EFFECTS OF UNIFORM MAGNETIC FIELD ON SQUEEZE FILM LUBRICATION IN HUMAN JOINTS

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The present paper deals with the study of squeeze film lubrication in human joints under the influence of uniform magnetic field of small intensity. The transportation of synovial fluid in porous cartilaginous matrix deformed due to load has been evaluated to observe the transverse flux entering into intra-articular gap. A theory of two-phase mechanically interacting mixture is applied, following slip-condition of Rudraiah [1], to represent the solid matrix and interstitial fluid of cartilage continuum so that a two-region flow model consisting of viscous fluid in the porous matrix and squeeze film lubrication with viscoelastic fluid between two approaching surfaces is considered. The effects of magnetic parameter and porosity factor on normal young, old and osteoarthritic synovial fluid on normal fluid velocity through cartilage, pressure and load carrying capacity for various parameters are discussed.

Key Words: Squeeze Film Lubrication; Human Joints; Synovial Fluid; Porous Cartilaginous Matrix

1. INTRODUCTION

The study of flow of fluids through porous media is of considerable interest due to its natural occurrence and importance in many problems of engineering and technology such as porous bearings, porous rollers, porous layer insulation consisting of solid and pores etc. In addition, these flows are applicable to bio-mathematics particularly in the study of blood flow in lungs, arteries, cartilage and so on. The flow in the porous medium is governed either by Darcy equation or by modified Darcy equation depending on the structure and the depth of the porous medium. It has been customary to assume that no slip boundary condition is valid at the surface of the porous material. Kenyon [2, 3] has studied theoretically the time dependent filtration of liquid through the wall of a soft, porous tube and indicated the possibilities of generation of fluid flow induced boundary layers in arterial walls. Wijesinghe et al. validated these theoretical results by experiments on porous polychrethane foam models. Jayaraman obtained the exact solutions for the displacement of the matrix and the interstitial fluid for large and small consolidation time using these theoretical results and observed significant changes in the pressure distribution when the characteristic load time is much smaller than the consolidation time. In addition, Jain and Jayaraman [6] have obtained a theoretical model for water flux through an arterial wall using Darcy velocity and Barry et al. [7] have studied a flow of a viscous fluid over a thin deformable porous layer fixed to the solid wall of the channel using binary mixture theory, Darcy law and the assumptions of linear elasticity.

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Beavers and Joseph [8] have postulated the slip boundary condition (called the BJ-slip condition) which admits non-zero tangential velocity at the porous surface. Beavers et al. [9] have performed experiment with a view to test the validity of BJ-slip boundary condition. Subsequently, Taylor [10] and Safirman [11] have provided theoretical basis via average argument for this slip velocity model. Since BJ-slip condition is valid only in a densely packed porous medium of large thickness, the variation of velocity in it can be ignored and the flow is assumed to be governed by Darcy equation. However, many engineering problems and bio-mathematical problems involve porous layers of shallow depth. In such porous layers, the velocity is no longer uniform and the distortion of velocity gives rise to usual viscous shear. Therefore, instead of using Darcy equation, one has to use a modified Darcy equation involving usual viscous force and Darcy resistance. Rudraiah [1] has derived a special type of modified BJ-slip condition introducing a typical characteristic length scale, which tends to BJ-slip condition in the limit when characteristic length tends to infinity. Rudraiah et al. [12-15] have made studies using Darcy equation, modified Darcy equation and BJ-slip condition.

The bio-mechanics of human joints plays a significant role in the study of human locomotion. The system of human joints consists of two matching bones covered with cartilage (a porous medium) and a lubricant (synovial fluid) between the intra-articular gap. The cartilage is basically a two-phase deformable porous material formed due to consolidation of the solid matrix by tissue deformation which can absorb or gives out fluid owing to the pressure difference and is treated as a mixture of two interacting continua which consists of a linear elastic solid and a viscous fluid. The presence of lubricant in the joint cavity prevents direct contact of bones with one another and forms uninterrupted fluid film between the surfaces of the bones. When two bones of the joints with lubricant in the joint cavity approach each other, the fluid is squeezed resulting in the built up of pressure which helps in avoiding the possible contact of surfaces. This is termed as 'squeeze film lubrication'.

Torzilli and Mow [16-17] have pointed out that cartilage is three layered porous medium consisting of a deep zone, a middle zone and a superficial tangential zone and synovial fluid behaves as a blood plasma which diffuses through the synovium into the joint cavity. Nigam et al. [8] investigated the effects of variation of porosity in the upper most layer of the cartilage which plays a predominant role in the self adjusting nature of the human joints taking a three layered porous medium. During the diffusion process varying amounts of hyaluronic acid and protein complex are added by the synovial cells to the plasma. Davies [19] investigated that nutrition of cells of articular cartilage, protection of cartilage from mechanical damages and removal of products of metabolism are the main functions of synovial fluid. Ogston and Stenier [20] found that synovia! fluid is non-Newtonian in character due to the presence of hyaluronic acid (a long-chain polymer) molecules and its viscosity decreases with increasing shear rate. The viscoelastic and lubricant character of synovial fluid has been supported by BlochandDintenfass [21], Maroudas [22], Dowson [23] and Mow [24]. The lubricate action of the synovial fluid depends mainly on material properties (chemical stability, degree of viscosity and elasticity) of the fluid.

Tandon and Rakesh [25] studied the lubrication mechanism occurring in knee joints replacement under restricted motion. Singh et al. [26] have studied a model for micropolar fluid film mechanism with reference to human joints. Rudraiah et al. [27] have investigated Hartmann flow over a permeable bed considering two zones where in the flow above the bed is governed by Hartmann equation and that below the bed is governed by modified Darcy law. Chandrasekhar et al. [28] have studied the effect of slip on porous walled squeeze films in the presence of a magnetic field. Rudraiah and Veerabhadraiah [29-30] have studied Couette flow past a permeable bed for temperature distribution and buoyancy effects respectively. Sinha [31] analysed the variation of pressure and load capacity with reference to load bearing human joints by introducing a continuously varying porosity model and observed the effect of the presence of uniform magnetic
field on squeeze film lubrication. Hou et al. [32] have presented an analysis of the squeeze film lubrication mechanism for articular cartilage. Recently, Hlavacek [33-35] have studied squeeze film lubrication of the human ankle joint with synovial fluid and articular cartilage. More recently, Singh et al. [36] have studied Micropolar fluid film lubrication in human joints under sinusoidal porosity variation in cartilage. Some of the dynamic interaction problems that may arise within diarthrodial joints between cartilage and synovial fluid can be found in Mow and Lai [37-38].

In the present investigation, following Rudraiah [1], the behaviour of normal young synovial fluid, normal old synovial fluid and osteoarthritic synovial fluid is studied to observe the effect of uniform magnetic field on normal fluid velocity through poro-elastic region, pressure and load carrying capacity.

2. MATHEMATICAL FORMULATION

The physical configuration of the problem considered in this paper consists of incompressible synovial fluid flowing between two parallel surfaces of the joint separated by a fluid film of thickness $h$ and both the surfaces (each of length $2L$) approach to each other with a squeezing velocity $V_0 (= dh/ht)$. The upper surface of the joint is considered to be a rigid surface while the lower surface supports a cartilaginous matrix of thickness $H$. A cartesian coordinate frame is embedded such that the $x$-axis is chosen on and along the upper rigid surface of the joint and $y$-axis normal to it so that cartilaginous surface is at $y = +h$. The flow region is divided into two regions namely the fluid film region ($0 < y < h$) and the cartilaginous region ($h < y < h + H$). Further, the presence of a magnetic field of uniform intensity $B_0$ is assumed normal to the flow. In addition, our analysis is based on the following assumption:

(i) The fluid (synovial fluid) is non-Newtonian, slightly conducting and incompressible.

(ii) The flow in the fluid film region and in the porous medium is driven by a common uniform pressure gradient $\frac{\partial p}{\partial x}$.

(iii) The flow is laminar and fully developed.

(iv) The flow in the fluid film region and in the porous medium (cartilage) is coupled through proper boundary conditions.

(v) Since the thickness of porous medium (cartilage) is finite, the slip condition of Rudraiah [1] is used.

(vi) The cartilaginous medium is homogeneous with uniformly and symmetrically distributed pores.

(vii) An uniform magnetic field is applied normal to the flow region and the intensity of the magnetic field is small.

Therefore, the governing equations of motion for fluid film region following Shercliff [36] and Pinkus and Sternlicht [40] are:

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \tau_{xy} \right) - Mu \quad ... \ (1)$$

$$\frac{\partial p}{\partial y} = 0 \quad ... \ (2)$$
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(continuity equation)} \quad \ldots \quad (3)
\]

where

\[
\tau_{xy} = \eta \left[ 1 - (\lambda_1 - \lambda_2) \mu \left( \frac{\partial u}{\partial y} \right)^2 \right]
\]

Ignoring inertial forces (Mow, et al., [41]), the coupled equations of motion for the deformable cartilage matrix and the mobile portion of the fluid contained in its pores, following Torzilli and Mow [42] are:

Fluid: \[ \text{div} (\tau_f) + \frac{1}{k} \left( \frac{dU}{dt} - \mathbf{V} \right) = 0 \quad \ldots \quad (4) \]

Matrix: \[ \text{div} (\tau_m) - \frac{1}{k} \left( \frac{dU}{dt} - \mathbf{V} \right) = 0 \quad \ldots \quad (5) \]

where

\[
\tau_f = E (\nabla \cdot \mathbf{U}) - \bar{p} \mathbf{I}
\]

and

\[
\tau_m = \bar{p} \mathbf{I} + A (\nabla \cdot \mathbf{U})
\]

The boundary conditions for fluid film region (Rudraiah [1]), cartilaginous region and matching condition are:

\[
u = 0 \quad \text{at} \quad y = 0 \quad ; \quad \frac{\partial u}{\partial y} = \lambda \delta \left[ u_b \lambda \phi \coth H + \frac{Q \lambda \phi}{\sinh H} \right] \quad \text{at} \quad y = h \quad \ldots \quad (6a)
\]

\[
u = 0 \quad \text{at} \quad y = 0 \quad ; \quad \nu = \nu_0 - \frac{dh}{dt} (= V_0 \text{ say}) \quad \text{at} \quad y = h \quad \ldots \quad (6b)
\]

\[
p = \bar{p} \quad \text{at} \quad y = h \quad ; \quad \frac{dp}{dy} = 0 \quad \text{at} \quad y = h + H \quad \ldots \quad (6c)
\]

\[
p = 0 \quad \text{at} \quad x = L \quad ; \quad p = 0 \quad \text{at} \quad x = -L \quad \ldots \quad (6d)
\]

The divergence of the addition of (4) and (5) (Tandon et al., [43], Gupta and Singh [44]) yields the following Laplace equation:

\[ \nabla^2 F = 0 \quad \ldots \quad (7) \]

where \( F(= \nabla \cdot \mathbf{U}) \) is cartilage dilation and is characterized by a single linear equation in terms of corresponding average bulk modulus \( K \). Therefore, following Hori and Mockros [45], we have

\[ F = F_0 + \frac{\bar{p}}{K} \quad \ldots \quad (8) \]
where $F_0$ is the cartilage dilation at zero pressure.

From eqs. (7) and (8), we obtain

$$\nabla^2 \bar{p} = 0$$

... (9)

3. MATHEMATICAL ANALYSIS

Following Gupta et al. [46], we define

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + ...$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + ...$$

$$D(x) = D_0(x) + \epsilon D_1(x) + \epsilon^2 D_2(x) + ...$$

... (10)

where $\epsilon$ is a small parameter such that

$$\epsilon = \frac{(\lambda_1 - \lambda_2) \mu}{\tau^2}.$$ 

The eq. (1) can be written as

$$\frac{dp}{dx} y + D(x) = \eta \frac{\partial u}{\partial y} \left[ 1 - (\lambda_1 - \lambda_2) \mu \left( \frac{\partial u}{\partial y} \right)^2 \right] - Mu$$

... (11)

Substituting (10) in (11) and comparing the terms of zero and first order in $\epsilon$, we get the following system of coupled differential equations:

$$\frac{dp_0}{dx} y + D_0 = \eta \frac{du_0}{dy} - Mu_0$$

... (12)

$$\frac{dp_1}{dx} y + D_1 = \eta \left[ \frac{du_1}{dy} - \tau^2 \left( \frac{du_0}{dy} \right)^3 \right] - Mu_1$$

... (13)

The equation of continuity (3) transform to:

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

... (14)

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0$$

... (15)

The solutions of the eqs. (12) and (13) under the corresponding boundary conditions are:
\[ u_0 = \frac{dp_0}{dx} \left[ \frac{1}{M} y - A_1 \exp \left( \frac{My}{\eta} \right) + A_1 \right] + A_2 \exp \left( \frac{My}{\eta} \right) - A_2 \] \hfill \ldots (16)

\[ u_1 = A_3 \exp \left( \frac{3My}{\eta} \right) + A_4 \exp \left( \frac{2My}{\eta} \right) + A_5 \exp \left( \frac{My}{\eta} \right) \]

\[ - \frac{1}{M} \frac{dp_1}{dx} \left( y + \eta \right) + \left( A_9 + \frac{\eta}{M^2} \frac{dp_1}{dx} \exp \left( -\frac{Mh}{\eta} \right) \right) \exp \left( \frac{My}{\eta} \right) \]

\[ + \frac{1}{M} \left[ \frac{\eta}{M} \frac{dp_1}{dx} \left\{ 1 - \exp \left( -\frac{Mh}{\eta} \right) \right\} + A_{10} \right] + A_6 \] \hfill \ldots (17)

where

\[
A_1 = \left[ \frac{\eta \sqrt{\lambda k}}{\mu \sinh (\delta H)} - \frac{\eta}{M} \right] \exp \left( -\frac{Mh}{\eta} \right),
\]

\[
A_2 = \frac{\lambda \phi \eta u_B}{M} \sqrt{\frac{\lambda}{k}} \exp \left( -\frac{Mh}{\eta} \right) \coth (\delta H),
\]

\[
A_3 = k_0 + k_1 \left( \frac{dp_0}{dx} \right)^3 + k_2 \left( \frac{dp_0}{dx} \right)^{2} - k_3 \left( \frac{dp_0}{dx} \right),
\]

\[
A_4 = k_4 \left( \frac{dp_0}{dx} \right)^3 + k_5 \left( \frac{dp_0}{dx} \right)^{2} - k_6 \left( \frac{dp_0}{dx} \right),
\]

\[
A_5 = -k_7 \left( \frac{dp_0}{dx} \right)^3 + k_8 \left( \frac{dp_0}{dx} \right)^{2},
\]

\[
A_6 = k_9 \left( \frac{dp_0}{dx} \right)^3, \quad A_7 = -k_{10} \left( \frac{dp_1}{dx} \right)^3,
\]

\[
A_8 = k_{11} \left( \frac{dp_0}{dx} \right)^3 + k_{12} \left( \frac{dp_0}{dx} \right)^{2} + k_{13} \left( \frac{dp_0}{dx} \right) + b_0,
\]

\[
A_9 = k_{14} \left( \frac{dp_0}{dx} \right) - k_{15} \left( \frac{dp_0}{dx} \right)^{2} - k_{16} \left( \frac{dp_0}{dx} \right)^{3} - k_{17} \left( \frac{dp_0}{dx} \right) - b_1,
\]

and

\[
A_{10} = k_{21} \left( \frac{dp_1}{dx} \right) + k_{18} \left( \frac{dp_0}{dx} \right)^{3} + k_{19} \left( \frac{dp_0}{dx} \right)^{2} + k_{20} \left( \frac{dp_0}{dx} \right) + k_{22}.
\]
4. Flow Rate, Normal Component of Velocity and Load Carrying Capacity

The flow rate $Q_1$ is given by

$$Q_1 = \int_0^h u dy = \int_0^h (u_0 + \varepsilon u_1) dy = A_{11} \left( \frac{dp_0}{dx} \right) + A_{12}$$

$$+ \varepsilon \left[ k_{30} \left( \frac{dp_0}{dx} \right)^3 + k_{31} \left( \frac{dp_0}{dx} \right)^2 + k_{32} \left( \frac{dp_0}{dx} \right) + k_{33} \left( \frac{dp_1}{dx} \right) + k_{34} \right]$$

... (18)

where

$$A_{11} = \frac{h^2}{2M} - \frac{\eta A_1}{M} \left\{ \exp \left( \frac{Mh}{\eta} \right) - 1 \right\} + A_1 h,$$

$$A_{12} = \frac{\eta A_2}{M} \left\{ \exp \left( \frac{Mh}{\eta} \right) - 1 \right\} - A_2 h,$$

$$b_0 = \frac{3Mk_0}{\eta} \exp \left( \frac{3Mh}{\eta} \right), \quad b_1 = \frac{b_0 \eta}{M} \exp \left( -\frac{Mh}{\eta} \right),$$

$$b_2 = \frac{\eta^2 T^2}{2M}, \quad b_3 = \frac{3\eta^2 T^2}{M^2}, \quad b_4 = \frac{3\eta^2 T^2}{M^2},$$

$$k_0 = b_2 \left( \frac{A_2 M}{\eta} \right)^3, \quad k_1 = -b_2 \left( \frac{A_1 M}{\eta} \right)^3,$$

$$k_2 = 3b_2 \left( \frac{A_1 M}{\eta} \right)^2 \left( \frac{A_2 M}{\eta} \right), \quad k_3 = 3b_2 \left( \frac{A_2 M}{\eta} \right)^2 \left( \frac{A_1 M}{\eta} \right),$$

$$k_4 = b_3 \left( \frac{A_1 M}{\eta} \right)^2, \quad k_5 = b_3 \left( \frac{A_1 M}{\eta} \right) \left( \frac{A_2 M}{\eta} \right),$$

$$k_6 = b_3 \left( \frac{A_2 M}{\eta} \right)^2, \quad k_7 = b_4 \left( \frac{A_1 M}{\eta} \right), \quad k_8 = b_4 \left( \frac{A_2 M}{\eta} \right),$$

$$k_9 = -\frac{nT^2}{M^4}, \quad k_{10} = \frac{\sqrt{\lambda k}}{\mu} \frac{1}{\sinh(\delta H)},$$

$$k_{11} = \frac{3Mk_1}{\eta} \exp \left( \frac{3Mh}{\eta} \right) + \frac{2Mk_4}{\eta} \exp \left( \frac{2Mh}{\eta} \right) - k_7 \exp \left( \frac{Mh}{\eta} \right) \left( \frac{Mh}{\eta} + 1 \right).$$
\[ k_{12} = \frac{3Mk_2}{\eta} \exp\left(\frac{3Mh}{\eta}\right) - \frac{2Mk_5}{\eta} \exp\left(\frac{2Mh}{\eta}\right) + k_8 \exp\left(\frac{Mh}{\eta}\right) \left(\frac{Mh}{\eta} + 1\right) \]

\[ k_{13} = -\frac{3Mk_3}{\eta} \exp\left(\frac{3Mh}{\eta}\right) + \frac{2Mk_6}{\eta} \exp\left(\frac{2Mh}{\eta}\right) \]

\[ k_{14} = -\frac{\eta k_{10}}{M} \exp\left(-\frac{Mh}{\eta}\right), \quad k_{15} = \frac{\eta k_{11}}{M} \exp\left(-\frac{Mh}{\eta}\right) \]

\[ k_{16} = \frac{\eta k_{12}}{M} \exp\left(-\frac{Mh}{\eta}\right), \quad k_{17} = \frac{\eta k_{13}}{M} \exp\left(-\frac{Mh}{\eta}\right) \]

\[ k_{18} = -Mk_{15} + Mk_1 + Mk_4 + Mk_9, \quad k_{19} = -Mk_{16} + Mk_2 - Mk_5 \]

\[ k_{20} = -Mk_{17} - Mk_3 + Mk_6, \quad k_{21} = Mk_{14}, \quad k_{22} = Mk_0 - Mb_1 \]

\[ k_{23} = \frac{\eta}{3M} \left\{ \exp\left(\frac{3Mh}{\eta}\right) - 1 \right\}, \quad k_{24} = \frac{\eta}{2M} \left\{ \exp\left(\frac{2Mh}{\eta}\right) - 1 \right\} \]

\[ k_{25} = \frac{\eta}{M} \left[ h \exp\left(\frac{Mh}{\eta}\right) - \eta \left\{ \exp\left(\frac{Mh}{\eta}\right) - 1 \right\} \right], \quad k_{26} = \frac{1}{M} \left( \frac{h^2}{2} + \frac{\eta h}{M} \right) \]

\[ k_{27} = \frac{\eta}{M} \left\{ \exp\left(\frac{Mh}{\eta}\right) - 1 \right\}, \quad k_{28} = \frac{\eta^2}{M^3} \exp\left(-\frac{Mh}{\eta}\right) \left\{ \exp\left(\frac{Mh}{\eta}\right) - 1 \right\} \]

\[ k_{29} = \frac{\eta^2}{M^3} \exp\left(1 - \exp\left(-\frac{Mh}{\eta}\right)\right) \]

\[ k_{30} = k_{1,23} + k_{4,24} - k_{7,25} - k_{15,27} - \frac{h}{M} k_{18} + h k_9 \]

\[ k_{31} = k_{2,23} - k_{5,24} + k_{8,25} - k_{16,27} - \frac{h}{M} k_{19} \]

\[ k_{32} = -k_{3,23} + k_{6,24} - k_{17,27} - \frac{h}{M} k_{20} \]

\[ k_{33} = k_{28} - k_{26} + k_{29} + k_{14,27} - \frac{h}{M} k_{21} \]

and

\[ k_{34} = k_{0,23} - b_1 k_{27} - \frac{h}{M} k_{22} \]
From (5), (7) and (8), we get

$$\mathbf{v} - \frac{d\mathbf{u}}{dt} = k \left( \frac{E}{K} - 1 \right) \nabla \bar{p}$$

... (19)

The normal component of relative fluid velocity $v_n$ at the cartilage surface is given by:

$$v_n = k \left( \frac{E}{K} - 1 \right) \frac{d\bar{p}}{dy} \bigg|_{y=h}$$

... (20)

The mass conservation of the fluid between the cartilage surfaces shown in (3), using (18) and boundary conditions (6.b), yields:

$$\frac{dQ}{dx} = V_0 - v_n$$

where

$$V_0 = -\frac{dh}{dt}$$

... (21)

Using Morgan and Cameron [47] approximation (the thickness of porous matrix is small) and boundary conditions (6.c) in equation (9) and integrating with respect to $y$ from $h$ to $h+H$, we obtain:

$$\frac{d\bar{p}}{dy} \bigg|_{y=h} = H \frac{\partial^2 \bar{p}}{\partial x^2}$$

... (22)

From eqs. (18), (20), (21) and (22), we get

$$A_{11} \frac{d^2 p_0}{dx^2} + \epsilon \frac{d}{dx} \left[ k_{30} \left( \frac{dp_0}{dx} \right)^3 + k_{31} \left( \frac{dp_0}{dx} \right)^2 + k_{32} \left( \frac{dp_0}{dx} \right) + k_{33} \left( \frac{dp_1}{dx} \right) + k_{34} \right]$$

$$= V_0 - k \left( \frac{E}{K} - 1 \right) H \left[ \frac{d^2 p_0}{dx^2} + \epsilon \frac{d^2 p_1}{dx^2} \right].$$

... (23)

The eq. (23) can be transformed to following two equations

$$\left[ A_{11} + k \left( \frac{E}{K} - 1 \right) H \right] \frac{d^2 p_0}{dx^2} = V_0$$

... (24)

$$\left[ k_{33} + k \left( \frac{E}{K} - 1 \right) H \right] \frac{d^2 p_1}{dx^2}$$
\begin{equation}
\frac{d}{dx} \left[ k_{30} \left( \frac{dp_0}{dx} \right)^3 + k_{31} \left( \frac{dp_0}{dx} \right)^2 + k_{32} \left( \frac{dp_0}{dx} \right) + k_{34} \right] = ... \tag{25}
\end{equation}

The solutions of the equations (24) and (25) under the boundary conditions (6.d) are

\begin{equation}
p_0 = \frac{V_0}{2A_13} \left( x^2 - L^2 \right) \tag{26}
\end{equation}

\begin{equation}
p_1 = \frac{A_{15}}{12} \left( x^4 - L^4 \right) + \frac{A_{16}}{6} \left( x^2 - L^2 \right)x + \frac{A_{17}}{2} \left( x^2 - L^2 \right) \tag{27}
\end{equation}

where

\begin{align*}
A_{11} &= \frac{h^2}{2M} - \frac{\eta A_1}{M} \left\{ \exp \left( \frac{Mh}{\eta} \right) - 1 \right\} + A_1 h, \\
A_{12} &= \frac{\eta A_2}{M} \left\{ \exp \left( \frac{Mh}{\eta} \right) - 1 \right\} - A_2 h, \\
A_{13} &= \left[ A_{11} + k \left( \frac{E}{K} - 1 \right) H \right], \quad A_{14} = \left[ k \left( 1 - \frac{E}{K} \right) H - k_{33} \right], \\
A_{15} &= \frac{3k_{30} V_0^3}{A_{14} A_{13}^3}, \quad A_{16} = \frac{2k_{31} V_0^2}{A_{14} A_{13}^2},
\end{align*}

and

\begin{equation}
A_{17} = \frac{k_{32} V_0}{A_{14} A_{13}}.
\end{equation}

The load carrying capacity $W$ per unit width of the joint is given by

\begin{equation}
W = \int_{-L}^{L} p \, dx = \int_{0}^{L} (p_0 + \varepsilon p_1) \, dx = -\frac{2V_0 L^3}{A_{13}} \left[ \frac{2A_{15} L^5}{15} + \frac{A_{16} L^4}{12} + \frac{2A_{17} L^3}{3} \right] \tag{28}
\end{equation}

Using (20), (22), (26) and (27) the normal velocity ($v_n$) at the cartilage surface is:

\begin{equation}
v_n = Hk \left( \frac{E}{K} - 1 \right) \left\{ \frac{V_0}{A_{13}} - \varepsilon \left( A_{15} x^2 + A_{16} x + A_{17} \right) \right\}. \tag{29}
\end{equation}
In order to get the physical insight of the problem, normal fluid velocity through the cartilage, pressure distribution and load carrying capacity of the joint are observed taking different numerical values of the parameters encountered into the equations. The material constants $\lambda_1, \lambda_2$ and $\eta$ are taken from the experimental data of Balazs and Gibbs [48] shown in Table 1. The parameters $H, K$ and $E$ are of Collins [49] and Sinha [31], justified on laboratory scale, are taken as $H = 0.2, K = 7.5$ and $E = 9.9975$ and the values of $\phi$ and $\lambda$ are of Rudraiah [1]. The remaining parameters are chosen arbitrarily.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Category of the fluid</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Human normal young synovial fluid (A)</td>
<td>0.42</td>
<td>0.033</td>
<td>8.20</td>
</tr>
<tr>
<td>2.</td>
<td>Human normal old synovial fluid (B)</td>
<td>0.22</td>
<td>0.016</td>
<td>2.16</td>
</tr>
<tr>
<td>3.</td>
<td>Human osteoarthritic fluid (C)</td>
<td>0.088</td>
<td>0.011</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Figure 1 shows the variation of normal fluid velocity ($v_n$) versus intra-articular gap $h$ for different values of $\sqrt{k/\lambda}$ at $x = 0.0$ and $M = 0.1$. It is observed that the maximum and minimum normal velocity is attained by all the three categories of the fluid samples in the region $10^{-6} < h < 10^{-5}$ and $10^{-4} < h < 10^{-3}$ and becomes zero in the region $10^{-5} < h < 10^{-4}$. It is also observed that an increase in $\sqrt{k/\lambda}$ is favourable to the normal velocity in the region $10^{-7} < h < 10^{-5}$ while reverse effect is noted in the region $10^{-4} < h < 10^{-2}$. Besides, it is apparent that exudation process is faster while imbibition process is slower for normal young synovial fluid (A) in comparison to old (B) and osteoarthritic (C) synovial fluids. Further, in the region $10^{-5} < h < 10^{-4}$, the fluid starts to enter in the cartilage and increases the concentration of hyaluronic acid molecules in the joint cavity. Due to the concentration of hyaluronic acid molecules in the cavity, macromolecules come closer and coalesce. This starts formulation of gel on cartilage and rigid surfaces. However the formulation of mucin gel on the surfaces starts when the gap between the cartilage surfaces is of the same order or less than the cartilages surfaces asperities. These results are similar to Sinha [31] and Tandon et al. [43]. Figure 2 indicates variation of normal velocity with $h$ for magnetic parameter $M = 0.2$; and $M = 0.3$ at $\sqrt{k/\lambda} = 0.15$ and $x = 0.0$. It is observed that an increase in $M$ decreases the normal velocity in the region $10^{-7} < h < 10^{-5}$ while increases in the region $10^{-4} < h < 10^{-2}$. In these regions, the maximum and minimum velocities are noted for all the three samples of synovial fluid.
fluids. The comparison of Fig. 1 and Fig. 2 shows that the effect of increase in magnetic parameter is reverse to the effect of increase in porosity parameter.

Figure 3 points out the variation of pressure $p$ versus $x/L$ for fluid film thickness $\frac{h}{h_0} = 0.4$ and $\frac{h}{h_0} = 0.8$ of synovial fluid samples for $M = 0.1$ and $M = 0.3$ at $\sqrt{k/\lambda} = 0.05$. It is observed that an increase in the fluid film thickness or magnetic parameter decreases the pressure in all the synovial fluid samples. For normal young synovial fluid (A), the fluid pressure is maximum while it is minimum for osteoarthritic fluid (C) inside the cavity. These observations are in agreement with those of Prakash and Sinha [50] and Singh et al. [51]. Figure 4 shows the variation of pressure $p$ versus $x/L$ for fluid film thickness $\frac{h}{h_0} = 0.3$ and $\frac{h}{h_0} = 0.6$ at $M = 0.2$ choosing $\sqrt{k/\lambda} = 0.15$ and $\sqrt{k/\lambda} = 0.12$. It is observed that the pressure decreases when the fluid film thickness increases or the magnitude of $\sqrt{k/\lambda}$ decreases. A comparison of Fig. 3 and Fig. 4 indicates that an increase in magnetic parameter results in a reduction in the distance $x/L$ for minimum pressure. This shows that magnetotheraphy maintain desired pressure in the joint after suitable adjustment of the magnetic parameter.

Figure 5 indicates the variation of load carrying capacity $W$ versus porosity parameter $\sqrt{k/\lambda}$ for $M = 0.1$ and $M = 0.3$ at $\frac{h}{h_0} = 0.2$ for three samples of synovial fluid. It is observed that in each sample an increase in the porosity of cartilaginous matrix surfaces decreases the load carrying capacity for a given magnitude of magnetic induction and vice versa. These findings are in agreement with the suggestions of the medical doctors that the persons with heavy weight must avoid fats in their meals to reduce the abnormal formation of gel and magnetotheraphy can be of significant use in case of diseased joints. Figure 6 shows variation of load carrying capacity versus $\frac{h}{h_0}$ for $M = 0.1$ and $M = 0.3$ at $\sqrt{k/\lambda} = 0.08$. It is observed that the load carrying capacity of all the three categories of the synovial fluid decreases as $M$ increases. It is also noted that for normal young (A) synovial fluid, the load carrying capacity is always higher than the normal old (B) synovial fluid and osteoarthritic (C) synovial fluid. Thus we conclude that minimum is the ratio $\left(\frac{h}{h_0}\right)$, maximum is the load carrying capacity for all samples of synovial fluids.
Fig. 1. Variation of normal velocity with $h$ ($M = 0.1$)

Fig. 2. Variation of normal velocity with $h$ ($\sqrt{\frac{k}{\lambda}} = 0.15$)
Fig. 3. Variation of pressure distribution with $\frac{X}{L} (\frac{\sqrt{k}}{\lambda} = 0.05)$

Fig. 4. Variation of pressure distribution with $\frac{X}{L} (M = 0.2)$
FIG. 5. Variation of load carrying capacity with $\sqrt{h/\lambda}$.

FIG. 6. Variation of load carrying capacity with $h/h_0$. 
NOMENCLATURE

\( F \) cartilage dilation

\( F_0 \) dilation at zero pressure

\( h \) intra-articular gap

\( h_0 \) half of the intra-articular gap

\( H \) thickness of cartilage matrix

\( u_B \) slip velocity at the permeable surface

\( \phi \) porosity

\( \lambda \) viscosity factor [such that \( -\eta y = \) a Rudraiah [1]]

\( \eta \) steady shear viscosity

\( M \) magnetic parameter

\( \lambda_1 \) relaxation time

\( \lambda_2 \) retardation time

\( B_0 \) uniform magnetic field intensity

\( Q \) Darcy velocity

\( k \) permeability of the cartilaginous matrix

\( \delta = \sqrt{\frac{k}{\lambda}} \) permeability factor

\( K \) bulk modules

\( 2L \) characteristic length of the joint

\( p \) fluid pressure

\( \mathbf{p} \) hydrostatic pressure of pore fluid

\( Q_1 \) volumetric flow rate

\( t \) time

\( T \) characteristic time

\( u, v \) velocity components

\( x, y \) co-ordinate axes

\( A, E \) elastic parameters of the cartilage

\( \mu \) Newtonian viscosity coefficient

\( W \) load capacity

\( v_n \) normal component of relative fluid velocity at the cartilage surface

\( \mathbf{I} \) unit vector
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U displacement vector in the cartilage

V fluid velocity vector in the cartilage

$\tau_{AV}$ stress component in joint cavity

$\tau_m$ stress tensor for solid phase

$\tau_f$ stress tensor for fluid phase

REFERENCES