GENERALIZED JORDAN TRIPLE HIGHER DERIVATIONS ON PRIME RINGS

YONG-SOO JUNG

Department of Mathematics, Chungnam National University, Taejon 305-764, Korea
E-mail: ysjung@math.cnu.ac.kr

(Received 19 January 2005; after final revision 2 August 2005;
accepted 15 December 2005)

We prove that every generalized Jordan triple higher derivation on a 2-torsion free prime ring is a generalized higher derivation.

Key Words: Higher Derivations; Jordan Triple Higher Derivations; Generalized Jordan Triple Higher Derivations

1. INTRODUCTION

Throughout this paper, $R$ will represent an associative ring. Let $a, b, c \in R$, the element $abc - cba$ will be denoted by $[a, b, c]$.

A derivation (resp. Jordan derivation) is an additive map $\delta : R \rightarrow R$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$ (resp. $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in R$). An additive map $\delta : R \rightarrow R$ is called a Jordan triple derivation if $\delta(aba) = \delta(a)ba + a\delta(b)a + ab\delta(a)$ for all $a, b \in R$.

It is obvious that every derivation is a Jordan derivation. But the converse is in general not true. Herstein [8] proved that the converse is true on 2-torsion free prime rings and latter on. Brešar [5] extended this result to 2-torsion free semiprime rings. In [4], he also generalized his result by proving that every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Of course, it turns out that every Jordan derivation on a 2-torsion free ring is a Jordan triple derivation [4].

In [3], Brešar defined the following concept. An additive map $f : R \rightarrow R$ is called a generalized derivation if there exists a derivation $\delta : R \rightarrow R$ such that $f(ab) = f(a)b + a\delta(b)$ for all $a, b \in R$. This notion is found in [11] and other properties of generalized derivations were given by Hvala [9] and Quadri et al. [12]. We call an additive map $f : R \rightarrow R$ a generalized Jordan derivation.
if there exists a Jordan derivation $\delta: R \to R$ such that $f(a^2) = f(a) a + a \delta(a)$ for all $a \in R$ (see [2] and [10]).

For example, let $a, b \in R$ be such that one of them is not zero. Define $f(x) = ax + xb$ for all $x \in R$. Then for all $x, y \in R$, we have $f(x + y) = f(x) + f(y)$ and

$$f(x^2) = ax^2 + x^2 b$$

$$= (ax + xb) x + x (b x - x (-b))$$

$$= f(x) x + x \delta(x),$$

where $\delta$ defined by $\delta(x) = -bx - x (-b)$ for all $x \in R$ is a Jordan derivation on $R$. That is, $f$ is a generalized Jordan derivation on $R$.

Recently, Ashraf and Rehman [1] showed that in a 2-torsion free ring $R$ which has a commutator nonzero divisor, every generalized Jordan derivation on $R$ is a generalized derivation, and Jing and Lu [10] proved this result on a 2-torsion-free prime ring. Furthermore, they [10] proved that every generalized Jordan triple derivation on a 2-torsion free prime ring is a generalized derivation, where an additive map $f: R \to R$ is said to be a generalized Jordan triple derivation if there exists a Jordan triple derivation $\delta: R \to R$ satisfying $f(aba) = f(a) ba + a \delta(b) a + ab \delta(a)$ for all $a, b \in R$.

On the other hand, higher derivations have been studied in rings (mainly in commutative rings), but also in noncommutative rings. In particular, Ferrero and Haetinger [7] extended the above Brešars results in [4] and [5] to higher derivations.

Cortes and Haetinger [6] improved the Ashraf and Rehman's result [1] to a generalized Jordan higher derivation, that is, if $R$ is a 2-torsion free ring which has a commutator right nonzero divisor and $U$ is a square closed Lie ideal of $R$, then every generalized Jordan higher derivation of $U$ into $R$ is a generalized higher derivation of $U$ into $R$.

As pointed in [6, Examples, p. 8-9], we remark that the assumption that a ring has a commutator nonzero divisor and the assumption that a ring is semiprime are independent each other.

Let $R$ be a 2-torsion-free prime ring. Then we see that every generalized higher derivation on $R$ is a generalized Jordan higher derivation in view of the fact that every generalized derivation is a Jordan generalized derivation, and every generalized Jordan higher derivation on $R$ is a generalized Jordan triple higher derivation as in the proof of [7, Theorem 1.3].

In this paper we investigate that, by showing that every generalized Jordan triple higher derivation on a 2-torsion free prime ring is a generalized higher derivation (which generalizes the Jing and Lu's result [10, Theorem 3.5]). the notions of generalized higher derivation, generalized
Jordan higher derivation and generalized Jordan triple higher derivation on a 2-torsion free prime ring are equivalent to each other.

2. DEFINITIONS AND TECHNICAL LEMMAS

Let $\mathbb{N}_0$ be the set of all nonnegative integers.

Definition 2.1 — Let $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ be a family of additive maps on a ring $R$ such that $\delta_0 = id_R$. $\Delta$ is said to be:

(i) a higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a) \delta_j(b) \quad \text{for all } a, b \in R;$$

(ii) a Jordan higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(a^2) = \sum_{i+j=n} \delta_i(a) \delta_j(a) \quad \text{for all } a \in R;$$

(iii) a Jordan triple higher derivation if for each $n \in \mathbb{N}_0$,

$$\delta_n(aba) = \sum_{i+j+k=n} \delta_i(a) \delta_j(b) \delta_k(a) \quad \text{for all } a, b \in R;$$

Definition 2.2 — Let $F = (f_i)_{i \in \mathbb{N}_0}$ be a family of additive maps on a ring $R$ such that $f_0 = id_R$. $F$ is said to be:

(i) a generalized higher derivation if there exists a higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(ab) = \sum_{i+j=n} f_i(a) \delta_j(b) \quad \text{for all } a, b \in R;$$

(ii) a generalized Jordan higher derivation if there exists a Jordan higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,

$$f_n(a^2) = \sum_{i+j=n} f_i(a) \delta_j(a) \quad \text{for all } a \in R;$$
(iii) a generalized Jordan triple higher derivation if there exists a Jordan triple higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$,
\[
f_n(aba) = \sum_{i+j+k=n} f_i(a) \delta_j(b) \delta_k(c) \quad \text{for all } a, b \in R.
\]

We precede the proof of our main theorem by a series of lemmas.

Lemma 2.3 ([4, Lemma 1.2]) — Let $G_1, G_2, \ldots, G_n$ be additive groups and $R$ be a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \ldots \times G_n \rightarrow R$ and $T : G_1 \times G_2 \times \ldots \times G_n \rightarrow R$ are additive in each argument. If we have
\[
S(a_1, a_2, \ldots, a_n) xT(a_1, a_2, \ldots, a_n) = 0
\]
for all $x \in R, a_i \in G_i, i = 1, 2, \ldots, n$, then we get
\[
S(a_1, a_2, \ldots, a_n) xT(b_1, b_2, \ldots, b_n) = 0
\]
for all $x \in R, a_i, b_i \in G_i, i = 1, 2, \ldots, n$.

Given a generalized Jordan triple higher derivation $F = (f_i)_{i \in \mathbb{N}_0}$ on a ring $R$, and a Jordan triple higher derivation $\Delta = (\delta_i)_{i \in \mathbb{N}_0}$ satisfying the definition 2.2(iii), we put, for all $a, b, c \in R$ and $n \in \mathbb{N}_0$,
\[
P_n(a, b, c) = f_n(abc) - \sum_{i+j+k=n} f_i(a) \delta_j(b) \delta_k(c).
\]

The following result remarks two elementary properties of the just defined expression. Whereas linearity in each argument (ii) is an immediate consequence of the definition, the equality (i) can be easily obtained from the definition of generalized Jordan triple higher derivation by linearity.

Lemma 2.4 — Let $R$ be a ring equipped with a generalized Jordan triple higher derivation $F$. Then, with the above notation, for all $a, b, c \in R$ and $n \in \mathbb{N}_0$, we have

(i) $P_n(a, b, c) + P_n(c, b, a) = 0$;

(ii) $P_n(a, b, c)$ is additive in each argument.
Lemma 2.5 — Let $R$ be a semiprime ring equipped with a generalized Jordan triple higher derivation $F$. Then, with the above notation, if $n \in \mathbb{N}_0$ is such that $P_m(a, b, c) = 0$ for all $a, b, c \in R$ and $m < n$, we have

$$P_n(a, b, c) x [a, b, c] = 0 \text{ for all } a, b, c \in R.$$ 

PROOF: Since $R$ is semiprime, it follows from [7, Theorem 1.2] that $\Delta = (\delta^i_{i \in \mathbb{N}_0}$ is a higher derivation on $R$. Hence it is easy to see that

$$\delta_n(abc) = \sum_{i+j+k=n} \delta_i(a) \delta_j(b) \delta_k(c) \text{ for all } a, b, c \in R.$$

Let $W = f_n(abxcba + cbaxabc)$. We compute $W$ in two ways. On the one hand we have by Lemma 2.4(i)

$$W = \sum_{i+j+k=n} f_i(a) \delta_j(bcxcb) \delta_k(a) \sum_{i+j+k=n} f_i(c) \delta_j(baxab) \delta_k(c)$$

$$= \sum_{i+j+k=n} f_i(a) \left( \sum_{p+q+r=j} \delta_p(b) \delta_q(cxc) \delta_r(b) \right) \delta_k(a)$$

$$+ \sum_{i+j+k=n} f_i(c) \left( \sum_{p+q+r=j} \delta_p(b) \delta_q(axa) \delta_r(b) \right) \delta_k(c)$$

$$= \sum_{i+p+q+r+k=n} f_i(a) \delta_p(b) \delta_q(cxc) \delta_r(b) \delta_k(a)$$

$$= \sum_{i+p+q+r+k=n} f_i(c) \delta_p(b) \delta_q(axa) \delta_r(b) \delta_k(c)$$

$$\sum_{i+p+q+r+k=n} f_i(a) \delta_p(b) \left( \sum_{l+s+t=q} \delta_l(c) \delta_s(x) \delta_t(c) \right) \delta_r(b) \delta_k(a)$$

$$+ \sum_{i+p+q+r+k=n} f_i(a) \delta_p(b) \left( \sum_{l+s+t=q} \delta_l(a) \delta_s(x) \delta_t(a) \right) \delta_r(b) \delta_k(c)$$

$$= \sum_{i+p+l+s+t+r+k=n} f_i(a) \delta_p(b) \delta_l(c) \delta_s(x) \delta_t(c) \delta_r(b) \delta_k(a)$$

$$+ \sum_{i+p+l+s+t+r+k=n} f_i(c) \delta_p(b) \delta_l(a) \delta_s(x) \delta_t(a) \delta_r(b) \delta_k(c)$$
\[
= \sum_{i+p+l+s+t+r+k=n} f_i(a) \delta_p(b) \delta_i(c) \delta_s(x) \delta_t(c) \delta_r(b) \delta_k(a) + \sum_{i+p+l=n} f_i(a) \delta_p(b) \delta_i(c) \times cba
\]

\[
+ \sum_{i+p+l+s+t+r+k=n} f_i(c) \delta_p(b) \delta_i(a) \delta_s(x) \delta_t(a) \delta_r(b) \delta_k(c)
\]

\[
+ \sum_{i+p+l=n} f_i(c) \delta_p(b) \delta_i(a) \times abc.
\]

On the other hand, using the assumption \( P_m(a, b, c) = 0 \) for all \( a, b, c \) and \( m < n \), we get

\[
W = f_n((abc) x (cba)) + f_n((cba) x (abc))
\]

\[
= \sum_{u+v+w=n} f_u(abc) \delta_v(x) \delta_w(cba) + \sum_{u+v+w=n} f_u(cba) \delta_v(x) \delta_w(abc)
\]

\[
= \sum_{u+v+w=n} f_u(abc) \delta_v(x) \left( \sum_{e+g+h=w} \delta_e(c) \delta_g(b) \delta_h(a) \right)
\]

\[
+ \sum_{u+v+w=n} f_u(cba) \delta_v(x) \left( \sum_{e+g+h=w} \delta_e(a) \delta_g(b) \delta_h(c) \right)
\]

\[
= \sum_{u+v+e+g+h=n} f_u(abc) \delta_v(x) \delta_e(c) \delta_g(b) \delta_h(a)
\]

\[
+ \sum_{u+v+e+g+h=n} f_u(cba) \delta_v(x) \delta_e(a) \delta_g(b) \delta_h(c)
\]

\[
= \sum_{u+v+e+g+h=n} f_u(abc) \delta_v(x) \delta_e(c) \delta_g(b) \delta_h(a) + f_n(abc) \times cba
\]

\[
+ \sum_{u+v+e+g+h=n} f_u(cba) \delta_v(x) \delta_e(a) \delta_g(b) \delta_h(c) + f_n(cba) \times abc
\]

\[
= \sum_{u+v+e+g+h=n} \left( \sum_{\alpha+\beta+\gamma=u} f_\alpha(a) \delta_\beta(b) \delta_\gamma(c) \right) \delta_v(x) \delta_e(c) \delta_g(b) \delta_h(a)
\]

\[
+ f_n(cba) \times abc.
\]
\[= \sum_{u+v+e+g+h=n}^{u+v+g+h \neq 0} \left( \sum_{\alpha+\beta+\gamma=0}^{\alpha+\beta+\gamma=0} f_{\alpha}(x) \delta_{\beta}(b) \delta_{\gamma}(a) \right) \delta_{\nu}(x) \delta_{\epsilon}(a) \delta_{g}(b) \delta_{h}(c) + f_n(cba) xabc \]

\[= \sum_{\alpha+\beta+\gamma+u+e+g+h=n}^{\alpha+\beta+\gamma+u+e+g+h \neq 0} f_{\alpha}(a) \delta_{\beta}(b) \delta_{\gamma}(c) \delta_{\nu}(x) \delta_{\epsilon}(a) \delta_{g}(b) \delta_{h}(a) + f_n(abc) xcba \]

\[= \sum_{\alpha+\beta+\gamma+u+e+g+h=n}^{\alpha+\beta+\gamma+u+e+g+h \neq 0} f_{\alpha}(c) \delta_{\beta}(b) \delta_{\gamma}(a) \delta_{\nu}(x) \delta_{\epsilon}(a) \delta_{g}(b) \delta_{h}(c) + f_n(cba) xabc. \]

Comparing the last terms of two expressions so obtained for \( W \), we obtain

\[P_n(a, b, c) xcba + P_n(c, b, a) xabc = 0 \text{ for all } a, b, c, x \in R.\]

Since \( P_n(a, b, c) = -P_n(c, b, a) \) by Lemma 2.4(i), we see that

\[P_n(a, b, c) x [a, b, c] = 0 \text{ for all } a, b, c, x \in R.\]

This completes the proof of the lemma.

### 3. THE PROOF OF MAIN THEOREM

Now we are ready to prove our main theorem.

The proof of our main theorem is based on [7] to use the corresponding property for Jordan triple higher derivations and on the result in [10], which is essentially, a first step in our induction argument.

**Theorem** — Let \( R \) be a 2-torsion free prime ring. Then every generalized Jordan triple higher derivation on \( R \) is a generalized higher derivation.

**Proof:** We first intend to prove that \( P_n(a, b, c) = 0 \) for all \( a, b, c \in R \) and \( n \in \mathbb{N}_0 \) by induction.

In case \( n = 0 \), we get \( P_0(a, b, c) = 0 \) for all \( a, b, c \in R \). If \( n = 1 \), then it follows from [10, Theorem 3.5] that \( P_1(a, b, c) = 0 \) for all \( a, b, c \in R \). Thus, we assume that \( P_m(a, b, c) = 0 \) for all \( a, b, c \in R \) and \( m < n \). Then, from Lemma 2.3 and Lemma 2.5, we see that
$$P_n(a, b, c) x [w, y, z] = 0 \text{ for all } a, b, c, w, x, y, z \in R.$$  \hfill (1)

We have two cases:

**Case I:** There exist \(w_0, y_0, z_0 \in R\) satisfying \([w_0, y_0, z_0] \neq 0\) \hfill (1).

The primeness of \(R\) with Lemma 2.5 yields \(P_n(a, b, c) = 0\) for all \(a, b, c \in R\) and \(n \in \mathbb{N}\).

**Case II:** \([w, y, z] = 0\) is fulfilled for all \(w, y, z \in R\) in (1).

It is easily proved that \(R\) is then commutative.

Set \(W = f_n(c^3 ba + abc^3)\). We recall that \(\Delta = (\delta_n)_{n \in \mathbb{N}_0}\) a generalized higher derivation by the semiprimeness of \(R\). Let us compute \(W\) in two ways. On the one hand, applying Lemma 2.4(i), we get

\[ W = f_n(c^3 ba + abc^3) \]

\[ = \sum_{i+j+k=n} f_i(c^3) \delta_j(b) \delta_k(a) + \sum_{i+j+k=n} f_i(a) \delta_j(b) \delta_k(c^3) \]

\[ = \sum_{i+j+k=n} \left( \sum_{h+u+v=i} f_h(c) \delta_v(c) \delta_u(c) \delta_j(b) \delta_k(a) \right) \]

\[ + \sum_{i+j+k=n} f_i(a) \delta_j(b) \left( \sum_{\alpha+\beta+\gamma=k} \delta_\alpha(c) \delta_\beta(c) \delta_\gamma(c) \right) \]

\[ = \sum_{h+u+v+j+k=n} f_h(c) \delta_u(c) \delta_v(c) \delta_j(b) \delta_k(a) \]

\[ + \sum_{i+j+\alpha+\beta+\gamma=n} f_i(a) \delta_j(b) \delta_\alpha(c) \delta_\beta(c) \delta_\gamma(c) \]

\[ = \sum_{h+u+v+j+k=n} f_h(c) \delta_u(c) \delta_v(c) \delta_j(b) \delta_k(a) \]

\[ + \sum_{i+j+\alpha+\beta+\gamma=n} f_i(a) \delta_j(b) \delta_\alpha(c) \delta_\beta(c) \delta_\gamma(c) \]

\[ + \sum_{i+j+\alpha=n} f_i(a) \delta_j(b) \delta_\alpha(c) c^2. \]

On the other hand, we have, by Lemma 2.4(i), the induction assumption, and commutativity of \(R\),
\[ W = f_n \left( \sum_{i+j+k=n} f_i(c) \delta_j(c) \delta_k(cba) \right) + \sum_{i+j+k=n} f_i \left( \sum_{p+q+r=k} \delta_p(c) \delta_q(b) \delta_r(a) \right) \]

\[ = \sum_{i+j+k=n} f_i(c) \delta_j(c) \left( \sum_{p+q+r=k} \delta_p(c) \delta_q(b) \delta_r(a) \right) \]

\[ + \sum_{i+j+k=n} \left( \sum_{l+s+t=i} f_l(a) \delta_s(b) \delta_t(c) \right) \delta_j(c) \delta_k(c) + f_n(abc) c^2 \]

\[ = \sum_{i+j+p+q+r=n} f_i(c) \delta_j(c) \delta_p(c) \delta_q(b) \delta_r(a) \]

\[ + \sum_{l+s+t+j+k=n} f_l(a) \delta_s(b) \delta_t(c) \delta_j(c) \delta_k(c) + f_n(abc) c^2. \]

The comparison of two expressions for \( W \) yields

\[ \left( f_n(abc) - \sum_{i+j+\alpha=n} f_i(a) \delta_j(b) \delta_\alpha(c) \right) c^2 = 0, \]

that is,

\[ P_n(a, b, c) c^2 = 0 \text{ for all } a, b, c \in R. \]

Then we obtain

\[ P_n(a, b, c) cx + P_n(a, b, c) c = P_n(a, b, c) c^2 xP_n(a, b, c) = 0 \text{ for all } a, b, c, x \in R \]

which implies that

\[ P_n(a, b, c) c = 0 \text{ for all } a, b, c \in R \]

\[ \text{... (2)} \]

according to the semiprimeness of \( R \).

Linearizing (2) (by using Lemma 2.4(ii)) leads to

\[ P_n(a, b, c) y + P_n(a, b, y) c = 0 \text{ for all } a, b, c, y \in R. \]

Then it follows from (3) that

\[ P_n(c, b, c) yz P_n(a, b, c) y \]
\[ = -P_n(c, b, y) cz P_n(a, b, c) y \]
\[ = -P_n(c, b, y) ycz P_n(a, b, c) \]
\[ = 0 \]

for all \( a, b, c, y, z \in R \) which gives \( P_n(a, b, c) y = 0 \) for all \( a, b, c, y \in R \). Hence we have \( P_n(a, b, c) = 0 \) for all \( a, b, c \in R \) and \( n \in \mathbb{N}_0 \) by the primeness of \( R \).

It remains to prove that \( F = (f_n)_{n \in \mathbb{N}_0} \) is a generalized higher derivation.

Let \( \Theta_n(a, b) = f_n(ab) - \sum_{i+j=n} f_i(a) \delta_j(b) \) for all \( a, b \in R \) and \( n \in \mathbb{N}_0 \). It is trivial that \( \Theta_n(a, b) \) is additive in each argument. If \( n = 0 \), then \( \Theta_0(a, b) = 0 \). If \( n = 1 \), then we see from [10: Theorem 3.5] that \( \Theta_1(a, b) = 0 \). By induction, we assume that \( \Theta_m(a, b) = 0 \) for all \( a, b \in R \) and \( m < n \). Let \( W = f_n(abxba) \) for all \( a, b, x \in R \). On the one hand we have

\[
W = f_n(a(bxa)b)
\]
\[
= \sum_{i+j+k=n} f_i(a) \delta_j(bxa) \delta_k(b)
\]
\[
= \sum_{i+j+k=n} f_i(a) \left( \sum_{l+s+t=j} \delta_l(b) \delta_s(x) \delta_t(a) \right) \delta_k(b)
\]
\[
= \sum_{i+l+s+t+k=n} f_i(a) \delta_l(b) \delta_s(x) \delta_t(a) \delta_k(b)
\]
\[
= \sum_{i+l+s+t+k=n} f_i(a) \delta_l(b) \delta_s(x) \delta_t(a) \delta_k(b)
\]
\[
+ \sum_{i+l=n} f_i(a) \delta_l(b) xab.
\]

On the other hand we obtain, by using inductive hypothesis,

\[
W = f_n((ab)x(ab))
\]
\[
= \sum_{i+j+k=n} f_i(ab) \delta_j(x) \delta_k(ab)
\]
\[
\begin{align*}
&= \sum_{i+j+k=n} f_i(ab) \delta_j(x) \left( \sum_{u+v=k} \delta_u(a) \delta_v(b) \right) \\
&= \sum_{i+j+u+v=n} f_i(ab) \delta_j(x) \delta_u(a) \delta_v(b) \\
&= \sum_{i+j+u+v=n, j+u+v \neq 0} f_i(ab) \delta_j(x) \delta_u(a) \delta_v(b) + f_n(ab) xab \\
&= \sum_{p+q+j+u+v=n, j+u+v \neq 0} \left( \sum_{p+q=i} f_p(a) \delta_q(b) \right) \delta_j(x) \delta_u(a) \delta_v(b) + f_n(ab) xab.
\end{align*}
\]

Comparing the last terms of two expressions for \( W \), we have
\[
\left( f_n(ab) - \sum_{i+i=n} f_i(a) \delta_i(b) \right) xab = 0,
\]
i.e.,
\[
\Theta_n(a, b) xab = 0 \text{ for all } a, b, x \in R. \quad \cdots (3)
\]

Furthermore, the linearization of (3) is
\[
\Theta_n(a, b) xab = -\Theta_n(y, b) xab,
\]
and so (3) yields
\[
(\Theta_n(a, b) xyb) z (\Theta_n(a, b) xyb) = -\Theta_n(a, b) x (ybz \Theta_n(y, b)) xab = 0
\]
for all \( a, b, x, y, z \in R \). Hence we obtain
\[
\Theta_n(a, b) xyb = 0 \text{ for all } a, b, x, y \in R. \quad \cdots (4)
\]

Linearizing (4) and using the same approach as just done, we also have
\[
\Theta_n(a, b) xyw = 0 \text{ for all } a, b, x, y, w \in R.
\]

In particular, we see that
\[
\Theta_n(a, b) xy \Theta_n(a, b) x = 0 \text{ for all } a, b, x, y \in R.
\]
Therefore, we get
\[ \Theta_n (a, b) x = 0 \text{ for all } a, b, x \in R. \]

From the primeness of \( R \), we conclude that
\[ \Theta_n (a, b) = 0 \text{ for all } a, b \in R, \]
that is, \( F = (f_i)_{i \in \mathbb{N}_0} \) is a generalized higher derivation and completes the proof.

\[ \square \]

ACKNOWLEDGEMENT

The author would like to thank the referees for their careful reading and valuable suggestions.

REFERENCES