ANALYSIS OF A BULK QUEUE WITH STATE DEPENDENT ARRIVALS AND MULTIPLE VACATIONS

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An $M^I/G(a, b)/1$ queueing system with state dependent arrivals and multiple vacations is discussed in this paper. The arrivals occur in bulk with a rate $\lambda$, when the server is busy and with a rate $\lambda_0$, when the server is on vacation. This assumption is quite meaningful, because in practice the arrival rate need not be constant always. The service starts only when there are at least ‘a’ customers waiting for service. After a service (or a vacation) if the number of customers waiting is $\xi(\xi \geq a)$ then the server serves a batch of $\min(\xi, b)$ customers, where $b \geq a$. After finishing a service if the queue length is less than ‘a’, then the server leaves for a vacation of random length. When he returns again if the queue length is less than ‘a’, then he leaves for another vacation and so on, until he finally finds at least ‘a’ customers waiting for service. The probability generating function of the number of customers in the queue at an arbitrary time epoch is obtained. Expressions for the queue length distribution, expected length of queue and expected length of idle and busy periods are derived. A cost model for the queueing system is discussed with a numerical illustration.

Key Words: Laplace-stieltjes Transformation, Multiple Vacations, State Dependent Arrivals, Steady State Solution

1. INTRODUCTION

Server vacation models are useful for the system in which the server wants to utilize his idle time for different purposes. Application of vacation models can be found in production systems, designing local area networks and data communication systems. To cite a live example of the bulk queueing models with state dependent arrivals and multiple vacations, consider the environment of a theme park situated in a metropolitan city. The theme park contains so many counters for recreation and entertainment. Total fee will be collected at the entry itself. On exchange of token, people arbitrarily (or on their own interest) select any counter for recreation. Consider the giant wheel. There exists minimum ‘a’ and maximum ‘b’ capacities (in terms of numbers) to operate the giant wheel. When the number of customers waiting in the queue to make use of the giant wheel is less than ‘a’, then the machine will not be operated and the operator will be looking in to other works associated with
the giant wheel, such as lubricating the moving parts, tightening the small nuts, cleaning the seats etc., When the number of customers waiting in the queue to use giant wheel is more than ‘a’ say $\xi$, then a batch of $\min(\xi, b)$ customers will be taken for the trip. When the giant wheel is operated normally, many people want to utilize it and hence the arrival rate will be high. If it is not functioning or when the operator is on vacation, people normally go for some other entertainment to optimize their time. Hence the arrival rate is low when it is not functioning.

Queueing problems with server vacations have been analyzed by various authors with several combinations. For a detailed survey on queueing systems with server vacations, one can refer to Refs. (4,15). Neuts [12] has studied general bulk service queueing system. Lee [10] has analyzed an $M^{x}/G(a, b)/l$ queue with single vacation. Borthakur and Medhi [2] have studied a queueing system with arrival and services in batches of variable size. They have derived the queue length distribution for the $M^{x}/G(a, b)/l$ model. Characteristics of $M^{x}/G/l$ system with N-policy are analyzed by Ho Woo Lee et al., [6]. They have analyzed an $M^{x}/G/l$ queueing system with N-policy and multiple vacations using supplementary variable technique. Soon seok Lee et al., [13] considered an $M^{x}/G/l$ with N-policy and single vacation. Krishna Reddy et al., [8] have discussed a bulk queue with N-policy multiple vacations and setup time. Arumuganathan [1] has analyzed a bulk queue with bulk service and multiple vacations. Queueing models with controllable service rates have been discussed by Gray et al., [5]. They have discussed an $M/G/l$ model with controllable service rates. This paper is concentrated on vacation systems with state dependent arrivals.

This paper deals with the analysis of a single server queueing system with Poisson bulk state dependent arrivals, general bulk service and multiple vacations. The arrival rate is high ($\lambda$) when the server is busy and low ($\lambda_0$) when the server is on vacation. The service is done according to Neut’s general bulk service rule. After finishing a service if the queue length is less than ‘a’, then the server leaves for a vacation of random length. When he returns again if the queue length is less than ‘a’, then he leaves for another vacation and so on, until he finally finds at least ‘a’ customers waiting for service. After a vacation if he finds more than ‘a’ customers for service (say) $\xi$, then he serves a batch of $\min(\xi, b)$ customers, where $b \geq a$. We denote the queueing system as an $M^{x}/G(a, b)/l$ queueing system with state dependent arrivals and multiple vacations. For such a queueing system, the steady state queue length distribution at an arbitrary time is obtained. Expression for the expected length of the queue is derived. Expressions for the expected length of idle and busy periods are derived, using conditional expectation concept as discussed in [14]. Numerical solution for a particular case of the model is also presented.
2. Notations

\( \lambda \) arrival rate when the server is busy

\( \lambda_0 \) arrival rate when the server is on vacation

\( X \) group size random variable

\( g_k \) \( Pr \left( X = k \right) \)

\( X(z) \) probability generating function (PGF) of \( X \)

\( S(\cdot) \) cumulative distribution function (cdf) of services times.

\( V(\cdot) \) cumulative distribution function (cdf) of vacation times.

\( s(x) \) the probability density function of \( S \)

\( v(x) \) the probability density function of \( V \)

\( \mathcal{Z}(\theta) \) Laplace-Stieltjes transform (LST) of \( S \)

\( \mathcal{V}(\theta) \) Laplace-Stieltjes transform (LST) of \( V \)

\( S^0(t) \) remaining service time of a batch in service at time \( t \)

\( V^0(t) \) remaining vacation time of the server on vacation at time \( t \)

\( N_q(t) \) queue length at time \( t \)

\( N_s(t) \) number of customers in service at time \( t \)

\( Y(t) = 0 \) if the server is on vacation

\( = 1 \) if the server is busy.

\[ Z(t) = j \] if the server is on \( j \)th vacation starting from idle period.

\[ P \left[ \begin{array}{c} N_q(t) = i, N_s(t) = j, x \leq S^0(t) \leq x + dt, Y(t) = 1 \end{array} \right] \quad a \leq i \leq b, j \geq 0 \]

\[ Q_j(n, t) dt = Pr \left\{ N_q(t) = n, x \leq V^0(t) \leq x + dt, Y(t) = 0, Z(t) = j \right\} \quad n \geq 0, j \geq 1. \]

3. Queue Length Distribution

The supplementary variable technique was introduced by Cox [3] and was followed by Lee [9]. Lee introduced an effective technique for solving queueing models using supplementary variable. Now the following equations are obtained for the queueing system, using supplementary variable technique:

\[ P_{i0} (x - \Delta t, t + \Delta t) = P_{i00} (x, t) (1 - \lambda \Delta t) \]
\[ + \sum_{m=a}^{b} P_{m}^{ij} (0, t) s(x) \Delta t + \sum_{i=1}^{\infty} Q_{li} (0, t) s(x) \Delta t \quad (a \leq i \leq b) \]

\[ P_{ij} (x - \Delta t, t + \Delta t) = P_{ij} (x, t) (1 - \lambda \Delta t) \]

\[ + \sum_{k=1}^{j} P_{i, j-k} (x, t) \lambda g_{k} \Delta t \quad (a \leq i \leq b - 1), \quad (j \geq 1) \]

\[ P_{b,j} (x - \Delta T, t + \Delta t) = P_{b,j} (x, t) (1 - \lambda \Delta t) \]

\[ + \sum_{m=a}^{b} P_{m, b+j} (0, t) s(x) \Delta t \]

\[ + \sum_{k=1}^{j} P_{b, j-k} (x, t) \lambda g_{k} \Delta t + \sum_{l=1}^{\infty} Q_{l, b+j} (0, t) s(x) \Delta t \quad (j \geq 1) \]

\[ Q_{10} (x - \Delta t, t + \Delta t) = Q_{10} (x, t) (1 - \lambda_{0} \Delta t) + \sum_{m=a}^{b} P_{m0} (0, t) v(x) \Delta t \]

\[ Q_{1n} (x - \Delta t, t + \Delta t) = Q_{1n} (x, t) (1 - \lambda_{0} \Delta t) + \sum_{m=a}^{b} P_{mn} (0, t) v(x) \Delta t \]

\[ + \sum_{k=1}^{n} Q_{1, n-k} (x, t) \lambda_{0} g_{k} \Delta t \quad (1 \leq n \leq a - 1) \]

\[ Q_{1n} (x - \Delta t, t + \Delta t) = Q_{1n} (x, t) (1 - \lambda_{0} \Delta t) + \sum_{k=1}^{n} Q_{1, n-k} (x, t) \lambda_{0} g_{k} \Delta t \quad (n \geq a) \]

\[ Q_{j0} (x, t, t + \Delta t) = Q_{j0} (x, t) (1 - \lambda_{0} \Delta t) + Q_{j-1,0} (0, t) v(x) \Delta t \quad (j \geq 2) \]

\[ Q_{jn} (x - \Delta t, t + \Delta t) = Q_{jn} (x, t) (1 - \lambda_{0} \Delta t) + Q_{j-1,n} (0, t) v(x) \Delta t \]

\[ + \sum_{k=1}^{n} Q_{j, n-k} (x, t) \lambda_{0} g_{k} \Delta t \quad (j \geq 2), \quad (1 \leq n \leq a - 1) \]

\[ Q_{jn} (x - \Delta t, t + \Delta t) = Q_{jn} (x, t) (1 - \lambda_{0} \Delta t) + \sum_{k=1}^{n} Q_{j, n-k} (x, t) \lambda_{0} g_{k} \Delta t \]
From the above equations the steady state queue length equations are obtained as follows:

\[ -\frac{d}{dx} P_{i0}(x) = -\lambda P_{i0}(x) + \sum_{m=a}^{b} P_{mi}(0) s(x) + \sum_{l=1}^{\infty} Q_{li}(0) s(x) \]

\[ (a \leq i \leq b) \quad \ldots \quad (1) \]

\[ -\frac{d}{dx} P_{ij}(x) = -\lambda P_{ij}(x) + \lambda \sum_{k=1}^{j} P_{i,j-k}(x) g_k \]

\[ (a \leq i \leq b-1), \quad (j \geq 1) \quad \ldots \quad (2) \]

\[ -\frac{d}{dx} P_{bj}(x) = -\lambda P_{bj}(x) + \sum_{m=a}^{b} P_{m,b+j}(0) s(x) \]

\[ + \sum_{i=1}^{\infty} Q_{l,b+j}(0) s(x) + \lambda \sum_{k=1}^{j} P_{b,j-k}(x) g_k \]

\[ (j \geq 1) \quad \ldots \quad (3) \]

\[ -\frac{d}{dx} Q_{10}(x) = -\lambda_0 Q_{10}(x) + \sum_{m=a}^{b} P_{m0}(0) v(x) \]

\[ \ldots \quad (4) \]

\[ -\frac{d}{dx} Q_{1n}(x) = -\lambda_0 Q_{1n}(x) + \sum_{m=a}^{b} P_{mn}(0) v(x) \]

\[ + \lambda_0 \sum_{k=1}^{n} Q_{1,n-k}(x) g_k \]

\[ (1 \leq n \leq a-1) \quad \ldots \quad (5) \]

\[ -\frac{d}{dx} Q_{1n}(x) = -\lambda_0 Q_{1n}(x) + \lambda_0 \sum_{k=1}^{n} Q_{1,n-k}(x) g_k \]

\[ (n \geq a) \quad \ldots \quad (6) \]

\[ -\frac{d}{dx} Q_{j0}(x) = -\lambda_0 Q_{j0}(x) + Q_{j-1,0}(0) v(x) \]

\[ (j \geq 2) \quad \ldots \quad (7) \]

\[ -\frac{d}{dx} Q_{jn}(x) = -\lambda_0 Q_{jn}(x) + Q_{j-1,n}(0) v(x) + \lambda_0 \sum_{k=1}^{n} Q_{j,n-k}(x) g_k \]

\[ (j \geq 2), \quad (1 \leq n \leq a-1) \quad \ldots \quad (8) \]
\[- \frac{d}{dx} Q_{jn} (x) = - \lambda_0 Q_{jn} (x) + \lambda_0 \sum_{k=1}^{n} Q_{j,n-k} (x) g_k \quad (j \geq 2), \quad (n \geq a) \quad \ldots (9)\]

The Laplace stieltjes transforms of \( P_{in} (x) \) and \( Q_{jn} (x) \) are defined as

\[
\mathcal{P}_{in} (\theta) = \int_{0}^{\infty} e^{-\theta x} P_{in} (x) \, dx \quad \text{and} \quad \mathcal{Q}_{jn} (\theta) = \int_{0}^{\infty} e^{-\theta x} Q_{jn} (x) \, dx
\]

Taking Laplace Stieltjes transform on both sides of the eqs. (1)-(9), we have

\[
\theta \mathcal{P}_{ij} (\theta) - P_{ij} (0) = \lambda \mathcal{P}_{ij} (\theta) - \sum_{m=a}^{b} P_{mi} (0) \mathcal{S} (\theta) - \mathcal{S} (\theta) \sum_{l=1}^{\infty} Q_{jl} (0)
\quad (a \leq i \leq b) \quad \ldots (10)
\]

\[
\theta \mathcal{P}_{ij} (\theta) - P_{ij} (0) = \lambda \mathcal{P}_{ij} (\theta) - \lambda \sum_{k=1}^{j} \mathcal{P}_{i,j-k} (\theta) g_k
\quad (a \leq i \leq b - 1), \quad (j \geq 1) \quad \ldots (11)
\]

\[
\theta \mathcal{P}_{bij} (\theta) - P_{bij} (0) = \lambda \mathcal{P}_{bij} (\theta) - \lambda \sum_{k=1}^{j} \mathcal{P}_{i,j-k} (\theta) g_k
\quad (a \leq i \leq b - 1)
\quad (j \geq 1)
\quad \ldots (12)
\]

\[
\theta \mathcal{Q}_{in} (\theta) - Q_{10} (0) = \lambda_0 \mathcal{Q}_{10} (\theta) - \sum_{m=a}^{b} P_{m0} (0) \mathcal{V} (\theta) \quad \ldots (13)
\]

\[
\theta \mathcal{Q}_{10} (\theta) - Q_{10} (0) = \lambda_0 \mathcal{Q}_{10} (\theta) - \sum_{m=a}^{b} P_{mn} (0) \mathcal{V} (\theta) - \lambda_0
\quad (1 \leq m \leq a)
\quad \ldots (14)
\]

\[
\theta \mathcal{Q}_{jn} (\theta) - Q_{jn} (0) = \lambda_0 \mathcal{Q}_{jn} (\theta) - \lambda_0 \sum_{k=1}^{n} \mathcal{Q}_{j,n-k} (\theta) g_k \quad (n \geq a) \quad \ldots (15)
\]

\[
\theta \mathcal{Q}_{j0} (\theta - Q_{j0} (0)) = \lambda_0 \mathcal{Q}_{j0} (\theta) - Q_{j-1,0} (0) \mathcal{V} (\theta) \quad (j \geq 2) \quad \ldots (16)
\]
\[ \theta \mathcal{Q}_{jn} (\theta) Q_{jn} (0) = \lambda_0 \mathcal{Q}_{jn} (\theta) - Q_{j-1, n} (0) \mathcal{V}_{1n} (\theta) - \lambda_0 \sum_{k=1}^{n} \mathcal{Q}_{j, n-k} (\theta) g_k \]

\[ (1 \leq n \leq a - 1), \ (j \geq 2) \quad ... \ (17) \]

\[ \theta \mathcal{Q}_{jn} (\theta) - Q_{jn} (0) = \lambda_0 \mathcal{Q}_{jn} (\theta) - \lambda_0 \sum_{k=1}^{n} \mathcal{Q}_{j, n-k} (\theta) g_k \]

\[ (n \geq a), \ (j \geq 2) \quad ... \ (18) \]

We define the following probability generating functions:

\[
\left[ \begin{array}{c}
\mathcal{P}_i (z, \theta) = \sum_{j=0}^{\infty} \mathcal{P}_{ij} (\theta) z^j \quad \text{and} \quad P_i (z, 0) = \sum_{j=0}^{\infty} P_{ij} (0) z^j \quad (a \leq i \leq b) \\
\mathcal{Q}_j (z, \theta) = \sum_{n=0}^{\infty} \mathcal{Q}_{jn} (\theta) z^n \quad \text{and} \quad Q_j (z, 0) = \sum_{n=0}^{\infty} Q_{jn} (0) z^n \quad (j \geq 1)
\end{array} \right] \quad ... \ (19) \]

Multiplying the eqs. (13) by \( z^0 \), (14) by \( z^n \) (\( 1 \leq n \leq a - 1 \)) and [15] by \( z^n \) (\( n \geq a \)) and summing up from \( n = 0 \) to \( n = \infty \) and using (19) we get

\[
(\theta - \lambda_0 + \lambda_0 X (z)) \mathcal{Q}_1 (z, \theta) = Q_1 (z, 0) - \mathcal{V} (\theta)
\]

\[
\left[ \sum_{m=a}^{b} P_{m0} (0) \sum_{n=1}^{a-1} \sum_{m=a}^{b} P_{mn} (0) z^n \right] \quad ... \ (20) \]

Multiplying the eqs. (16) by \( z^0 \), (17) by \( z^n \) (\( 1 \leq n \leq a - 1 \)) and (18) by \( z^n \) (\( n \geq a \)) and summing up from \( n = 0 \) to \( n = \infty \) and using (19) we get

\[
(\theta - \lambda_0 + \lambda_0 X (z)) \mathcal{Q}_j (z, \theta) = Q_j (z, 0) - \mathcal{V} (\theta) \sum_{n=0}^{a-1} Q_{j-1, n} (0) z^n \quad (j \geq 2) \quad ... \ (21) \]

Multiplying the eqs. (10) by \( z^0 \), (11) by \( z^j \) (\( j \geq 1 \)) and summing up from \( j = 0 \) to \( j = \infty \) and using (19) we get

\[
(\theta - \lambda + \lambda X (z)) \mathcal{P}_i (z, \theta) = P_i (z, 0) - \mathcal{S} (\theta)
\]
\[
\left[ \sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0) \right] (a \leq i \leq b-1) \quad \ldots \quad (22)
\]

Multiplying the eq. (10) with \( i = b \) by \( z^0 \), the eq. (12) by \( z^j (j \geq 1) \) and summing up from \( j = 0 \) to \( j = \infty \) and using (19) we get

\[
z^b (\theta - \lambda + \lambda X (z)) P_b (z, 0) = z^b P_b (z, 0) - \mathfrak{F} (\theta)
\]

\[
\left[ \sum_{m=a}^{b} P_{m}(z, 0) - \sum_{m=a}^{b} \sum_{j=0}^{b-1} P_{mj}(0) z^j \right]
\]

\[- \mathfrak{F} (\theta) \sum_{l=1}^{\infty} \left( Q_{l}(z, 0) - \sum_{j=0}^{b-1} Q_{lj}(0) z^j \right) \quad \ldots \quad (23)
\]

Substituting \( \theta = \lambda_0 - \lambda_0 X (z) \) in eqs. (20) and (21) and \( \theta = \lambda - \lambda X (z) \) in (22) and (23), we have

\[
Q_1 (z, 0) = \mathfrak{V} (\lambda_0 - \lambda_0 X (z)) \left[ \sum_{m=a}^{b} P_{m0}(0) + \sum_{n=1}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n \right] \quad \ldots \quad (24)
\]

\[
Q_j (z, 0) = \mathfrak{V} (\lambda_0 - \lambda_0 X (z)) \sum_{n=0}^{a-1} Q_{j-1, n}(0) z^n \quad (j \geq 2) \quad \ldots \quad (25)
\]

\[
P_i (z, 0) = \mathfrak{F} (\lambda - \lambda X (z)) \left[ \sum_{m=a}^{b} P_{mi}(z, 0) + \sum_{l=1}^{\infty} Q_{li}(0) \right]
\]

\[(a \leq i \leq b-1) \quad \ldots \quad (26)\]

\[
z^b P_b (z, 0) = \mathfrak{F} (\lambda - \lambda X (z))
\]

\[
\left[ \sum_{m=a}^{b-1} P_{m}(z, 0) + P_b (z, 0) - \sum_{m=a}^{b-1} \sum_{j=0}^{b-1} P_{mj}(0) z^j + \sum_{l=1}^{\infty} \left( Q_1 (z, 0) - \sum_{j=0}^{b-1} Q_{lj}(0) z^j \right) \right]
\]

which gives

\[(z^b - \mathfrak{F} (\lambda - \lambda X (z))) P_b (z, 0) = (\mathfrak{F} (\lambda - \lambda X (z)))\]
\[
\begin{align*}
&
\sum_{m=a}^{b-1} P_m(z,0) - \sum_{m=a}^{b-1} \sum_{j=0}^{b-1} P_{mj}(0) z^j - \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}(0) z^j \\
&
\left[ \sum_{m=a}^{b-1} P_m(z,0) - \sum_{m=a}^{b-1} \sum_{j=0}^{b-1} P_{mj}(0) z^j - \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}(0) z^j \right] \\
&
\ldots (27)
\end{align*}
\]

From eqs. (20) and (24), we have

\[
\overline{\mathcal{V}}(\lambda_0 - \lambda X(z)) - \overline{Q}(\theta) \left[ \sum_{m=a}^{b} P_{m0}(0) + \sum_{n=1}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n \right] \\
\overline{Q}_1(z, \theta) = \frac{\overline{V}(\lambda_0 - \lambda_0 X(z)) - \overline{Q}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n}{(\theta - \lambda_0 + \lambda_0 X(z))} \quad (j \geq 2) \quad \ldots (28)
\]

From eqs. (21) and (25), we have

\[
\overline{Q}_j(z, \theta) = \frac{\left( \overline{V}(\lambda_0 - \lambda_0 X(z)) - \overline{V}(\theta) \right) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n}{(\theta - \lambda_0 + \lambda_0 X(z))} \quad (j \geq 2) \quad \ldots (29)
\]

From eqs. (22) and (26), we have

\[
\overline{P}_i(z, \theta) = \frac{\left( \overline{F}(\lambda - \lambda X(z)) - \overline{F}(\theta) \right) \sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0)}{(\theta - \lambda + \lambda X(z))} \quad (a \leq i \leq b - 1) \quad \ldots (30)
\]

\[
\overline{P}_b(z, \theta) = \frac{\left( \overline{F}(\lambda - \lambda X(z)) - \overline{F}(\theta) \right) f(z)}{(\theta - \lambda + \lambda X(z)) \left( z^b - \overline{F}(\lambda + \lambda X(z)) \right)} \quad \ldots (31)
\]

where \( f(z) \) is given by

\[
f(z) \left[ \sum_{i=a}^{b-1} P_i(z,0) - \sum_{i=a}^{b-1} \sum_{j=0}^{b-1} P_{ij}(0) z^j + \sum_{l=1}^{\infty} Q_l(z,0) - \sum_{j=0}^{b-1} Q_{lj}(0) z^j \right] \\
\ldots (27)
\]

Substituting the expression for \( P_i(z,0) \) from (26) and \( Q_l(z,0) \) from (24) and (25) in \( f(z) \) we have

\[
f(z) = \overline{F}(\lambda - \lambda X(z)) \left[ \sum_{i=a}^{b} \sum_{m=a}^{b} P_{mi}(0) + \sum_{i=a}^{b} \sum_{l=1}^{\infty} Q_{li}(0) \right] \\
- \sum_{i=a}^{b} \sum_{j=a}^{b-1} P_{ij}(0) z^j + \overline{V}(\lambda_0 - \lambda_0 X(z)) - 1)
\]

\[
\ldots (31)
\]
\[
\sum_{n=0}^{a-1} \left( \sum_{m=a}^{b} P_{mn}(0) z^n + \sum_{l=2}^{\infty} Q_{l-1,n}(0) z^n \right) - \sum_{l=1}^{\infty} \sum_{j=a}^{b-1} Q_{ij}(0) z^j.
\]

Let \( P(z) \) be the PGF of the queue at an arbitrary time epoch.

Then

\[
P(z) = \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \sum_{j=1}^{\infty} \tilde{Q}_j(z,0) \quad \ldots \text{(32)}
\]

Using the eqs. (28) through (31) in (32) and by assuming

\[
P_i = \sum_{m=a}^{b} P_{mi}(0), \quad q_i = \sum_{l=1}^{\infty} Q_{li}(0)
\]

and

\[
c_i = p_i + q_i \quad (i = 0, 1, 2, \ldots, b-1)
\]

as the probabilities of number of customers in the queue at departure and vacation completion epochs, after some algebra we get PGF of queue length distribution at an arbitrary time epoch.

\[
P(z) = \frac{(\mathcal{S}(\lambda - \lambda X(z)) - 1)(-\lambda_0 - \lambda_0 X(z)) \sum_{i=a}^{b-1} (z^b - z^i) c_i + (\mathcal{V}(\lambda_0 - \lambda_0 X(z)) - 1)}{(-\lambda + \lambda X(z))(z^b - (\mathcal{S}(\lambda - \lambda X(z)))}
\]

\[\ldots \text{(34)}\]

**Particular case**

When both the arrival rates are same, then \( \lambda = \lambda_0 \) and the eq. (34) reduces to the following form

\[
P(z) = \frac{\left[ (\mathcal{S}(\lambda - \lambda X(z)) - 1) \sum_{i=a}^{b-1} (z^b - z^i) c_i + (z^b - 1) \mathcal{V}(\lambda - \lambda X(z)) - 1 \right]}{(-\lambda + \lambda X(z))(z^b - (\mathcal{S}(\lambda - \lambda X(z)))}
\]

\[\ldots \text{(35)}\]
Eq. (35) gives the queue length distribution of an $M^X/G(a, b)/1$ queueing system with multiple vacations. The result exactly coincides with the queue length distribution of Arumuganathan et al. (1).

Eq. (34) gives the PGF of the number of customers in the queue, but involves 'b' unknowns $c_0, c_1, ..., c_{b-1}$. To find these constants, Rouche's theorem on complex variables can be used. By Rouche's theorem it follows that $z^b - \lambda^X(z) \lambda (\lambda - \lambda X(z))$ has $(b-1)$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator of (34) must vanish at these points, which gives 'b' equations in 'b' unknowns. These equations can be solved by any suitable numerical technique. Thus (34) gives the PGF of the queue length at an arbitrary time.

**Remark**: The probability generating function has to satisfy $P(1) = 1$. In order to satisfy the condition, applying L'Hospital's rule and evaluating $\lim_{z \to 1} P(z)$ and equating the expression to 1 we have to satisfy

$$E(S) \sum_{i=a}^{b-1} (b-i)c_i + bE(V) \sum_{i=0}^{a-1} c_i = (b - \lambda E(X) E(S))$$

$$+ \left\lbrace (\lambda - \lambda_0) E(X) E(S) E(V) \sum_{i=0}^{a-1} c_i \right\rbrace$$

where $c_i = p_i + q_i$, $i = 0$ to $b-1$. Since $p_i, q_i$ are probabilities, it follows that the left hand side of the above expression must be positive. The second term in the left hand side of the above expression is always positive. Thus $P(1) = 1$ is satisfied only if

$$b - \lambda E(X) E(S) > 0.$$  If we let $\rho = \lambda E(X) E(S)/b$ then $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

4. EXPECTED QUEUE LENGTH

Then mean queue length $E(Q)$ at an arbitrary time epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by
\[
\begin{align*}
E(Q) &= \frac{\sum_{i=a}^{b-1} \left( b (b-1) - i (i-1) \right) c_i + f_2(x,s) \sum_{i=a}^{b-1} (b-i) c_i + f_3(x,s,v) \sum_{i=0}^{a-1} c_i + f_4(x,s,v) \sum_{i=0}^{a-1} i c_i}{2 \lambda (E(x))^2 (b - S1)^2}
\end{align*}
\]

The functions \( f_1 \) to \( f_4 \) are given by

\[
\begin{align*}
f_1 &= T1 \cdot S1; \quad f_2 = T2 \cdot S2; \quad f_3 = (T3 - T4); \\
f_4 &= 2(b-S1) \cdot V1 \cdot (S1 \cdot \lambda_0 \cdot X1 + \lambda_1 \cdot T1)/\lambda_0
\end{align*}
\]

where

\[
\begin{align*}
T1 &= X1 \cdot (b - S1); \quad T2 = X1 \cdot (b (b-1) - S2) + X2 \cdot (b - S1); \\
T3 &= [(b-S1) \cdot (V2 \cdot S1 \cdot X1 \cdot \lambda_0 + V1) \cdot S2 \cdot X1 \cdot \lambda_0 + V2 \cdot \lambda \cdot T1 + V1 \cdot S1 \cdot X2 \cdot \lambda_0 \\
&+ V1 \cdot \lambda \cdot T2)]/\lambda_0 \\
T4 &= \frac{\lambda \cdot V1 \cdot T1 + V1 \cdot S1 \cdot \lambda_0 \cdot X1 \cdot (X2 \cdot T1 + X1 \cdot T2)}{\lambda_0 \cdot X1^2}
\end{align*}
\]

\[
\begin{align*}
S1 &= \lambda \cdot X1 \cdot E(S); \quad S2 = \lambda \cdot X2 \cdot E(S) + \lambda^2 \cdot X1^2 \cdot E(S^2); \\
V1 &= \lambda_0 \cdot X1 \cdot E(V); \quad V2 = \lambda_0 \cdot X2 \cdot E(V) + \lambda_0^2 \cdot X1^2 \cdot E(V^2); \\
X1 &= E(X); \text{ and } X2 = X^{11}(1) = E(X^2) - E(X).
\end{align*}
\]

5. **Expected Length of Idle Period**

Idle period can be defined as the period starting from the service completion epoch of a service to the service initiation epoch of the next service inclusive of the multiple vacations between the services. It is easily found using the conditional expectation concept, as discussed in (14).

Let \( I \) be the random variable 'idle period'. Let \( U \) be a random variable defined by

\[
U = 0, \text{ if the server finds at least '}a' \text{ customers after the first vacation} \\
= 1, \text{ if the server finds less than '}a' \text{ customers after the first vacation}
\]
Case (i);

\[ E (V) \]

First Vacation

Service Completion epoch \( Q \geq a \)
Idle period initiation epoch \( (U = 0) \)
Idle period completed

Case (ii);

\[ \begin{align*}
E (V) & \quad E (I) \\
\mid \mid & \\
First \mbox{ Vacation} & \\
\end{align*} \]

Service Completion epoch \( Q < a \)
Idle period initiation epoch \( (U = 1) \)
Idle period restarts

Then, the expected length of idle period, \( E (I) \) is given by

\[ E (I) = E (I/\ U = 0) \ P (U = 0) + E (I/\ U = 1) \ P (U = 1) \]

\[ = E (V) \ P (U = 0) + (E (V) + E (I)) \ P (U = 1), \]

where \( E (V) \) is the mean vacation time.

Solving for

\[ E (I) \quad \mbox{we get} \quad E (I) = E (V) / P (U = 0) \]

... (36)

Using \( \sum_{m=a}^{b} P_{m} (0) = p_{n} \), from the eq. (24), we get

\[ Q_{1} (z, 0) = V (\lambda_{0} - \lambda_{0} X (z)) \sum_{n=0}^{a-1} P_{n} z^{n} \]

Using the fact that \( Q_{1n} (0) \) is the probability that \( n \) customers in the queue after the first vacation, from (19), we have

\[ \sum_{n=0}^{\infty} Q_{1n} (0) z^{n} = V (\lambda_{0} - \lambda_{0} X (z)) \sum_{n=0}^{a-1} P_{n} z^{n} = \sum_{n=0}^{\infty} \alpha_{n} z^{n} \left( \sum_{n=0}^{a-1} p_{n} z^{n} \right) \]
\[
Q_{1n}(0) = \sum_{i=0}^{n} \alpha_{i} p_{n-i}
\]

Now,

\[
P(U=0) = 1 - \sum_{n=0}^{a-1} Q_{1n}(0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_{i} p_{n-i}
\]  \(\ldots (37)\)

Where \(\alpha_{i}\) is the probability that \('i'\) customers arrive during a vacation and \(p_{n}\) is the probability of \('n'\) customers in the queue at a departure epoch.

From (36) and (37) the expected length of idle period is obtained as

\[
E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_{i} p_{n-i}}
\]  \(\ldots (38)\)

6. EXPECTED LENGTH OF BUSY PERIOD

Let \(B\) be the random variable 'busy period'. We define a random variable, \(J\) as

\(J = 0\) if the server finds less than \('a'\) customers after the first service.

\(J = 1\) if the server finds at least \('a'\) customers after the first service.

Now, expected length of busy period, \(E(B)\) is given by

\[
E(B) = E(B/J = 0) P(J = 0) + E(B/J = 1) P(J = 1)
\]

\[
= E(S) P(J = 0) + (E(S) + E(B)) P(J = 1)
\]

where \(E(S)\) is the mean service time solving for \(E(B)\), we have

\[
E(B) = E(S) / P(J = 0)
\]

\[
E(B) = E(S) / \sum_{i=0}^{a-1} p_{i}
\]  \(\ldots (39)\)
7. COST MODEL

In this section, we find the total average cost with the following assumptions:

\[ C_s = \text{Start up cost}; \quad C_h = \text{Holsing cost per customer}; \]

\[ C_o = \text{Opening cost per unit time} \quad C_r = \text{Reward cost per unit time due to vacation}. \]

Since the length of the cycle is the sum of the idle and busy periods from eqs. (38) and (39), the expected length of cycle, \( E(T_c) \) is obtained as

\[
E(T_c) = E(I) + E(B) - \frac{E(V)}{a-1} \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i p_{n-i} \sum_{n=0}^{a-1} p_i
\]

... (40)

Now, the total average cost per unit time is obtained as

\[
\text{Total cost} = (c_s - c_r E(I)) \frac{1}{E(T_c)} + c_h E(Q) + C_0 \rho
\]

... (41)

Where

\[
\rho = \frac{\lambda E(X) E(S)}{b}.
\]

8. NUMERICAL EXAMPLE

We wish to arrive at an optimal policy regarding threshold value, which will minimize the total average cost assuming the following costs:

Rs. 4.00 Start up cost; Rs.0.50 Holding cost per customer;

Rs. 5.00 Operating cost per unit time; Rs.2.00 Reward per unit time due to vacation;

The above system can be modeled as \( M^X/G(a, b)/l \) queue with the following assumptions:

1. Service time distribution is \( k \)-Erlang with \( k = 2 \) and service rate \( \mu = 11 \).
2. Arrival is Poisson with arrival rates \( \lambda = 10 \) and \( \lambda_0 = 4 \).
3. Batch size distribution of the arrival is geometric with mean = 2.
4. Vacation time is exponential with parameter \( \alpha = 10 \).

The unknown probabilities of the queue length of an \( M^X/G(a, b)/l \) with state dependent arrivals and multiple vacations are computed using numerical techniques. Using Bairstow's method (7) the zeros of the function \( z^b - \mathbb{S}(\lambda - \lambda X(z)) \) are obtained. The Gauss Elimination method is used.
to solve the simultaneous equations. MATLAB (11) software is used for programming. The numerical results for various threshold values with $b = 10$ are presented in Table 1. A graph, total average cost versus threshold value is also presented (Fig. 1).

![Graph showing total average cost versus threshold value](image)

**Fig. 1.** Total average cost versus threshold value

<table>
<thead>
<tr>
<th>Threshold value</th>
<th>$E(Q)$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
<th>$E(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8164</td>
<td>0.2930</td>
<td>0.1972</td>
<td>10.1918</td>
</tr>
<tr>
<td>2</td>
<td>2.8706</td>
<td>0.6474</td>
<td>0.1682</td>
<td>6.5700</td>
</tr>
<tr>
<td>3</td>
<td>3.1106</td>
<td>1.3421</td>
<td>0.1624</td>
<td>4.2481</td>
</tr>
<tr>
<td>4</td>
<td>3.4685</td>
<td>1.7948</td>
<td>0.1655</td>
<td>3.7619*</td>
</tr>
<tr>
<td>5</td>
<td>3.9036</td>
<td>1.4240</td>
<td>0.1730</td>
<td>4.4914</td>
</tr>
<tr>
<td>6</td>
<td>4.3943</td>
<td>0.9861</td>
<td>0.1830</td>
<td>5.7499</td>
</tr>
<tr>
<td>7</td>
<td>4.9327</td>
<td>0.7190</td>
<td>0.1942</td>
<td>7.0900</td>
</tr>
<tr>
<td>8</td>
<td>5.5238</td>
<td>0.5668</td>
<td>0.2054</td>
<td>8.2919</td>
</tr>
<tr>
<td>9</td>
<td>6.1875</td>
<td>0.4806</td>
<td>0.2153</td>
<td>9.2787</td>
</tr>
<tr>
<td>10</td>
<td>6.9627</td>
<td>0.4381</td>
<td>0.2218</td>
<td>10.0371</td>
</tr>
</tbody>
</table>

- $a$ - threshold value, $E(Q)$ - expected queue length, $E(I)$ - expected length of idle period, $E(B)$ - expected length of busy period, $E(C)$ - total average cost.

The normal cost evaluation procedure adopted in various manufacturing industries reveals the following:

(i) if the threshold value is 1 for a given ‘$b$’, then the service procedure will start even for a single unit waiting for service and definitely the operating cost will increase.
(ii) if the threshold value is more than the optimal value, then the holding cost per customer will increase. So to trade-off between these two, the authors selected the optimal value indicated in the Table 1.

CONCLUSION

An $M^X/G(a, b)/1$ model with state dependent arrivals and multiple vacations is discussed. Expressions for queue length distribution, expected length of queue and expected length of idle and busy periods are derived. A cost model is discussed with numerical illustration. From the Table 1 and the Fig. 1 it is clear that the threshold (minimum batch size) value may be taken as 4 to minimize the Total average cost.

REFERENCES