ELASTIC-PLASTIC TRANSITION IN A TRANSVERSELY ISOTROPIC DISC WITH VARIABLE THICKNESS UNDER INTERNAL PRESSURE

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Elastic-plastic stresses have been derived for transversely isotropic disc having variable thickness under internal pressure using Seth’s transition theory. Results obtained have been discussed numerically and depicted graphically. The yielding occurs at the internal or external surface of the disc depending upon thickness ratio. The disc having variable thickness and made of transversely isotropic material yields at a less pressure as compared to disc made of isotropic material. Transversely isotropic disc requires high percentage increase in pressure to become fully plastic from initial yielding as compared to isotropic material.

Key Words: Elastic-Plastic Transition; Internal Pressure; Transversely Isotropic; Disc; Variable Thickness

LIST OF SYMBOLS

\( a \) and \( b \) : Internal and external radii of disc.
\( d \) : A constant.
\( r \) : Radial distance.
\( u, v, w \) : Displacement components.
\( x, y, z \) : Cartesian co-ordinates.
\( r, \theta, z \) : Polar co-ordinates.
\( A_1, A_2, C_2 \) : Constants.
\( e_{ij} \) and \( T_{ij} \) : Strain and stress tensor.
\( e_{kk} \) : First strain invariant.
\( \beta \) : Function of \( r \) only.
\( h \) : Variable thickness.
\( Y \) : Yield stress.
\( P \) : Function of \( \beta \) only.
\( C_{ij} \) : Material constants.
Poisson’s ratio.

**DIMENSIONLESS QUANTITIES**

\[ R = r/b, \quad R = a/b \]

\[ \sigma_r = \frac{T_{rr}}{Y} \] - Radial stress component.

\[ \sigma_\theta = \frac{T_{\theta \theta}}{Y} \] - Circumferential stress component.

\[ \sigma_z = \frac{T_{zz}}{Y} \] - Axial stress component.

**1. INTRODUCTION**

It is well known that disc with variable thickness are frequently found in mechanical engineering. Axial symmetric solutions of the isotropic and orthotropic discs, including variable thickness, variable density, and other cases, can be found in most of the standard elasticity textbooks. A literature survey indicates that several workers have analyzed circular discs with constant material properties under various conditions. Durban [1] found an exact solution for the initially pressurized elastic-plastic strain-hardening annular plate. Chaudhuri [2] obtained the stresses in a non-homogeneous rotating annulus by varying Poisson’s ratio of the material. Giiven [3&4] studied the plane state of stress in elastic-plastic annular discs with variable thickness subjected to internal & external pressure assuming Tresca’s yield condition, its associated flow rule and strain-hardening. In analyzing the problem, these authors used some simplifying assumptions. First, the deformation is small enough to make infinitesimal strain theory applicable. Second, simplifications were made regarding the constitutive equations of the material like incompressibility of the material and an yield criterion. Incompressibility of the material is one of the most important assumptions which simplifies the problem. In fact, in most of the cases, it is not possible to find a solution in closed form without this assumption. Seth’s transition theory does not require these assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. Seth’s transition theory utilizes the concept of generalized strain measure and asymptotic solution through the critical points of differential system defining the deformed field and has successfully been applied to a large number of problems [5,6,&8-15] in plasticity. Seth [5] has defined the generalized principal strain measure as

\[
e_{ii}^A = \int_0^\frac{\Delta A}{2} \left( 1 - 2 e_{ii}^A \right)^{n-1} - 1 \cdot \frac{1}{n} \left[ 1 - \left( 1 - 2 e_{ii}^A \right)^{n/2} \right],
\]

where ‘n’ is the measure and \( e_{ii}^A \) is principal Almansi finite strain components.

In this paper an attempt has been made to study the behaviour of transversely isotropic disc having variable thickness under internal pressure, using Seth’s transition theory. The thickness \( h \) is assumed to vary in the radial direction as,
\[ h = h_0 \left( \frac{r}{b} \right)^{-K} . \] 

where \( h_0 \) and \( K \) are real constants.

2. GOVERNING EQUATIONS

We consider a thin disc of non-constant thickness of transversely isotropic material with internal and external radii 'a' and 'b' respectively subjected to internal pressure \( p \). The disc is thin and it is effectively in a state of plane stress.

The displacement components in cylindrical co-ordinates are given (5) by,

\[ u = r (1 - \beta), \quad v = 0, \quad w = dz \]  \hspace{1cm} \text{... (2.1)}

where \( \beta \) is a function of \( r = \sqrt{x^2 + y^2} \) only and \( d \) is a constant.

The finite components of strain (6) are

\[ \varepsilon_{ii}^A = \frac{1}{2} \left[ 1 - (\beta + r \beta')^2 \right], \]

\[ \varepsilon_{\theta\theta}^A = \frac{1}{2} \left[ 1 - \beta^2 \right], \]

\[ \varepsilon_{zz}^A = \frac{1}{2} \left[ 1 - (1 - d)^2 \right], \]

\[ \varepsilon_{r\theta}^A = \varepsilon_{\theta z}^A = \varepsilon_{x r}^A = 0, \]  \hspace{1cm} \text{... (2.2)}

where

\[ \beta' = \frac{dB}{dr}. \]

Substituting eq. (2.2) in eq. (1.1), the generalized components of strain are

\[ \varepsilon_{rr} = \frac{1}{n} \left[ 1 - (\beta + r \beta')^n \right], \]

\[ \varepsilon_{\theta\theta} = \frac{1}{n} \left[ 1 - \beta^n \right], \]

\[ \varepsilon_{zz} = \frac{1}{n} \left[ 1 - (1 - d)^n \right], \]

\[ \varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{x r} = 0, \]  \hspace{1cm} \text{... (2.3)}

The stress-strain relations for transversely isotropic materials (7) are
\[ T_{rr} = C_{11} e_{rr} + (C_{11} - 2C_{66}) e_{\theta\theta} + C_{13} e_{zz}, \]
\[ T_{r\theta} = (C_{11} - 2C_{66}) e_{rr} + C_{11} e_{\theta\theta} + C_{13} e_{zz}, \]
\[ T_{zz} = T_{\theta\theta} = T_{zr} = T_{rr} = T_{zz} = 0, \quad \ldots (2.4) \]

where \( C_{ij} \) are material constants.

Using eqs. (2.3) in eqs. (2.4), we have

\[ T_{rr} = \frac{A}{n} \left( 2 - \beta^n \left( 1 + (1 + P)^n \right) \right) - 2 \frac{C_{66}}{n} \left( 1 - \beta^n \right), \]
\[ T_{\theta\theta} = \frac{A}{n} \left( 2 - \beta^n \left( 1 + (1 + P)^n \right) \right) - 2 \frac{C_{66}}{n} \left( 1 - \beta^n (1 + P)^n \right), \]
\[ T_{zr} = T_{\theta z} = T_{r\theta} = T_{zz} = 0, \quad \ldots (2.5) \]

where

\[ A = C_{11} - \frac{C_{13}^2}{C_{33}}. \]

The equations of equilibrium are all satisfied except

\[ \frac{d}{dr} (hr T_{rr}) - h T_{r\theta} = 0. \quad \ldots (2.6) \]

Substituting eq. (2.5) in eq. (2.6), we get a non-linear differential equation in \( \beta \) as,

\[
p \beta^{n+1} (1 + P)^{n-1} \frac{dP}{d \beta} = \left[ \frac{2C_{66}}{nA} \beta^n [1 - (1 + P)^n] - \frac{2C_{66} k' r}{nAh} (1 - \beta^n) - p \beta^n [1 + (1 + P)^n] \right]
\[
+ \frac{2C_{66}}{A} \beta'' + \frac{2 k' r}{nh} - \frac{k' r}{nA} \beta^n ((1 + P)^n + 1) \], \quad \ldots (2.7)
\]

where \( r \beta' = \beta P \) and \( k' = \frac{dh}{dr} \).

The transition points of \( \beta \) in eq. (2.7) are \( P = -1 \) and \( P = \pm \infty \).

The boundary conditions are

\[ T_{rr} = -p \text{ at } r = a, \]
\[ T_{rr} = 0 \text{ at } r = b. \] ... (2.8)

3. SOLUTION OF PROBLEM

It has been shown (8-15) that the asymptotic solution through the principal stress leads from elastic to plastic state at the transition point \( P \rightarrow \pm \infty \).

We define the transition function \( R \) as

\[ R = T_{\theta \theta} = \frac{2}{n} (A - C_{66}) - \frac{A}{n} \beta^n [1 + (1 + P)^n] + \frac{2C_{66}}{n} \beta^n [(1 + P)^n] \] ... (3.1)

Taking logarithmic differentiation of eq. (3.1) w. r. t. \( r \), we get

\[ \frac{d}{dr} (\log R) \]

\[ = - \left[ \frac{AP\beta^n [1 + (1 + P)^n] - 2C_{66} P \beta^n (1 + P)^n + (A - 2C_{66}) P \beta^n (1 + P)^{n-1} \frac{dP}{d\beta}}{rR} \right] \] ... (3.2)

Substituting \( \frac{dP}{d\beta} \) from eq. (2.7) in eq. (3.2) and taking the asymptotic value as \( P \rightarrow \pm \infty \), we obtained after integration,

\[ R = \frac{A_1}{h} r^{-C_2}, \] ... (3.3)

where \( A_1 \) is a constant of integration and \( C_2 = \frac{2C_{66}}{A} \).

Using eq. (3.3) in eq. (3.1), we have

\[ T_{\theta \theta} = \frac{A_1}{h} r^{-C_2}, \] ... (3.4)

Substituting eq. (3.4) in eq. (2.6), we have

\[ T_{rr} = \frac{A_1 r^{-C_2}}{h (1 - C_2)} + \frac{A_2}{rh} \] ... (3.5)

where \( A_2 \) is a constant of integration.
Using boundary conditions (2.8) in eq. (3.5), we get

\[
A_1 = \frac{pah(a)(1-C_2)}{b^{1-C_2}\left[1-\left(\frac{a}{b}\right)^{1-C_2}\right]}
\]

and

\[
A_2 = \frac{-pah(a)}{\left[1-\left(\frac{a}{b}\right)^{1-C_2}\right]}
\] ... (3.6)

Substituting \(A_1\) and \(A_2\) in eqs. (3.4) and (3.5), we get

\[
T_{rr} = \frac{p R_o^{1-K} R^{K-1}}{1-R_o^{1-C_2}} \left[R^{1-C_2} - 1\right]
\]

\[
T_{\theta\theta} = \frac{p (1-C_2) R_o^{1-K} R^{K-C_2}}{1-R_o^{1-C_2}},
\] ... (3.7)

where \(R = r/b\) and \(R_o = a/b\).

From eq. (3.7), we have

\[
T_{\theta\theta} - T_{rr} = \frac{p R_o^{1-K} R^{K-1}}{1-R_o^{1-C_2}} \left[1-C_2 R^{1-C_2}\right].
\] ... (3.8)

**INITIAL YIELDING**

The maximum value of \(T_{\theta\theta} - T_{rr}\) occurs at radius at \(R = \left[\frac{K-1}{C_2(K-C_2)}\right]^{\frac{1}{1-C_2}} = R_1\) (say) for \(K > 1\). Therefore, yielding of the disc takes place at radius \(R = R_1\) depending upon the values of \(K\) and \(C_2\). For example, if we take \(K = 1.38912\) and \(C_2 = 0.69\) (Beryl Material), yielding starts at the internal surface and for \(K = 1.69\), yielding starts at the external surface (see Table II). For yielding at \(R = R_1\), eq. (3.8) becomes,
\[
\begin{align*}
\left| T_\theta \theta - T_{rr} \right|_{r=R_1} &= \left| \frac{K-1}{1-C_2} pR_o^{1-K} (1-C_2) \left( \frac{K-1}{C_2} \right) \right| \frac{1-C_2}{1-R_o} (K-C_2) \\
&= Y \text{ (say)} \quad \text{(3.9)}
\end{align*}
\]

The pressure required for initial yielding is given by

\[
P_1 = \frac{P}{Y} = \left( 1-R_o^{1-C_2} \right) (K-C_2) \left[ \frac{K-1}{C_2} \right]^{1-C_2} \frac{K-C_2}{1-R_o^{1-K} \left( \frac{K-1}{C_2} \right)^{1-C_2}} = \left( 1-C_2 \right) R_o^{1-K} \left[ \frac{K-1}{C_2} \right]^{1-C_2} \quad \text{(3.10)}
\]

Using eq. (3.9) in eq. (3.7), we get the transitional stresses as

\[
\sigma_r = \frac{P_1 R_o^{1-K} R^{K-1}}{1-R_o^{1-C_2}} \left[ R^{1-C_2} - 1 \right],
\]

\[
\sigma_\theta = \frac{P_1 \left( 1-C_2 \right) R_o^{1-K} R^{K-C_2}}{1-R_o^{1-C_2}}, \quad \text{(3.11)}
\]

where \( \sigma_r = \frac{T_{rr}}{Y} \) and \( \sigma_\theta = \frac{T_\theta \theta}{Y} \).

Stresses for fully-plastic state (11) are obtained from eq. (3.8)-(3.11) by taking \( C_2 \to 0 \). There are two plastic zones:

1. Inner-plastic zone: \( R_o \leq R \leq R_1 \).
2. Outer-plastic zone: \( R_1 \leq R \leq 1 \).

For inner-plastic zone, eq. (3.8) becomes

\[
\left| T_\theta \theta - T_{rr} \right|_{R=R_o} = \left| \frac{P}{1-R_o} \right| = Y^* \quad \text{(say)}, \quad \text{(3.12)}
\]

and the required pressure is

\[
P_1^* = \frac{P}{Y^*} = 1-R_o.
\]

Using eq. (3.12) in eq. (3.7), we get stresses for inner-plastic zone as
\[ \sigma_r^* = \frac{T_{rr}}{Y^*} = \frac{p_1 R_o^{1-K} R^{K-1}}{1-R_o} [R-1], \]

\[ \sigma_\theta^* = \frac{T_{\theta \theta}}{Y^*} = \frac{p_1 R_o^{1-K} R^K}{1-R_o}. \]

For outer-plastic zone, eq. (3.8) becomes

\[
\left| T_{\theta \theta} - T_{rr} \right|_{R=1} = \left| \frac{p R_o^{1-K}}{1-R_o} \right| \equiv Y^{**} \text{ (say)},
\]

and the required pressure is

\[ P_1^{**} = \frac{p}{Y^{**}} = \frac{1-R_o}{R_o^{1-K}}. \]

Using eq. (3.15) in eq. (3.7), we get stresses for outer-plastic zone as

\[ \sigma_r^{**} = \frac{T_{rr}}{Y^{**}} = \frac{p_1 R_o^{1-K} R^{K-1}}{1-R_o} [R-1], \]

\[ \sigma_\theta^{**} = \frac{T_{\theta \theta}}{Y^{**}} = \frac{p_1 R_o^{1-K} R^K}{1-R_o}. \]

**ISOTROPIC MATERIAL**

For isotropic materials, the material constants reduce to two only (16), i.e.

\[ C_{11} = C_{22} = C_{33}, \]

\[ C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = (C_{11} - 2C_{66}). \]

In terms of constant \( \lambda \) and \( \mu \), these can be written as

\[ C_{12} = \lambda, C_{66} = \frac{1}{2} (2C_{11} - C_{12}) \equiv \mu \text{ and } C_{11} = \lambda + 2\mu. \]

Elastic-plastic transitional stresses are obtained by using eq. (3.18) in eq. (3.7), as
\[ T_{rr} = \frac{pR_o^{1-K}R^{K-1}}{1-C} \left[ \frac{1-C}{R^{2-C} - 1} \right], \]

\[ T_{\theta\theta} = \frac{1-C}{2-C} \frac{pR_o^{1-K}R^{K-1}}{1-C} \cdot \frac{R^{1-C}}{R^{2-C}} \]

... (3.19)

where

\[ C = \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\sigma}{1 - \sigma}. \]

From eq. (3.19), we have

\[ T_{\theta\theta} - T_{rr} = \frac{pR_o^{1-K}R^{K-1}}{1-C} \left[ 1 - \frac{1}{2-C} \frac{R^{1-C}}{(R^{2-C})} \right]. \]

... (3.20)

From eq. (3.20), yielding of the disc will take place at

\[ R = \left[ \frac{(K-1)(2-C)^2}{K(2-C) - 1} \right]^{\frac{2-C}{1-C}} = R_2 \text{ (say)} \]

for \( K > 1 \). Therefore

\[ \left| T_{\theta\theta} - T_{rr} \right|_{R=R_2} \]

\[ = \left| \frac{pR_o^{1-K}[(K-1)(2-C)^2]}{1-C} \cdot \left[ \frac{1-C}{(2-C)(K-1)} + 1 \right] \right| = Y \text{ (say)}, \]

... (3.21)

and the pressure required for yielding is
\[ p_1 \equiv p_y = \frac{1 - C}{1 - R_o^{2 - C}} \left[ 1 - \frac{1}{K (2 - C) - 1}\right] \frac{2 - C (K - 1)}{1 - C}. \]  

... (3.22)

Eq. (3.19) for inner-plastic zone becomes

\[ \left| T_{\theta \theta} - T_{rr} \right|_{R = R_o} = \left| \frac{p (2 - \sqrt{R_o})}{2 (1 - \sqrt{R_o})} \right| \equiv Y^* \text{ (say)}, \]  

... (3.23)

and the required pressure is

\[ p_1^* = \frac{p}{Y^*} = \frac{2 (1 - \sqrt{R_o})}{2 - \sqrt{R_o}}. \]  

... (3.24)

Using eq. (3.24) in eq. (3.19), we get stresses for inner-plastic zone as

\[ \sigma_r^* \equiv \frac{T_{rr}}{Y^*} = \frac{p_1^* R^{1 - K} R^{K - 1}}{1 - \sqrt{R_o}} \left[ \sqrt{R} - 1 \right], \]  

\[ \sigma_\theta^* \equiv \frac{T_{\theta \theta}}{Y^*} = \frac{p_1^* R^{1 - K} R^{K - \frac{1}{2}}}{2 (1 - \sqrt{R_o})}. \]  

... (3.25)

For outer-plastic zone \((C \to 0)\), eq. (3.20) becomes

\[ \left| T_{\theta \theta} - T_{rr} \right|_{R = 1} = \left| \frac{p R^{1 - K}}{2 (1 - \sqrt{R_o})} \right| \equiv Y^{**} \text{ (say)}, \]  

... (3.26)

and the required pressure is

\[ p_1^{**} = \frac{p}{Y^{**}} = \frac{2 (1 - \sqrt{R_o})}{1 - K R_o}. \]  

... (3.27)

Using eq. (3.26) in eq. (3.19), we get stresses for outer-plastic zone as

\[ \sigma_r^{**} = \frac{T_{rr}}{Y^{**}} = \frac{p_1^{**} R^{1 - K} R^{K - 1}}{1 - R_o^{1 - K}} \left[ \sqrt{R} - 1 \right], \]
\[
\sigma^{**} = \frac{T_{\theta \theta}}{Y^{**}} = \frac{1}{2} P_{1} \frac{R_{o}}{1 - \sqrt{R_{o}}} R^{1 - K} \left(1 - \frac{1}{2} \right)
\]

Eqs. (3.25) and (3.28) are same as obtained by Gupta and Ruchika (17).

**NUMERICAL ILLUSTRATION**

![Diagram of disc with variable thickness](image)

**Fig. 1** Disc having variable thickness. \( h = h_{o} \left(\frac{r}{b}\right)^{-K} \).

As a numerical example, elastic constants \( C_{ij} \) have been given in Table I for isotropic material (19) (Brass, \( \sigma = 0.33 \)) and transversely isotropic material (7) (Beryl Material). In Table II, yielding of disc \( (R_{o} = a/b = 1/2) \) at different pressure has been given.

<table>
<thead>
<tr>
<th>Isotropic Material (( \sigma = 0.33 ))</th>
<th>( C_{44} )</th>
<th>( C_{11} )</th>
<th>( C_{12} )</th>
<th>( C_{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beryl Material</td>
<td>0.883</td>
<td>2.746</td>
<td>0.980</td>
<td>0.674</td>
</tr>
</tbody>
</table>

It can be seen that yielding occurs at radius \( R = R_{1} \) or \( R_{2} \) of the disc for Beryl and Isotropic Material depending upon values of \( K \). For Beryl Material and \( K = 1.38912 \), yielding occurs at the internal surface of the disc at pressure 2.29327 \( P \) and for \( K = 1.69 \) it occurs at the outer surface at pressure 2.58645 \( P \) i.e., as the values of \( K \) increases from \( K = 1.38912 \) to 1.69, yielding of disc shifted from internal surface towards the outer surface at a higher pressure. Similar is the case for isotropic material i.e., as the values of \( K \) increases from 1.37458 to 1.6667, the yielding of disc shifted from...
internal surface towards the outer surface. It is also seen that the disc having variable thickness and made of isotropic material yields at higher pressure as compare to disc made of transversely isotropic material. Transversely isotropic material (Beryl material) requires high percentage increase in pressure to become fully plastic from initial yielding as compared to isotropic material (Brass).

In Fig. 2, curves have been drawn between pressure required for initial yielding of the disc made of transversely isotropic material/isotropic material having variable thickness and radii ratio \(a/b\). It is seen that disc made of transversely isotropic material (Beryl Material) requires less pressure to yield as compare to isotropic material. High pressure is required for yielding as the thickness ratio \(a/b\) of the disc increases.

In Fig. 3, curves have been drawn between transitional stresses and radii ratio \(r/b\) for Beryl Material: In Fig. 4, curves have been drawn between stresses and radii ratio \(r/b\) for fully-plastic state. It is seen that circumferential stress is maximum at outer surface of the disc for different values of \(K\). When the yielding takes place at the internal surface of the disc made of transversely isotropic material/isotropic material (\(K = 1.38912/K = 1.37458\)), the circumferential stress for fully-plastic state is maximum at external surface for transversely isotropic material as compare to isotropic material and for other values of \(K\), we have two plastic zones, inner and outer as shown in the Fig. 4.

**CONCLUSION**

For a transversely isotropic disc with variable thickness, under internal pressure, it is seen that yielding occurs at the internal or external surface of the disc depending upon the thickness ratio \(h/h_0\). For smaller thickness ratio \((h/h_0)\), yielding occurs at the internal surface but as the thickness ratio increases, the yielding shifted towards the outer surface of the disc at a higher pressure. The disc having variable thickness and made of transversely isotropic material (Beryl material) requires high percentage increase in pressure to become fully plastic from initial yielding as compared to isotropic material (Brass).
FIG. 2. Pressure required for initial yielding for different values of $K$ and radii ratio $a/b$
FIG. 3. Elastic-plastic transitional stresses for Beryl Material with respect to radii ratio a/b
FIG. 4. Stresses for fully-plastic state at different values of $K$ for Beryl and Isotropic Material
REFERENCES