, we obtain

$$\sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta))\, b_n = \sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta)) \left(\frac{c+1}{n+c} \right) a_n$$

$$\leq \sum_{n=2}^{\infty} (\phi(n, \alpha) - \gamma\phi(n, \beta))\, a_n$$

$$\leq 1 - \gamma,$$

where $\phi(n, \alpha)$ and $\phi(n, \beta)$ are given by (2.3). Hence, in view of Theorem 1, we conclude that $F(z) \in \mathcal{T}(\alpha, \beta, \gamma)$.

REFERENCES

1. J. H. Choi, Y.C. Kim and S. Owa, Fractional calculus operator and its applications in the univalent functions, *Fract. Calc. Appl. Anal.*, **4** (2001), 367-78.
2. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of Integral Transforms, Vol. II, McGraw-Hill Book Company, New York, Toronto and London, 1954.
3. S. Owa, On the distortion theorems. I, *Kyungpook Math. J.*, **18** (1978), 53-59.
4. S. Owa, On new subclasses of analytic and univalent functions, *J. Korean Math. Soc.*, **22** (1985), 151-72.
5. S. Owa and C.Y. Shen, Generalized classes of starlike and convex functions of order α, *Internat. J. Math. & Math. Sci.*, **8** (1985), 455-67.
6. S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions. *Canad. J. Math.*, **39** (1987), 1057-77.
7. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integral and Derivatives, Theory and Applications*, Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Toronto and Melbourne, 1993.
8. H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109-16.
9. H. M. Srivastava and R. G. Buschman, *Theory and Applications of Convolution Integral Equations, Mathematics and Its Applications*, 79, Kluwer Academic Publishers, Dordrecht, Boston and London, 1992.
10. H. M. Srivastava, A. K. Mishra and M. K. Das, A nested class of analytic functions defined by fractional calculus, *Comm. Appl. Anal.*, **2** (1998), 321-32.
11. H. M. Srivastava, A. K. Mishra and M. K. Das, A unified operator in fractional calculus and its applications to a nested class of analytic functions with negative coefficients, *Complex Variables Theory Appl.*, **40** (1999), 119-32.
12. H. M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japan.*, **29** (1984), 383-89.
13. H. M. Srivastava and S. Owa (Editors), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.