ON p-MODULUS AND p-CAPACITY EQUALITIES IN METRIC MEASURE SPACES

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Suppose $X$ is a non-compact metric measure space, $E$ and $F$ are two disjoint closed subsets of $X$ and the boundary of $E$ is also compact. By means of the Alexandroff compactification $X^*$ of $X$ we obtain some sufficient conditions for the equalities of $p$-modulus $\text{mod}_p(E, F, X)$ and $p$-capacities with continuous or locally Lipschitz test-functions. Our conclusions answer partly the open question on $p$-modulus and $p$-capacity posed by Heinonen and Koskela.

Key Words: $p$-Capacity; $p$-Modulus; Metric Measure Space; Quasiconformal Mappings

1. INTRODUCTION AND PRELIMINARIES

Following Ahlfors and Beurling [1], Fuglede [2], Gehring [3], Väisälä [14], many other mathematicians have studied intensively the relations of modulus of curve families and capacities, see, for example, [4-11, 13-15] and the references therein. In particular, recently, by means of $p$-modulus and $p$-capacity on metric measure spaces, Heinonen and Koskela have extended the theory of quasiconformal mappings [4] and at the same time, Kinnunen and Martio and their colleagues have developed the potential theory, see, for example, [9] and the references therein.

All metric spaces $X$ in this paper are assumed to be rectifiably connected and all measures $\mu$ on $X$ are assumed to be locally finite and Borel regular with dense support. A metric space is called rectifiably connected if every pair of two points in it can be joined by a rectifiable curve. We shall denote by $(X, \mu)$ such a metric measure spaces [4].

By a curve we mean either a continuous map $\gamma$ of an interval $I \subset (-\infty, \infty)$ into $X$, or the image $\gamma(I)$ of such a map. We usually abuse notation by writing $\gamma = \gamma(I)$. For the properties of curves on $X$ we refer to [14].

The space $X$ is said to be (globally) quasiconvex if there is a constant $C > 0$ so that every pair of points $x$ and $y$ in $X$ can be joined by a curve $\gamma$ whose length satisfies $l(\gamma) \leq C |x - y|$, here
and hereafter we use the Polish distance notation $|x - y|$ in any metric space, and refer to $C$ as the quasiconvexity constant. Moreover, $X$ is locally quasiconvex if every point in $X$ has a neighbourhood that is quasiconvex. More generally, $X$ is said to be $\varphi$-convex if there is a cover of $X$ by open sets $\{U_\alpha\}$ together with homeomorphisms $\{\varphi_\alpha : [0, \infty) \rightarrow [0, \infty)\}$ such that every pair of two points $x$ and $y$ in $U_\alpha$ can be joined by a curve in $X$ whose length does not exceed $\varphi_\alpha (|x - y|)$. $X$ is said to be proper if its closed balls are compact. For details of the above concepts we refer to [4].

**Definition 1** — Let $\Gamma$ be a family of curves in $X$ and $p \geq 1$ be a real number. Let $\Phi(\Gamma)$ denote the set of all non-negative Borel functions $\rho$ satisfying $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable $\gamma \in \Gamma$ (such function $\rho$ is said to be $\Gamma$-admissible). The $p$-modulus of $\Gamma$ is defined as

$$
\text{mod}_p \Gamma = \inf \left\{ \int_X \rho^p \, d\mu : \rho \in \Phi(\Gamma) \right\}.
$$

... (1.1)

**Remark 1** : In Definition 1, as in [14] we can also consider only the sub-family $\Gamma_0$ of all locally rectifiable curves in $\Gamma$; if every $\gamma \in \Gamma$ is contained in some fixed Borel set $W$, then in (1.1) we can use $\Phi_0 = \{\rho \in \Phi(\Gamma) : \rho \chi_{X \setminus W} = 0\}$ instead of $\Phi(\Gamma)$, or only consider the integral on $W$.

In the following we will investigate the relation between $p$-modulus and $p$-capacity and always suppose that $X$ is non-compact because the corresponding problem is solved in [4] (see Theorem A below) when $X$ is compact. Denote by $X^*$ the Alexandroff compactification of $X$, i.e. $X^* = X \cup \{\infty\}$, $\infty \in X$ such that any neighbourhood of $\infty$ is of the form $(X \setminus K) \cup \{\infty\}$, where $K$ is a compact subset of $X$. By extending $\mu$ to $X^*$ such that $\mu(\{\infty\}) = 0$, we get a new space $(X^*, \mu)$. It is easy to see that for any point $x$ in $X$ there is a locally rectifiable curve $\gamma : [0, b) \rightarrow X$ such that $\gamma(0) = x$ and $\gamma \cap (X \setminus K) \neq \emptyset$ for all compact subsets $K$ of $X$. Then such a $\gamma$ is called a curve which starts at $x$ and tends to $\infty$.

Suppose that $U$ is an open subset of $X$, that $Y$ stands either for $U$ or for $X^*$, and that $p \geq 1$ is a real number.

**Definition 2**[4]: Suppose $u$ is a real-valued function on $Y$. We say a non-negative Borel function $\rho : Y \rightarrow [0, \infty]$ is the upper gradient of $u$ in $Y$ if
\[ |u(x) - u(y)| \leq \int_{\gamma_{xy}} \rho \, ds, \]

whenever \( \gamma_{xy} \) is a rectifiable curve joining two points \( x \) and \( y \) in \( Y \).

**Definition 3[^4]**: Suppose \( E \) and \( F \) are disjoint closed subsets of \( Y \). The triple \((E, F; Y)\) is called a condenser and its \( p \)-capacity is defined as

\[
\text{cap}_p (E, F; Y) := \inf_{\gamma} \int_{\gamma} \rho^p \, d\mu,
\]

where the infimum is taken over all upper gradients of all functions in \( A(E, F; Y) := \{ u : Y \to (-\infty, \infty) : u|_E \geq 1, u|_F \leq 0 \} \).

Let \( C(S) \) (\( LC(S) \)) be the family of all continuous (or locally Lipschitz, respectively) real-valued functions on a set \( S \). We denote \( A_C(E, F; Y) := C(Y) \cap A(E, F; Y) \); \( A_L(E, F; Y) := LC(Y) \cap A(E, F; Y) \).

Let \( F^* := F \cup \{ \infty \} \). If \( Y = X^* \) and \( E \) is also a compact subset of \( X \), then \( E \) and \( F^* \) are disjoint compact subsets of \( X^* \). Thus, we denote \( A_{C^*}(E, F^*; X^*) := C_0(X^*) \cap A_C(E, F; X^*) \), \( A_{L^*}(E, F^*; X^*) := C_0(X^*) \cap A_L(E, F; X^*) \), where \( C_0(X^*) := \{ u \in C(X^*) : \text{the support of } u \text{ is a compact subset of } X \} \).

We use the notation \( \text{cap}_{pC}(E, F; Y) \), \( \text{cap}_{pL}(E, F; Y) \), \( \text{cap}_{pC^*}(E, F^*; X^*) \) and \( \text{cap}_{pL^*}(E, F^*; X^*) \) for the quantity in (1.2) if the infimum is taken over all upper gradients of all functions in \( A_{C}(E, F; Y) \), \( A_{L}(E, F; Y) \), \( A_{C^*}(E, F^*; X^*) \) and \( A_{L^*}(E, F^*; X^*) \), respectively.

We trivially have

\[
\text{cap}_p (E, F; Y) \leq \text{cap}_{pC}(E, F; Y) \leq \text{cap}_{pL}(E, F; Y); \quad \text{... (1.3)}
\]

\[
\text{cap}_{pC}(E, F^*; X^*) \leq \text{cap}_{pC^*}(E, F^*; X^*); \quad \text{cap}_{pL}(E, F^*; X^*) \leq \text{cap}_{pL^*}(E, F^*; X^*).
\]

It is known that \( X = R^n \), then for any \( Y \) the equality in (1.3) holds, see [6, 13].

Heinonen and Koskela posed an open question in [4] for the case \( Y = U \): in what generality is there equality in (1.3)?
Just like in [4], the triple \((E, F; Y)\) will denote also the family of all curves in \(Y \bigcap X\) joining two closed subsets \(E\) and \(F\) of \(Y \bigcap X\), cf. above. The triple \((E, F^*; X^*)\) will denote the family of all curves in \(X\) joining \(E\) and \(F^*\) (a curve \(\gamma\) in \(X\) is said to be joining a point \(x \in X\) and \(\infty\) if \(\gamma\) is starts at \(x\) and tends to \(\infty\)). The following Theorems A and B are well-known:

**Theorem A** (Prop. 2.17 of [4]) — We always have

\[
cap_p (E, F; U) = \text{mod}_p (E, F; U).
\]

If \(X\) is \(\varphi\)-convex, \(E\) and \(F\) are two disjoint closed sets in \(X\) with compact boundaries, and \(X\) is proper, then

\[
\cap_{pC} (E \bigcap B, F \bigcap B; B) \leq \text{mod}_p (E, F; X)
\]

... \(1.4\)

for each ball \(B\) in \(X\). If moreover, \(X\) is locally quasiconvex, then \((1.4)\) holds with \(\text{cap}_{pL}\) on the leftside.

**Theorem B** (Th. 2.1 of [15]) — Suppose that all hypotheses in Theorem A are valid and that \(D = X \setminus \{E \bigcup F\}\) is bounded, then

\[
\cap_{pC} (E, F; X) = \text{cap}_{pL} (E, F; X) = \text{mod}_p (E, F; X).
\]

... \(1.5\)

On the other hand, Kallunki and Shanmugalingam [7] obtained the further result that the equality in \((1.3)\) holds with triple \((E, F; U)\) where \(U\) is any domain, but with the additional condition that \(X\) is doubling and supports a \((1, p)\)-Poincaré inequality.

In this paper, we discuss equalities of \(p\)-modulus and \(p\)-capacity for two disjoint closed sets \(E\) and \(F\) in \(X\) without the restriction that \(X \setminus \{E \cup F\}\) is bounded and that the boundary of \(F\) is compact. At first we establish an equality of \(p\)-modulus and \(p\)-capacity on the Alexandroff compactification \(X^*\), and then obtain some corresponding equalities in \(X\). Our conclusions answer partly the above question of Heinonen and Koskela.

2. **Basic Lemmas**

**Lemma 1**[4] — Suppose each \(\Gamma_i\) is a family of curves in \((X, \mu)\). Then the \(p\)-modulus has the following properties:

(1) \(\text{mod}_p (\phi) = 0\);
(2) That $\Gamma_1 \subset \Gamma_2$ implies that $\text{mod}_p \Gamma_1 \leq \text{mod}_p \Gamma_2$.

(3) $\text{mod}_p \left( \bigcup_{i=1}^{\infty} \Gamma_i \right) \leq \sum_{i=1}^{\infty} \text{mod}_p \Gamma_i$.

(4) If $\Gamma_1$ and $\Gamma_2$ satisfy that each curve $\gamma \in \Gamma_1$ has a subcurve $\gamma_0 \in \Gamma_2$, then

$$\text{mod}_p \Gamma_1 \leq \text{mod}_p \Gamma_2.$$

From Definition 3, the following lemma follows directly.

Lemma 2 — For the triple $(E, F, Y)$ in $(X^*, \mu)$, the $p$-capacity has the following properties:

(1) $\text{cap}_p (\emptyset) = 0$;

(2) If $E_i$ and $F_i$ are closed subsets of $Y$, $i = 1, 2$, $E_1 \subset E_2$, $F_1 \subset F_2$, $E_2 \cap F_2 = \emptyset$, then

$$\text{cap}_p (E_1, F_1 ; Y) \leq \text{cap}_p (E_2, F_2 ; Y);$$

(3) If $Y_1 \subset Y_2$, then $\text{cap}_p (E, F ; Y_1) \leq \text{cap}_p (E, F ; Y_2)$.

If $\text{cap}_p$ is replaced by $\text{cap}_{pC}$, $\text{cap}_{pL}$, $\text{cap}_{pC^*}$ and $\text{cap}_{pL^*}$, respectively, we have the similar conclusions (for $\text{cap}_{pC^*}$ and $\text{cap}_{pL^*}$, we suppose $Y = X^*$ and $E$ is also a compact subset of $X$, and replace $F$ with $F^* := F \cup \{\infty\}$ instead).

Proof: The conclusions follow directly from Definition 3.

An open set $G \subset X$ is said to be relatively compact if its closure $\overline{G}$ in $X$ is compact.

Lemma 3$^{[4,14]}$ — Suppose $(X, \mu)$ is a metric measure space, $G$ is a relatively compact, open set of $X$, and $E$ and $F$ are disjoint closed subsets of $\overline{G}$. Suppose $\{\alpha_n\}$ is a sequence of rectifiable curves in $\overline{G}$, $\alpha_n : [0, b_n) \to \overline{G}$ (the arc length parametrization of $\alpha_n$, where $b_n = l(\alpha_n)$) such that $\alpha_n(0) \in E$, $\alpha_n(b_n) \in F$. Then there exists a sub-sequence of $\{\alpha_n\}$ converging to a locally rectifiable curve $\alpha : [0, b) \to \overline{G}$ such that $\alpha(0) \in E$, $b = \liminf_{n \to \infty} b_n$, $\overline{\alpha} \cap F \neq \emptyset$. In particular, if $b < \infty$, then $\alpha$ can be extended to a rectifiable curve $\alpha : [0, b) \to \overline{G}$ such that $\alpha(b) \in F$. 
PROOF: Take the endpoint \( x_n = \alpha_n(0) \in E \) as the starting point of \( \alpha_n \). For every \( n \in \mathbb{N} \), consider the natural extension of \( \alpha_n : [0, b_n) \to \overline{G} \). Since \( \overline{G} \) is compact and the natural extension of each \( \alpha_n \) is 1-Lipschitz, \( \{ \alpha_n \} \) is a normal class. Applying the Arzela-Ascoli theorem in \([0, 1], ..., [0, n] \) successively and then the diagonal principle, we find a sub-sequence of \( \{ \alpha_n \} \) converging to a 1-Lipschitz mapping \( \alpha_n : [0, b_n) \to \overline{G} \), and \( \alpha \) is also a locally rectifiable curve. Since \( E \) and \( F \) are compact, the remaining conclusion is obvious.

Lemma 4\(^{[4]} \) — Suppose (i) \( (X, \mu) \) is a metric measure space, \( A \) is an open set or the closure of some open set in \( X \); (ii) \( \{ \alpha_n \} \) is a sequence of locally rectifiable curves in \( A \), \( \alpha_n : [0, b_n) \to A \) (the arc length parametrization of \( \alpha_n, b_n = l(\alpha_n), b_n \) may be finite or infinite), and \( \{ \alpha_n \} \) converges to a locally rectifiable curve \( \alpha : [0, b) \to A \) such that \( b = l(\alpha) \) (\( b \) may be finite or infinite); (iii) \( g : A \to [0, \infty], \) is a lower semi-continuous function, \( g_n := \min\{g, n\} \) for all \( n \in \mathbb{N} \). Then we have

\[
\int g \, ds \leq \liminf_{n \to \infty} \int_{\alpha_n} g_n \, ds \tag{2.1}
\]

PROOF: Since \( \alpha_n \to \alpha \), we have \( b_n \to b \). Fix a number \( n \in \mathbb{N} \). Then we have the following for an arbitrary real number \( \delta \in (0, b) \),

\[
\liminf_{n \to \infty} \int_{\alpha_n} g_n \, ds \leq \liminf_{n \to \infty} \int_0^{b_n} g_n \circ \alpha_n(t) \, dt \\
\geq \int_0^{\delta} g_n \circ \alpha_n(t) \, dt \geq \int_0^{\delta} \liminf_{n \to \infty} g_n \circ \alpha_n(t) \, dt \\
\geq \int_0^{\delta} g_n \circ \alpha(t) \, dt,
\]

because \( g_n \) is lower semi-continuous. Since \( \delta \in (0, b) \) is arbitrary, we obtain

\[
\liminf_{n \to \infty} \int_{\alpha_n} g_n \, ds \geq \int_0^{b} g_n \circ \alpha(t) \, dt = \int_{\alpha} g_n \, ds.
\]

The fact that \( \{g_n\} \) is increasing and converges to \( g \) as \( n \) tends to \( \infty \) implies
\[
\liminf_{n \to \infty} \int g_n ds \geq \liminf_{n \to \infty} \int g_{n_0} ds \geq \int g_{n_0} ds.
\]

Now, (2.1) follows from the monotone convergence theorem.

**Remark 2:** It is easy to see, the conclusion of the lemma is also valid if the convergence of \( \{\alpha_n\} \) to \( \alpha \) in the assumption (ii) is only in the following sense: there exists an exhaustive sequence \( \{W_k\} \) of relatively compact, open sets with \( \overline{W_k} \subset W_{k+1}, k \in \mathbb{N} \), \( X = \bigcup_{n=1}^{\infty} W_k \) such that \( \{\alpha_n\} \) converges to \( \alpha \) on each \( W_k \).

The next lemma is plain, but it is useful in the proof of our theorems.

**Lemma 5** --- Suppose \( X \) is a proper. Then for every \( p > 0 \), there exists a Borel function \( f : X \to (0, 1] \) such that \( f \in L^p(X) \).

**Proof:** If \( X \) is compact, then \( \mu(X) < \infty \) by the assumption that \( \mu \) is a regular measure. Thus the constant 1 is the desired function. If \( X \) is non-compact and proper, then the diameter of \( X \), denoted by \( \text{diam} X \), is infinite.

Fix a point \( x_0 \) in \( X \). By setting \( S_n = \{y \in X : n - 1 \leq 1y - x_0| < n\}, n \in \mathbb{N} \), we define a function \( g : X \to (0, 1] \) as follows

\[
g(x) = \frac{1}{2^n (\mu(S_n) + 1)} \quad \text{when} \quad x \in S_n.
\]

Thus, if we put \( f = g^{1/p} \), then \( f : X \to (0, 1] \) is obviously a Borel function and we can easily verify that \( f \in L^p(X) \). Indeed,

\[
\int_X f^p d\mu = \int_X g^p d\mu = \sum_{n=1}^{\infty} \int_{S_n} g d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n (\mu(S_n) + 1)}
\]

\[
\mu(S_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.
\]
3. EQUALITIES OF MODULUS AND CAPACITIES IN \( X^* \)

Suppose that \( H \) and \( F \) are two disjoint closed subsets of \( X \) and \( H \) is also compact. Let \( \Omega \) be the closure of \( X \setminus (H \cup F) \) in \( X \), i.e., \( \Omega \subset X \). For each \( x \in X \), denote \( \Gamma_{xF} := \{ \gamma : \gamma \) is a rectifiable curve joining \( x \) and \( F \) in \( X \} \), \( \Gamma_{x\infty} := \{ \gamma : \gamma \) locally rectifiable curves in \( X \) which starts from \( x \) and tends to \( \infty \) with \( \gamma \cap F = \emptyset \} \). Let \( \Gamma_X(\Gamma_1) \) be the family of all rectifiable curves joining \( H \) and \( F \) in \( X \) (in \( \Omega \), respectively), and \( \Gamma_{\infty X}(\Gamma_\infty) \) be the family of all locally rectifiable curves in \( X \) (in \( \Omega \setminus F \), respectively) each of which starts from some point of \( H \) and tends to \( \infty \), i.e.,

\[
\Gamma_X = \bigcup \left\{ \Gamma_{xF} : x \in H \right\}, \quad \Gamma_1 := \bigcup \left\{ \gamma \in \Gamma_{xF} : x \in \partial H, \gamma \subset \Omega \right\},
\]

\[
\Gamma_{\infty X} = \bigcup \left\{ \Gamma_{x\infty} : x \in H \right\}, \quad \Gamma_\infty := \left\{ \gamma \in \Gamma_{x\infty} : x \in \partial H, \gamma \subset \Omega \setminus F \right\},
\]

here and hereafter, by \( \partial S \) we denote the boundary of a set \( S \subset X \) in \( X \). Then \( \Gamma_X \subset (H, F; X), \Gamma_1 \subset (H, F; \Omega) \) and each curve \( \gamma \in \Gamma_X \) has a subcurve \( \gamma_0 \in \Gamma_1 \). Denote \( \Gamma = \Gamma_1 \cup \Gamma_\infty \). Then each curve \( \gamma \in \Gamma_{\infty X} \) has a subcurve \( \gamma_0 \in \Gamma \); each curve \( \gamma \in \Gamma \) is contained in \( \Omega \) with one endpoint on the boundary of \( H \). Moreover, when \( \gamma \in \Gamma_1 \), \( \gamma \) is compact with the other endpoint on the boundary of \( F \) when \( \gamma \in \Gamma_\infty \), and \( \gamma \) is non-compact with \( \gamma \cap F = \emptyset \) when \( \gamma \in \Gamma_\infty \). By Remark 1 and Lemma 1, we have

\[
\text{mod}_p(H, F, X) = \text{mod}_p(\Gamma_X) = \text{mod}_p(\Gamma_1) \leq \text{mod}_p(\Gamma_\infty) = \text{mod}_p(\Gamma_1 \cup \Gamma_\infty) = \text{mod}_p(\Gamma) \quad (3.1)
\]

To describe the sufficient conditions of our theorems, we introduce some new concepts at first.

**Definition 4** — The space \( X \) is said to have Property \( P \) if \( X \) has an exhaustive sequence \( \{W_n\} \) of relatively compact, open sets with \( \overline{W_n} \subset W_{n+1}, n \geq 1 \), \( X = \bigcup_{n=1}^{\infty} W_n \) such that each component of \( X \setminus \overline{W_n} \) is unbounded and rectifiably connected for every \( n \geq 1 \).

**Remark 3** : If \( X \) is proper and is globally quasiconvex, then \( X \) has property \( P \). In fact, it is easy to see the condition that \( X \) is proper ensures that \( X \) has an exhaustive sequence \( \{W_n\} \) of
relatively compact, open sets with $\overline{W}_n \subset W_{n+1}, n \geq 1, X = \bigcup_{n=1}^{\infty} W_n$ such that each component of $X \setminus \overline{W}_n$ unbounded. Suppose $U$ is a component of of $X \setminus \overline{W}_n$ for some $n \in \mathbb{N}$, we will verify $U$ is rectifiably connected.

At first, we show that each point $z \in U$ has a ball $B(z, r)$ with $r > 0$ such that for any $y \in B(z, r)$, there exists a rectifiable curve $\gamma_{zy}$ joining $z$ and $y$ with $\gamma_{zy} \subset U$. Indeed, if we take $r = r_z/(2C_z)$, where $r_z := \inf \{ |z - v| : v \in \overline{W}_n \}$ and $C$ is the quasiconvexity constant, then there exists a rectifiable curve $\gamma_{zy} \subset X$ such that $l(\gamma_{zy}) \leq C |z - y| / r_z \leq r_z / 2$ by the globally quasiconvexity. Thus $\gamma_{zy} \subset B(z, r_z)$ and $\gamma_{zy} \cap \overline{W}_n = \emptyset$, hence we have $\gamma_{zy} \subset U$ since $U$ is connected.

Then, it is easy to see, each rectifiably connected component $V$ of $U$ is an open set since for any $z \in V$, we have $B(z, r) \subset V$ with $r = r_z/(2C_z)$. Therefore, we have $U = V$ since $U$ is connected.

**Definition 5** — The triple $(H, F; X)$ is said to have Property $Q$, if $X$ has an exhaustive sequence $\{W_n\}$ of relatively compact, open sets with $\overline{W}_n \subset W_{n+1}, n \geq 1, X = \bigcup_{n=1}^{\infty} W_n$ such that for each $x \in X \setminus \overline{W}_n$ there exists $\gamma \in \Gamma_{xF} \cup \Gamma_{x\infty}$, such that $\gamma \subset X \setminus \overline{W}_n$.

**Remark 4** : If $X$ is proper and if $X \setminus (H \cup F)$ is bounded, then $(H, F; Y)$ has property $Q$. In fact, $X$ has a exhaustive sequence of concentric balls $\{B(x_0, n) : n \geq k\}$ with a fixed point $x_0 \in X$ and a number $k \in \mathbb{N}$ large enough such that $H \cup (X \setminus F) \subset B(x_0, k)$ since $X \setminus (H \cup F)$ is bounded and $H$ is compact. Thus $X \setminus \overline{B}_n \subset F \setminus \overline{B}_n$ for all $n \geq k$. Then for each $x \in X \setminus \overline{B}_n$, the constant curve $\gamma_{xx} \in F_{xF}$ is desired. That $(X, \mu)$ is proper implies that $\overline{B}_n$ is compact, and then $\{B(x_0, n) : n \geq k\}$ is the desired exhaustive sequence.

**Lemma 6** — If $X$ has Property $P$, then every triple $(H, F; X)$ has property $Q$.

**Proof** : Let $\{W_n\}$ be the exhaustive sequence in Definition 4. Fix a number $n \in \mathbb{N}$ and a point $x \in X \setminus \overline{W}_n$. Suppose $G$ is the component of $X \setminus \overline{W}_n$ containing $x$. Since $G$ is unbounded and rectifiably connected, there exists a locally rectifiable curve $\gamma$ in $G$ starts at $x$ and tends to $\infty$. If
\( \gamma \cap F \neq \emptyset \), then \( \gamma \) has a rectifiable subcurve \( \gamma_1 \) jointing \( x \) and \( F \), i.e. \( \gamma_1 \in \Gamma_xF \); otherwise we have \( \gamma \in \Gamma_{x\infty} \). Thus the conclusion follows.

**Theorem 1** — Suppose \( X \) is \( \varphi \)-convex and proper, \( H \) and \( F \) are two disjoint closed sets in \( X \) and \( H \) is also compact. If the triple \((H, F, X)\) has Property \( Q \), then we have the equality.

\[
\cap_{pC^*}(H, F^*, X^*) = \mod_p (H, F^*, X^*),
\]

where \( F^* = F \cup \{\infty\} \). If, moreover, \( X \) is locally quasiconvex, (3.2) holds with \( \cap_{pL_*} \) on the left-side.

**PROOF:** Suppose \( u \in A_{C^*}(H, F^*, X^*) \). Then \( u|_H \geq 1, u|_F \leq 0 \) and there exists a compact set \( K \) such that \( H \subset K \subset X \) and \( u|_{X^* \setminus K} = 0 \). If \( \rho \) is an upper gradient of \( u \) in \( X \), then it is easy to see the inequality \( \int \rho \, ds \geq 1 \) holds for all \( \gamma \in \Gamma = \Gamma_1 \cup \Gamma_{\infty} \), i.e., \( \rho \) is \( \Gamma \)-admissible. Then, by Definition 1 we have \( \mod_p (\Gamma) \leq \int_X \rho^p \, d\mu \). From this and definition 3 we obtain

\[
\mod_p (\Gamma) \leq \cap_{pC^*}(H, F^*, X^*).
\]

(3.3)

To verify the opposite inequality, one may assume \( \mod_p (\Gamma) < +\infty \), otherwise the conclusion holds already. Suppose \( \rho \in \Phi (\Gamma) \cap L^p(X) \). By Lemma 5, there exists a Borel function \( g : X \to (0, 1] \) with \( g \in L^p(X) \). For any \( m \in \mathbb{N} \), we have \( m^{-1} g + \rho \in \Phi (\Gamma) \cap L^p(X) \) and \( m^{-1} g + \rho > 0 \) in \( X \). Because the sequence \( \{m^{-1} g + \rho\} \) is decreasing and converges to \( \rho \) as \( m \to \infty \), one deduces

\[
\lim_{m \to \infty} \int_X \left( \frac{1}{m} g + \rho \right)^p \, d\mu = \int_X \rho^p \, d\mu
\]

by Lebesgue's theorem. Therefore we may use \( \{m^{-1} g + \rho\} \) instead of \( \rho \) to calculate the value of \( \mod_p (\Gamma) \). Hence we may assume that \( \rho > 0 \) in \( X \).

By the Vitali-Carathéodory theorem [12], we may further assume that \( \rho > 0 \) is lower semi-continuous in \( X \) because \( X \) is proper. Suppose \( \{W_n\} \) is the exhaustive sequence given in
Definition 5 with $H \subset W_1$, then for every $n \in \mathbb{N}$, there is a constant $\eta_n < 1$ such that

$$\rho |_{\Omega_n} \geq \eta_n > 0,$$

where $\Omega_n := \Omega \cap W_n$ is compact.

Fix an integer $n \in \mathbb{N}$ and consider the function $\rho_n$ such that $\rho_n(x) = \min\{\rho(x), n\}$ when $x \in \Omega_n$ and $\rho_n(x) = 0$ when $x \in X \setminus \Omega_n$. Define

$$u_n(x) = \inf_{\gamma_x} \int_{\gamma_x} \rho_n \, ds, \quad x \in X; \quad u_n(\infty) = 0,$$

where the infimum is taken over all $\gamma_x \in \Gamma_{x,F} \cup \Gamma_{x,\infty}$. The similar procedure as that in [4,15] deduces that $u_n$ is continuous in $X^*$ and $\rho_n$ is an upper gradient of $u_n$. Since the triple $(H, F; X)$ has property $Q$, for each $x \in (X \setminus W_n)$ there exists $\gamma \in \Gamma_{x,F} \cup \Gamma_{x,\infty}$ such that $\gamma \subset (X \setminus W_n) \subset X \setminus \Omega_n$. Hence we have $u_n(x) = 0$, i.e., the support of $u_n$ is a compact subset of $X$.

If, moreover, $X$ is locally quasiconvex, then $u_n$ is locally Lipschitz.

Set $m_n = \inf \{\rho(x) : x \in H\}$. Since the two compact sets $H$ and $(F \cap W_n) \cup \partial W_n$ are disjoint and $\rho |_{\Omega_n} \geq \eta_n > 0$, we have $m_n > 0$. If we set $\nu_n = u_n/m_n$, then $\nu_n \in A_{C^*}(H, F^*, X^*)$ (or $\nu_n \in A_{L^*}(H, F^*, X^*)$ when $X$ is locally quasiconvex), and $\rho_n/m_n$ is an upper gradient of $\nu_n$. Hence

$$\text{cap}_{pC^*}(H, F^*, X^*) \leq (m_n)^{-p} \int_X \rho_n^p \, d\mu \leq (m_n)^{-p} \int_X \rho^p \, d\mu$$

(or $\text{cap}_{pL^*}(H, F^*, X^*) \leq (m_n)^{-p} \int_X \rho_n^p \, d\mu \leq (m_n)^{-p} \int_X \rho^p \, d\mu$ when $X$ is locally quasiconvex).

Therefore, if the inequality $\limsup_{n \to \infty} m_n \geq 1$ is true, then the inequality $\text{cap}_{pC^*}(H, F^*, X^*)$ \leq \int_X \rho^p \, d\mu$ is valid, from which follows $\text{cap}_{pC^*}(H, F^*, X^*) \leq \text{mod}_p(\Gamma)$ (or $\text{cap}_{pC^*}(H, F^*, X^*) \leq \text{mod}_p(\Gamma)$) since $\rho \in \Phi(\Gamma) \cap L^p(X)$ is arbitrary.
Suppose on the contrary that the inequality \( \limsup_{n \to m} m_n \geq 1 \) is not true. Then there is a constant \( q < 1 \) and a sequence \( \{ \alpha_n \} \subset \Gamma_{xF} \cup \Gamma_{x^{\infty}} \) such that

\[
\int_{\alpha_n} \rho_n \, ds \leq q \quad \ldots \quad (3.4)
\]

for all \( n \) in \( \mathbb{N} \) and each \( \alpha_n \) starts from some point in \( H \). We may assume that each \( \alpha_n \) is in \( \Omega \), i.e. \( \alpha_n \in \Gamma_1 \cup \Gamma_{\infty} \), otherwise we can replace \( \alpha_n \) with a subcurve \( \alpha_n \) of \( \alpha_n \) in \( \Omega \) instead.

Now one of the two cases occurs.

**Case 1**: There exists an integer \( k \in \mathbb{N} \) such that \( \{ \alpha_n \} \) has a subsequence \( \{ \alpha_{n_i} \} \) satisfying that every \( \alpha_{n_i} \) is contained in \( \overline{W}_k \), and thus in \( \Omega_k \). Since there is a constant \( \eta_k \in (0, 1) \) such that \( \rho \mid \Omega_k \geq \eta_k \), by (3.4) we find a constant \( b \) such that \( l(\alpha_{n_i}) \leq b \) for all \( i \in \mathbb{N} \). Hence, by Lemma 3, there exists a subsequence of \( \{ \alpha_{n_i} \} \) converging to some rectifiable curve \( \beta \in \Gamma_1 \) in \( \Omega_k \).

**Case 2**: The situation of case 1 does not occur. Then, for any \( k \in \mathbb{N} \), there are infinite many curves in \( \{ \alpha_n \} \) which intersect \( \Omega \setminus W_k \).

For each \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \) with \( n \geq k \) we may assume that each \( \alpha_n \) intersects \( \Omega \setminus W_k \), otherwise we can replace \( \{ \alpha_n \}_{n \geq k} \) with a suitable subsequence of \( \{ \alpha_n \}_{n \geq k} \) instead. For each \( n \geq k \) denote

\[
t_{kn} := \min \left\{ t \in (0, \infty) : \alpha_n (t) \in \partial W_k \right\},
\]

and define the special subcurve \( \alpha_{kn} \) of \( \alpha_n \) in \( \overline{W}_k \). \( \alpha_{kn} : [0, \infty) \to \Omega_k \subset \overline{W}_k \) as follows: \( \alpha_{kn} (t) = \alpha_n (t) \) when \( t \) is in \( [0, t_{kn}] \) and \( \alpha_{kn} (t) = \text{const} \) when \( t \) in \( [t_{kn}, \infty) \). Thus the sequence \( \{ \alpha_{kn} \}_{n} \) consists of rectifiable curves in \( \overline{W}_1 \), and has a subsequence \( \{ \alpha_{1n} \}_{j} \) converging to a locally rectifiable curve \( \beta_1 : [0, \infty) \to \Omega_1 \subset \overline{W}_1 \) such that \( \beta_1 (0) \in H \) and \( \overline{B}_1 \cap (\partial W) \neq \emptyset \) by Lemma 3. This means that \( \{ \alpha_n \}_{n} \) has a subsequence \( \{ \alpha_{n(1,j)} \}_{j} \) such that the sequence of the special subcurves of all \( \alpha_{n(1,j)} \) in \( \overline{W}_1 \) converges to \( \beta_1 \).
Then, replacing \( \{ \alpha_n \} \) by \( \{ \alpha_n (1, j) \} \), we can find a subsequence \( \{ \alpha_n (2, j) \} \) of \( \{ \alpha_n (1, j) \} \) such that the sequence of the special sub curves of all \( \alpha_n (2, j) \) in \( \overline{W}_2 \) converges to a locally rectifiable curve \( \beta_2 : [0, \infty) \to \Omega_2 \subset \overline{W}_2, \beta_2 \Omega_1 = \beta_1, \beta_2 \cap (\partial W_2) \neq \phi \). By induction, for every \( k \in \mathbb{N}, k > 1 \), we can find a subsequence \( \{ \alpha_n (k, j) \} \) of \( \{ \alpha_n (k-1, j) \} \) such that the sequence of the special sub curves of all \( \alpha_n (k, j) \) in \( \overline{W}_k \) converges to a locally rectifiable curve \( \beta_k : [0, \infty) \to \Omega_k \subset \overline{W}_k, \beta_k \Omega_{k-1} = \beta_{k-1}, \beta_k \cap (\partial W_k) \neq \phi \).

Finally, we obtain the diagonal sequence \( \{ \alpha_n (k, k) \} \), which is a subsequence of \( \{ \alpha_n \} \) and converges (on each \( \overline{W}_k \)) to a locally rectifiable curve \( \beta : [0, \infty) \to \Omega \) satisfying \( \beta (0) \in H, \beta \Omega_k = \beta_k, \beta \cap (X \setminus K) \neq \phi \) for all compact subsets \( K \) of \( X \). This means that \( \beta \in \Gamma_\infty \).

By the above discussion, \( \{ \alpha_n \} \) has a subsequence converging (on each \( \overline{W}_k \)) to a curve \( \beta \in \Gamma \). Since the function \( \rho \) is lower semi-continuous in \( X \), by Lemma 4, Remark 2 and (3.4), we have

\[
\int_{\beta} \rho \, ds \leq \liminf_{n \to \infty} \int_{\alpha_n} \rho_n \, ds \leq q < 1,
\]

which is contrary to the assumption \( \rho \in \Phi (\Gamma) \). The proof is complete.

**Remark 5:** If the assumption in Theorem 1 that the triple \((H, F; X)\) has Property \( Q \) is replaced with that \( X \) has Property \( P \) instead, then we have the same conclusion by Lemma 6. It is also true for the other theorems in the next section. But it is not clear for the author what is the best condition for \( u_n \) to be in \( C_0 (X^*) \).

4. **EQUALITIES OF MODULUS AND CAPACITIES IN** \( X \)

Now we consider equalities of modulus and capacities in the space \( X \).

**Theorem 2.** — Suppose \( X \) is \( \varphi \)-convex and proper, \( H \) and \( F \) are two disjoint closed sets in \( X \) and \( H \) is also compact. If \((H, F, X)\) has Property \( Q \) and \( \text{mod}_p (\Gamma_\infty) = 0 \), then

\[
\text{cap}_{pL} (H, F; C) = \text{mod}_p (H, F; X), \quad \ldots \ (4.1)
\]

where \( \Gamma_\infty \) is the family of all locally rectifiable curves in \( \Omega \setminus F \) each of which starts from some boundary point of \( H \) and tends to \( \infty \).

If, moreover, \( X \) is locally quasiconvex, then (4.1) holds with \( \text{cap}_{pL} \) on the left-side.
PROOF: By (3.1) and (3.2),
\[
\text{cap}_{pC}(H, F; X) \leq \text{cap}_{pC}(H, F^*, X^*) = \text{mod}_p(H, F^*, X^*) = \text{mod}_p(\Gamma).
\] ... (4.2)

where \( \Gamma = (H, F; X) \cup \Gamma_\infty \). By Lemma 1,
\[
\text{mod}_p(H, F; X) \leq \text{mod}_p(\Gamma) \leq \text{mod}_p(H, F; X) + \text{mod}_p(\Gamma_\infty).
\]

From this inequality and (4.2) it follows that
\[
\text{cap}_{pC}(H, F; X) \leq \text{mod}_p(H, F; X) + \text{mod}_p(\Gamma_\infty).
\]

By Theorem A and (1.3) we see that (4.1) holds when \( \text{mod}_p(\Gamma_\infty) = 0 \).

If, moreover, \( X \) is locally quasiconvex, a similar procedure deduces that (4.1) holds with \( \text{cap}_{pL} \) on the left-side.

Corollary 3 — Suppose \( X \) is \( \varphi \)-convex and proper, \( H \) and \( F \) are two disjoint closed sets in \( X \) and \( H \) is also compact. If \( X \setminus (H \cup F) \) is bounded, then we have the equality (4.1) in \( (X, \mu) \).

If, moreover, \( X \) is locally quasiconvex, (4.1) holds with \( \text{cap}_{pL} \) on the left-side.

PROOF: If \( X \setminus (H \cup F) \) is bounded, then \( \text{mod}_p(\Gamma_\infty) = 0 \). On another hand, \((H, F, X)\) has property \( Q \) by Remark 4. By Theorem 2 we have the conclusion.

Lemma 7[13,15] — Suppose \( X \) is \( \varphi \)-convex and proper, \( E \) and \( F \) are two disjoint closed sets in \( X \). Then we have
\[
\text{cap}_{pC}(E, F; X) = \text{cap}_{pC}(\partial E, \partial F; X).
\]

If, moreover, \( X \) is locally quasiconvex, then we have \( \text{cap}_{pL}(E, F; X) = \text{cap}_{pL}(\partial E, \partial F; X) \).

PROOF: By Definition 3 we know \( \text{cap}_{pC}(E, F; X) \geq \text{cap}_{pC}(\partial E, \partial F; X) \) immediately. To prove the opposite inequality, assume \( u \in A_C(\partial E, \partial F; X) \) and \( \rho \) is an upper gradient of \( u \). Set
\[
u_1(x) = \min \{1, \max\{u(x), 0\}\}, \ x \in X.
\]

Then we have \( \nu_1 \in A_C(\partial E, \partial F; X), \ 0 \leq \nu_1(x) \leq 1 \) and \(| \nu_1(x) - \nu_1(y) | \leq | u(x) - u(y) | \) for any \( x, y \in X \), hence \( \rho \) is also an upper gradient of \( \nu_1 \). Define
\[ u_2(x) = \begin{cases} 
1, & x \in \text{int } E \\
0, & x \in \text{int } F \\
u_1(x), & x \in \text{int } \left( \text{int } E \cup \text{int } F \right).
\end{cases} \]

It is easy to verify that \( u_2 \in A_C(E, F; X) \) and that \( \rho \) is also the upper gradient of \( u_2 \), which implies \( \text{cap}_{pC}(E, F; X) \leq \text{cap}_{pC}(\partial E, \partial F; X) \).

If, moreover, \( X \) is locally quasiconvex, we can use, the same method to verify the equality \( \text{cap}_{pL}(E, F; X) = \text{cap}_{pL}(\partial E, \partial F; X) \).

**Theorem 3** — Suppose \( X \) is \( \varphi \)-convex and proper, \( E \) and \( F \) are two disjoint closed sets in \( X \) and \( \partial E \) is compact. If \( (\partial E, F; X) \) has Property \( Q \) and \( \text{mod}_p(\Gamma_\infty) = 0 \), then

\[ \text{cap}_{pC}(E, F; X) = \text{mod}_p(\partial E, \partial F; X), \quad \text{... (4.3)} \]

where \( \Gamma_\infty \) the family of all locally rectifiable curves in \( (X \setminus F) \cup \partial E \) each of which starts form some point of \( \partial E \) and tends to \( \infty \).

If, moreover, \( X \) is locally quasiconvex, (4.3) holds with \( \text{cap}_{pL} \) on the left-side.

**Proof**: In the case that \( X \) is \( \varphi \)-convex, by Lemma 7, we need only to prove

\[ \text{cap}_{pC}(\partial E, \partial F; X) = \text{mod}_p(\partial E, \partial F; X). \quad \text{... (4.4)} \]

By Lemma 1, \( \text{mod}_p(E, F; X) = \text{mod}_p(\partial E, \partial F; X) \). On the other hand, since \( \partial E \) is compact and \( \text{mod}_p(\Gamma_\infty) = 0 \), by Theorem 2 we have

\[ \text{cap}_{pC}(\partial E, \partial F; X) = \text{mod}_p(\partial E, \partial F; X). \]

Now, (4.4) follows and then we have (4.3).

If, moreover, \( X \) is locally quasiconvex, we can use the same method to verify the equality \( \text{cap}_{pL}(\partial E, \partial F; X) = \text{cap}_{pL}(E, F; X) \).

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