AN APPLICATION OF DIFFERENTIAL SUBORDINATIONS AND SOME CRITERIA FOR STARLIKENESS

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(Received 10 January 2005; accepted 9 January 2006)

Let $-1 \leq b < a \leq 1$, $0 < \gamma \leq 1$, and let $p(z)$ be analytic in the unit disk with $p(0) = 1$. By using the method of differential subordinations, we derive certain conditions connecting $p(z)$ and $z p'(z)$ under which the functions $p(z)$ are subordinate to $\left( \frac{1 + az}{1 + bz} \right)^\gamma$. Some useful consequences of these results are also given.

Key Words: $p$-Valently Starlike Function; Strongly Starlike Function; Subordination

1. INTRODUCTION

Let $f(z)$ and $g(z)$ be analytic in the unit disk $U = \{z : |z| < 1\}$. The function $f(z)$ is subordinate to $g(z)$ in $U$, written $f(z) \prec g(z)$, if $g(z)$ is univalent in $U$, $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $A_p (p \in N = \{1, 2, 3, \ldots\})$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m} \quad \ldots \quad (1.1)$$

which are analytic in $U$. A function $f(z) \in A_p$ is called $p$-valently starlike of order $\alpha$ in $U$ if it satisfies

...
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > p \alpha \quad (z \in U) \quad \cdots \quad (1.2) \]

for some \( \alpha (0 \leq \alpha < 1) \). We denote by \( S_p^*(\alpha) \) \((0 \leq \alpha < 1)\) the subclass of \( A_p \) consisting of all \( p \)-valently starlike functions of order \( \alpha \) in \( U \). For \(-1 \leq a \leq 1, -1 \leq b \leq 1, a \neq b \) and \( 0 < \gamma \leq 1 \), a function \( f(z) \in A_p \) is said to be in the class \( S_p^*(\gamma, a, b) \) if it satisfies

\[ \frac{zf'(z)}{f(z)} < p \left( \frac{1+az}{1+bz} \right) ^\gamma. \quad \cdots \quad (1.3) \]

It is easy to know that each function in the class \( S_p^*(\gamma, a, b) \) is \( p \)-valently starlike in \( U \). Also, we write

\[ S_p^*(\gamma, 1, -1) = S_p^*(\gamma), S_p^*(1, a, b) = S_p^*(a, b), S_p^*(1, -1) = S_p^*. \quad \cdots \quad (1.4) \]

Note that \( S_p^*(\gamma) \) is the class of strongly starlike \( p \)-valent functions of order \( \gamma \) in \( U \) and \( S_p^*(1-2\alpha, -1) = S_p^*(2\alpha-1, 1) = S_p^*(\alpha) \) \((0 \leq \alpha < 1)\).

Let \( P \) be the class of functions \( p(z) \) of the form

\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad \cdots \quad (1.5) \]

which are analytic in \( U \). If \( p(z) \in P \) satisfies \( \text{Re} p(z) > 0 \) \((z \in U)\), then we say that \( p(z) \) is the Carathéodory function. For Carathéodory functions, Miller [2] has shown some sufficient conditions applying the differential inequalities. Recently, Nunokawa et al. [6] have given some improvements of results by Miller [2]. In the present paper, using the method of differential subordinations, we derive certain conditions under which we have

\[ p(z) < \left( \frac{1+az}{1+bz} \right) ^\gamma, \]

where \(-1 \leq b < a \leq 1\) and \( 0 < \gamma \leq 1 \). In particular, we obtain some criteria for \( p \)-valently starlikeness and strongly starlikeness.

To prove our results, we need the following lemma due to Miller and Mocanu [3].

**Lemma** — Let \( g(z) \) be analytic and univalent in \( U \) and let \( \theta(w) \) and \( \varphi(w) \) be analytic in a domain \( D \) containing \( g(U) \), with \( \varphi(w) \neq 0 \) when \( w \in g(U) \). Set
\( Q(z) = zg'(z) \varphi(g(z)), \ h(z) = \theta(g(z)) + Q(z) \)

and suppose that

(i) \( Q(z) \) is starlike univalent in \( U \), and

(ii) \( \Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left\{ \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \ (z \in U) \).

If \( p(z) \) is analytic in \( U \), with \( p(0) = g(0), \ p(U) \subset D \) and

\[ \theta(p(z)) + zp'(z) \varphi(p(z)) \prec \theta(g(z)) + zg'(z) \varphi(g(z)) = h(z) \quad \text{... (1.6)} \]

then \( p(z) \prec g(z) \) and \( g(z) \) is the best dominant of (1.6).

2. MAIN RESULTS

**Theorem 1** — Let \( 0 < a \leq 1, \lambda > -\frac{1}{2} \) and \( \mu \) be a complex number with \( \Re \mu \geq 0 \). If \( p(z) \in P \) satisfies

\[ \lambda (p(z))^2 + \mu p(z) + zp'(z) \prec h(z), \quad \text{... (2.1)} \]

where

\[ h(z) = \frac{a^2(\lambda - \mu)z^2 + 2a(\lambda + 1)z + \lambda + \mu}{(1-az)^2}, \quad \text{... (2.2)} \]

then \( p(z) \prec \frac{1+az}{1-az} \) and \( \frac{1+az}{1-az} \) is the best dominant of (2.1).

**PROOF:** Set

\[ g(z) = \frac{1+az}{1-az}, \quad (0 < a \leq 1), \ \theta(w) = \lambda w^2 + \mu w, \ \varphi(w) = 1. \quad \text{... (2.3)} \]

Then \( g(z) \) is analytic and univalent in \( U \), \( g(0) = p(0) = 1 \), \( \theta(w) \) and \( \varphi(w) \) are analytic with \( \varphi(w) \neq 0 \) in the \( w \)-plane. The function

\[ Q(z) = zg'(z) \varphi(g(z)) = \frac{2az}{(1-az)^2} \quad \text{... (2.4)} \]

is starlike univalent in \( U \). Further, we have

\[ \theta(g(z)) + Q(z) = \lambda \left( \frac{1+az}{1-az} \right)^2 + \mu \frac{1+az}{1-az} + \frac{2az}{(1-az)^2} = h(z) \quad \text{... (2.5)} \]

and
\[
\frac{zh'(z)}{Q(z)} = (2\lambda + 1) \frac{1 + az}{1 - az} + \mu. \tag{2.6}
\]

Since \(\lambda > -\frac{1}{2}\) and \(\text{Re} \mu \geq 0\), it follows from (2.6) that

\[
\text{Re} \frac{zh'(z)}{Q(z)} = (2\lambda + 1) \frac{1 + az}{1 - az} + \text{Re} \mu > 0
\]

for \(z \in U\). Thus \(h(z)\) in (2.5) is close-to-convex and univalent in \(U\). From (2.1)-(2.5) we see that

\[
\theta(p(z)) + z\varphi'(z) \varphi(p(z)) \prec \theta(g(z)) + zg'(z) \varphi(g(z)) = h(z).
\]

Therefore, by applying the lemma, we conclude that \(p(z) \prec g(z)\) and \(g(z)\) is the best dominant of (2.1). The proof of the theorem is complete.

Remark 1: When \(a = 1, \lambda = \frac{\alpha}{\beta}, \beta > 0, \alpha > -\frac{\beta}{2}\) and \(\mu = 0\), Theorem 1 was also proved by Nanokawa et al. [6] using another method.

Theorem 2 — Let \(-1 \leq b < a \leq 1, m \in \mathbb{N} \setminus \{1\}, 0 < \gamma \leq \frac{1}{m - 1}, \lambda \geq 0\) and \(\mu\) be a complex number such that \(\text{Re} \mu \geq -m \lambda \left(\frac{1 - a}{1 - b}\right)^{(m - 1)\gamma}\). If \(p(z) \in P\) satisfies

\[
\lambda(p(z))^m + \mu p(z) + zp'(z) \prec h(z), \tag{2.7}
\]

where

\[
h(z) = \lambda \left(\frac{1 + az}{1 + bz}\right)^m + \mu \left(\frac{1 + az}{1 + bz}\right)^\gamma + \frac{\gamma(a - b)z}{(1 + az)^{1 - \gamma}(1 + bz)^{1 + \gamma}} \tag{2.8}
\]

then \(p(z) \prec \left(\frac{1 + az}{1 + bz}\right)^\gamma\) and \(\left(\frac{1 + az}{1 + bz}\right)^\gamma\) is the best dominant of (2.7).

Proof: We choose

\[
g(z) = \left(\frac{1 + az}{1 + bz}\right)^\gamma, \theta(w) = \lambda w^m + \mu w, \varphi(w) = 1
\]

in the lemma. In view of \(-1 \leq b < a \leq 1\) and \(0 < \gamma \leq \frac{1}{m - 1} \leq 1\), the function \(g(z)\) is analytic and convex univalent in \(U\) because
\[
\text{Re} \left\{ 1 + \frac{z g''(z)}{g'(z)} \right\} = \text{Re} \left\{ 1 + (\gamma-1) \frac{az}{1+az} - (\gamma+1) \frac{bz}{1+az} \right\}
\]
\[
= -1 + (1-\gamma) \text{Re} \frac{1}{1+az} + (1+\gamma) \text{Re} \frac{1}{1+bz}
\]
\[
> -1 + \frac{1-\gamma}{1+|a|} + \frac{1+\gamma}{1+|b|}
\]
\[
\geq 0 \ (z \in U).
\]

Thus
\[
Q(z) = z g'(z) \varphi(g(z)) = z g'(z) = \frac{\gamma(a-b)z}{(1+az)^{1-\gamma}(1+bz)^{1+\gamma}}
\]
is starlike univalent in \(U\). Further, we have
\[
\theta(g(z)) + Q(z) = \lambda \left( \frac{1+az}{1+bz} \right)^m + \mu \left( \frac{1+az}{1+bz} \right)^\gamma + \frac{\gamma(a-b)z}{(1+az)^{1-\gamma}(1+bz)^{1+\gamma}}
\]
\[
= h(z)
\]
and
\[
\frac{zh'(z)}{Q(z)} = m \lambda \left( \frac{1+az}{1+bz} \right)^{m-1} + \mu + \frac{zQ'(z)}{Q(z)}.
\]...
(2.9)

Note that \(\text{Re} \ (w^\beta) \geq (\text{Re} \ w)^\beta \ (0 < \beta \leq 1, \text{Re} \ w > 0)\). For \(\lambda \geq 0, 0 < \gamma \leq \frac{1}{m-1}\) and
\[
\text{Re} \ \mu \geq -m \lambda \left( \frac{1-a}{1-b} \right)^{(m-1)\gamma},
\]
it follows from (2.9) that
\[
\text{Re} \ \frac{zh'(z)}{Q(z)} \geq m \lambda \left( \text{Re} \left( \frac{1+az}{1+bz} \right) \right)^{(m-1)\gamma} + \text{Re} \mu + \frac{zQ'(z)}{Q(z)}
\]
\[
> m \lambda \left( \frac{1-a}{1-b} \right)^{(m-1)\gamma} + \text{Re} \mu
\]
\[
\geq 0 \ (z \in U).
\]

The other conditions of the lemma are seen to be satisfied. Hence \(p(z) \prec g(z)\) and \(g(z)\) is the best dominant of (2.7). The proof is complete.

From Theorem 2 we can get a number of interesting results.
Corollary 1. Let $-1 \leq b < a \leq 1$ and $0 < \gamma \leq 1$. If $f(z) \in A_p$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \frac{p}{(1 + az)^{1-\gamma} (1 + bz)^{1+\gamma}} \right) \prec \frac{\gamma(a - b)z}{1 + az} \prec \frac{p}{1 + az} \gamma(a - b)z$$

then $f(z) \in S_p^{*}(\gamma, a, b)$.

**Proof**: Let $p(z) = \frac{zf'(z)}{pf(z)}$. Then $p(z) \in P$ and (2.10) can be written as

$$zp'(z) \prec \frac{\gamma(a - b)z}{(1 + az)^{1-\gamma} (1 + bz)^{1+\gamma}}$$

Taking $\lambda = \mu = 0$ and $m = 2$ in Theorem 2 and using (2.11), we know that $p(z) < \left(\frac{1 + az}{1 + bz}\right)^{\gamma}$ and $\left(\frac{1 + az}{1 + bz}\right)^{\gamma}$ is the best dominant of (2.11). So $f(z) \in S_p^{*}(\gamma, a, b)$.

Corollary 2 — Let $0 < \gamma \leq 1$ and $\lambda \geq 0$. If $f(z) \in A_p$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\frac{zf'(z)}{pf(z)} \left(1 + \frac{zf''(z)}{f'(z)} + \left(\frac{\lambda}{p} - 1\right)\frac{zf'(z)}{f(z)} \right) \prec h(z),$$

where

$$h(z) = \lambda \left(\frac{1 + z}{1 - z}\right)^{2\gamma} + \frac{2\gamma z}{(1 + z)^{1-\gamma} (1 - z)^{1+\gamma}},$$

then $f(z) \in S_p^{*}(\gamma)$ and the order $\gamma$ is sharp with the extremal function

$$f(z) = z^p \exp \left\{ p \int_0^z \frac{\gamma}{1 + \frac{t}{t - 1}} dt \right\}.$$  

**Proof**: Setting $a = 1$, $b = -1$, $m = 2$, $0 < \gamma \leq 1$, $\lambda \geq 0$, $\mu = 0$, and $p(z) = \frac{zf'(z)}{pf(z)}$ in Theorem 2 and using (2.12), we have $f(z) \in S_p^{*}(\gamma)$.

For the function $f(z)$ defined by (2.13), it is easy to verify that

$$\frac{zf'(z)}{pf(z)} \left(1 + \frac{zf''(z)}{f'(z)} + \left(\frac{\lambda}{p} - 1\right)\frac{zf'(z)}{f(z)} \right) = h(z)$$

and
\[
\left| \arg \frac{zf'(z)}{f(z)} \right| = \gamma \left| \arg \frac{1+z}{1-z} \right| \to \frac{\gamma \pi}{2}
\]
as \(z \to i\). Hence the corollary is proved.

**Remark 2**: Owa and Obradovic [9] proved that if \(f(z) \in A_1\) satisfies \(f(z) \neq 0\) in \(0 < |z| < 1\) and
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right\} > -\frac{1}{2} \quad (z \in U),
\]
then \(f(z) \in S^*_1\). It is clear that Corollary 2 with \(p = \gamma = 1\) and \(\lambda = 0\) is better than the result in [9].

**Remark 3**: It was proved in [5, Theorem 1] that if \(f(z) \in A_1\) satisfies \(f(z) \neq 0\) \((0 < |z| < 1)\) and
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U),
\]
then \(f(z) \in S^*_1 (\gamma_1)\), where \(\gamma_1 \in (0, 1)\) is the root of the equation
\[
1 = \gamma + \frac{2}{\pi} \arctan q(\gamma),
\]
\[
q(\gamma) = \tan \frac{\gamma \pi}{2} + \frac{\gamma}{2 \cos \frac{\gamma \pi}{2}} \left( \frac{1-\gamma}{1-\gamma} + \frac{1+\gamma}{1+\gamma} \right).
\]

Now we show that Corollary 2 with \(p = \lambda = 1\) and \(\gamma = \gamma_1\) refines the above result. Let us put
\[
h_1(z) = \left( \frac{1+z}{1-z} \right)^{2\gamma_1} + \frac{2\gamma_1 z}{(1+z)^{1-\gamma_1} (1-z)^{1+\gamma_1}}.
\]
Then for \(0 < \theta < \pi\), we have
\[
\arg h_1(e^{i\theta}) = \gamma_1 \arg \left( \frac{1+e^{i\theta}}{1-e^{i\theta}} \right) + \arg \left( \frac{2\gamma_1 e^{i\theta}}{1-e^{2i\theta}} \right) + \arg \left( \frac{1+e^{i\theta}}{1-e^{i\theta}} \right) + \frac{\gamma_1}{2} \frac{i\gamma \pi}{x^2} + \frac{\gamma_1 i}{2} \left( x + \frac{1}{x} \right)
\]

\[
= \frac{\gamma_1 \pi}{2} + \arg \left( \frac{iy_{\gamma_1 \pi}}{x_1 e^{i\theta}} + \frac{\gamma_1 i}{2} \left( x + \frac{1}{x} \right) \right)
\]
\[ = \frac{\gamma_1 \pi}{2} + \arctan g(x), \]

where \( x = \cot \frac{\theta}{2} > 0 \) and

\[ g(x) = \tan \frac{\gamma_1 \pi}{2} + \frac{\gamma_1}{2 \cos \frac{\gamma_1 \pi}{2}} \left( x^{-\gamma_1} + \frac{1}{x^{1+\gamma_1}} \right). \]

It is easy to know that \( g(x) \) takes its minimum value at \( x = \sqrt{\frac{1 + \gamma_1}{1 - \gamma_1}} \). Hence, in view of \( h_1 (e^{-i\theta}) = h_1 (e^{-i\theta}) \), we deduce that

\[
\inf_{|z| = 1} |\arg h_1 (z)| = \min_{0 < \theta < \pi} \arg h_1 (e^{i\theta})
\]

\[ = \frac{\gamma_1 \pi}{2} + \arctan \left( g \left( \sqrt{\frac{1 + \gamma_1}{1 - \gamma_1}} \right) \right) \]

\[ = \frac{\pi}{2} \left( \gamma_1 + \frac{2}{\pi} \arctan q(\gamma_1) \right) \]

\[ = \frac{\pi}{2}, \]

and so \( h_1 (U) \) properly contains the right half plane \( \text{Re } w > 0 \). Thus we conclude that Corollary 2 with \( p = \lambda = 1 \) and \( \gamma = \gamma_1 \) is better than the result of [5, Theorem 1].

**Corollary 3** — If \( f(z) \in A_p \) satisfies \( f(z) \neq 0 \) (0 < |z| < 1) and

\[
\frac{zf''(z)}{pf(z)} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) < \frac{p (1 - 2\alpha)^2 z^2 + 2 (p + 1 - (2p + 1) \alpha) z + p}{(1 - z)^2} \quad \text{... (2.14)}
\]

for some \( \alpha (0 \leq \alpha < 1) \), then \( f(z) \in S^*_p (\alpha) \) and the order \( \alpha \) is sharp for the function \( f(z) = z^p (1 - z)^{-2p(1 - \alpha)}. \)

**Proof**: Letting \( a = 1 - 2\alpha, b = -1, m = 2, \gamma = 1, \gamma = p, \mu = 0, \) and \( p(z) = \frac{zf'(z)}{pf(z)} \) in Theorem 2, it follows from (2.14) that \( f(z) \in S^*_p (\alpha) \). Sharpness can be verified easily.

**Remark 4**: For \( \alpha = 0 \), (2.14) becomes
\[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \frac{p}{2} < p \left( p + \frac{1}{2} \right) \left( \frac{1 + z}{1 - z} \right)^2, \]

or equivalently

\[ \left| \arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} + \frac{p}{2} \right) \right\} \right| < \pi \quad (z \in U). \]

Therefore Corollary 3 with \( \alpha = 0 \) reduces to the result of [8, Theorem 2].

**Remark 5:** It was shown in [1] that \( f(z) \in A_1 \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[ \Re \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U), \]

then \( f(z) \in S^*_1 \left( \frac{1}{2} \right) \). For \( p = 1 \) and \( \alpha = \frac{1}{2} \), (2.14) becomes

\[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < h(z) = \frac{1 + z}{(1 - z)^2}. \]

Since \( h(U) = \left\{ w = u + iv : v^2 > -\frac{u}{2} \right\} \) properly contains the right half plane \( \Re w > 0 \), we see that Corollary 3 with \( p = 1 \) and \( \alpha = \frac{1}{2} \) is better than the result in [1].

**Corollary 4:** Let \(-1 \leq b < a \leq 1\) and \( \Re \mu \geq 0 \). If \( f(z) \in A_p \) satisfies \( f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[ (1 - \mu) \frac{zf'(z)}{zf''(z)} + \frac{zf'(z)f''(z)}{(f'(z))^2} < h(z), \quad \ldots \ (2.15) \]

where

\[ h(z) = \frac{b((pb - \mu)a)(1 + \mu)a)}{p(1 + bz)^2} z^2 + \left( (2p + 1 - \mu)b - (1 + \mu)a \right) z + p - \mu, \]

then \( f(z) \in S^*_p (b, a) \).

**PROOF:** If we let \( \frac{zf'(z)}{zf''(z)} = \frac{pf(z)}{zf''(z)} \), then \( p(z) \in P \) and (2.15) can be expressed as

\[ \mu p(z) + zp'(z) < p - ph(z) = \mu \left( \frac{1 + az}{1 + bz} \right) + \frac{(a - b)z}{(1 + bz)^2}. \]

Hence, taking \( m = 2, \gamma = 1, \lambda = 0 \) and \( \Re \mu \geq 0 \) in Theorem 2, we have \( p(z) < \frac{1 + az}{1 + bz} \) and
so \( f(z) \in S^*_p (b, a) \).

**Remark 6**: Setting \( p = 1, 0 = b < a \leq 1 \) and \( \mu = 0 \) in Corollary 4, we get the result obtained by Singh [11, Theorem 1.2], which refines the result of Silverman [10, Theorem 1].

**Remark 7**: Putting \( p = a = 1, b = -1 \) and \( \mu = 1 \) in Corollary 4, we have the result obtained by Tuneski [12, Theorem 2.1].

Setting \( p = 1, 0 = b < a \leq 1 \) and \( \mu = 1 \), Corollary 4 leads to

**Corollary 5** — Let \( 0 < a \leq 1 \). If \( f(z) \in A_1 \) satisfies \( f'(z) \neq 0 \) and

\[
\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < 2a \quad (z \in U),
\]

then \( f(z) \in S^*_1 \left( \frac{1}{1 + a} \right) \) and the order \( \frac{1}{1 + a} \) is sharp with the extremal function \( f(z) = \frac{z}{1 + az} \).

**Remark 8**: Sharpness is immediate. Also, Tuneski [12] proved that if \( f(z) \in A_1 \) satisfies \( f'(z) \neq 0 \) and

\[
\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < 2 \quad (z \in U),
\]

then \( f(z) \in S^*_1 \). Obviously, Corollary 5 with \( a = 1 \) improves the result of [12].

Setting \( a = 0, b = -c \) and \( \mu = 1 \), Corollary 4 yields

**Corollary 6** — If \( f(z) \in A_p \) satisfies \( f'(z) \neq 0 \) \((0 < |z| < 1)\) and

\[
\frac{f(z)f''(z)}{(f'(z))^2} < 1 - \frac{1}{p(1-cz)^2}
\]

for some \( c \) \((0 < c \leq 1)\), then

\[
\left| \frac{zf''(z)}{f(z)} - p \right| < pc \quad (z \in U). \quad \text{... (2.16)}
\]

The bound \( pc \) in (2.16) is sharp for \( f(z) = z^p e^{-pcz} \).

For \( \mu = 0 \), Corollary 4 reduces to

**Corollary 7** — Let \(-1 \leq b < a \leq 1 \). If \( f(z) \in A_p \) satisfies \( f'(z) \neq 0 \) \((0 < |z| < 1)\) and

\[
\frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{pb^2 z^2 + ((2p + 1) b - a) z + p}{p(1+bx)^2}, \quad \text{... (2.17)}
\]

then \( f(z) \in S^*_p (b, a) \).
Remark 9: For \( a = 1 \) and \( b = -1 \), (2.17) becomes

\[
\frac{f(z)}{zf''(z)} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) - \left( 1 + \frac{1}{2p} \right) < -\frac{1}{2p} \left( \frac{1+z}{1-z} \right)^2,
\]

which is equivalent to

\[
\left| \arg \left( \frac{f(z)}{zf''(z)} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) - \left( 1 + \frac{1}{2p} \right) \right) \right| > 0 \quad (z \in U).
\]

Thus we see that Corollary 7 with \( a = 1 \) and \( b = -1 \) coincides with the result of [8, Theorem 1]. Similarly, for \( a = 0 \) and \( b = -1 \), (2.17) is equivalent to

\[
\left| \arg \left( \frac{f(z)}{zf''(z)} \left( 1 + \frac{zf'''(z)}{f'(z)} \right) - \left( 1 + \frac{1}{4p} \right) \right) \right| > 0 \quad (z \in U).
\]

Hence Corollary 7 with \( a = 0 \) and \( b = -1 \) coincides with the result of [7, Theorem].

Corollary 8 — Let \( 0 < \gamma < 1 \) and \( \lambda \geq 0 \). If \( f(z) \in A_p \) satisfies \( f(z) \neq 0 \) (\( 0 < |z| < 1 \)) and

\[
\frac{zf'(z)}{pf(z)} \left( \lambda \frac{zf'(z)}{pf(z)} + \gamma \cot \frac{\gamma \pi}{2} + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \left( \frac{1+z}{1-z} \right)^{2\gamma},
\]

then \( f(z) \in S^*_p(\gamma) \) and the bound \( 2\gamma \) in (2.18) is sharp.

Proof: For \( a = 1 \), \( b = -1 \), \( m = 2 \), \( 0 < \gamma < 1 \). \( \lambda \geq 0 \), \( \mu = \gamma \cot \frac{\gamma \pi}{2} > 0 \) and \( p(z) = \frac{zf'(z)}{pf(z)} \), (2.7) in Theorem 2 becomes

\[
\frac{zf'(z)}{pf(z)} \left( \lambda \frac{zf'(z)}{pf(z)} + \gamma \cot \frac{\gamma \pi}{2} + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < h(z),
\]

where

\[
h(z) = \left( \frac{1+z}{1-z} \right)^{\gamma} \left( \lambda \left( \frac{1+z}{1-z} \right)^{\gamma} + \gamma \cot \frac{\gamma \pi}{2} + \frac{2\gamma z}{1-z^2} \right).
\]

Letting \( 0 < \theta < \pi \) and \( x = \cot \frac{\theta}{2} > 0 \), then we have

\[
\arg h(e^{i\theta}) = \frac{\gamma \pi}{2} + \arg \left\{ \frac{i \gamma \pi}{2} \left( \lambda x^\gamma e^{\frac{i\gamma \pi}{2}} + \gamma \cot \frac{\gamma \pi}{2} + \frac{\gamma i}{2} \left( x + \frac{1}{x} \right) \right) \right\}
\]
\[ = \frac{\gamma \pi}{2} + \arctan \left( \tan \frac{\gamma \pi}{2} \left( \frac{\lambda x^\gamma \sin \frac{\gamma \pi}{2} + \frac{\gamma}{2} \left( x + \frac{1}{x} \right)}{\lambda x^\gamma \sin \frac{\gamma \pi}{2} + \gamma} \right) \right) \]

\[ \geq \frac{\gamma \pi}{2} + \frac{\gamma \pi}{2} = \gamma \pi, \]

and so

\[
\inf_{|z|=1} \arg h(z) = \min_{0 < \theta < \pi} \arg h(e^{i\theta}) = \gamma \pi. \quad \text{(2.20)}
\]

If \( f(z) \) satisfies (2.18), then it follows from (2.20) that the subordination (2.19) holds. Thus an application of Theorem 2 yields \( f(z) \in \mathbb{S}_p^\gamma (\gamma) \).

For the function \( f(z) \) defined by (2.13), we have

\[
\frac{zf''(z)}{pf(z)} \left( \frac{\lambda zf'(z)}{pf(z)} + \gamma \cot \frac{\gamma \pi}{2} + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) = h(z)
\]

and it follows from (2.20) that the bound \( 2\gamma \) in (2.18) is sharp.

**Theorem 3** — Let \(-1 \leq b < a \leq 1\), \( m \in \mathbb{N}, 0 < \gamma \leq \frac{1}{m} \), \( \lambda \geq 0 \) and \( \mu \geq 0 \). If \( p(z) \in P \) satisfies \( p(z) \neq 0 \) \((0 < |z| < 1)\) and

\[
\lambda (p(z))^m + \mu p(z) + \frac{zp'(z)}{p(z)} \prec h(z) \quad \text{(2.21)}
\]

where

\[
h(z) = \lambda \left( \frac{1 + az}{1 + bz} \right)^m + \mu \left( 1 + \frac{az}{1 + bz} \right)^\gamma + \frac{\gamma(a-b)z}{(1+az)(1+bz)},
\]

then \( p(z) \prec \left( \frac{1 + az}{1 + bz} \right)^\gamma \) and \( \left( \frac{1 + az}{1 + bz} \right)^\gamma \) is the best dominant of (2.21).

**PROOF:** We choose

\[
g(z) = \left( \frac{1 + az}{1 + bz} \right)^\gamma, \quad \theta(w) = \lambda w^m + \mu w, \quad \varphi(w) = \frac{1}{w}
\]

in the lemma. Noting that

\[
\text{Re} \, g(z) > \left( \frac{1-a}{1-b} \right)^\gamma \geq 0 \quad (z \in U)
\]
for $0 < \gamma \leq \frac{1}{m} \leq 1$, the function $\varphi(w)$ is analytic in $D = \{w : w \neq 0\}$ containing $g(U)$. The function

$$Q(z) = zg'(z) \varphi(g(z)) = \frac{\gamma(a-b)z}{(1+az)(1+bz)}$$

is starlike univalent in $U$ because

$$Re \left( \frac{zQ'(z)}{Q(z)} \right) = -1 + Re \left( \frac{1}{1+az} \right) + Re \left( \frac{1}{1+bz} \right)$$

$$> -1 + \frac{1}{1+|a|} + \frac{1}{1+|b|} \geq 0$$

for $z \in U$. Further we have

$$\theta(g(z)) + Q(z) = \lambda \left( \frac{1+az}{1+bz} \right)^m \gamma + \mu \left( \frac{1+az}{1+bz} \right)^\gamma + \frac{\gamma(a-b)z}{(1+az)(1+bz)} = h(z).$$

and

$$Re \left( \frac{zh'(z)}{Q(z)} \right) = m \lambda \left( \frac{1-a}{1-b} \right)^m \gamma + \mu \left( \frac{1-a}{1-b} \right)^\gamma + Re \left( \frac{zQ'(z)}{Q(z)} \right)$$

$$> m \lambda \left( \frac{1-a}{1-b} \right)^m \gamma + \mu \left( \frac{1-a}{1-b} \right)^\gamma \geq 0 \quad (z \in U)$$

for $0 < m \gamma \leq 1$, $\lambda \geq 0$ and $\mu \geq 0$. The other conditions of the lemma can be checked to be satisfied. Therefore $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (2.21).

**Remark 10**: Note that the univalent function

$$w = \frac{\alpha(1+z)}{1-z} + \frac{2\beta z}{1-z^2} \quad (\alpha > 0, \beta > 0)$$

maps $U$ onto the $w$-plane slit along the half-lines $Rew = 0$, $Imw \geq \sqrt{\beta(2\alpha + \beta/2)}$ and $Rew = 0$, $Imw \leq -\sqrt{\beta(2\alpha + \beta)}$. For $a = 1$, $b = -1$, $\gamma = 1$, $\lambda = 0$, $\mu = \frac{\alpha}{\beta}$, $\alpha > 0$ and $\beta > 0$, Theorem 3 reduces to the result obtained by Nunokawa et al. [6, Theorem 2] using another method.

**Corollary 9**: Let $-1 \leq b < a \leq 1$, $0 < \gamma \leq 1$ and $\beta > 0$. If $f(z) \in A_\beta$ satisfies $f(z)f'(z) \neq 0$ ($0 < |z| < 1$) and

$$(1-\beta) \frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec p \left( \frac{1+az}{1+bz} \right)^\gamma + \frac{\beta \gamma(a-b)z}{(1+az)(1+bz)}, \quad \cdots (2.22)$$
then \( f(z) \in S_p^* (\gamma, a, b) \).

**Proof:** Setting \( m = 1, \lambda = \frac{p}{\beta} > 0, \mu = 0 \) and \( p(z) = \frac{zf'(z)}{pf(z)} \) in Theorem 3 and using (2.22), the desired result follows at once.

**Corollary 10** — Let \( 0 < a \leq 1 \) and \( \beta > 0 \). If \( f(z) \in A_p \) satisfies \( f(z)f'(z) \neq 0 \) \( (0 < |z| < 1) \) and

\[
(1 - \beta) \left( \frac{zf'(z)}{f(z)} \right) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) - p < a \left( p + \frac{\beta}{1 + a} \right) \quad (z \in U), \quad \ldots \ (2.23)
\]

then

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < pa \quad (z \in U). \quad \ldots \ (2.24)
\]

**Proof:** For \( 0 = b < a \leq 1, \gamma = 1 \) and \( \beta > 0 \), (2.22) in Corollary 9 becomes

\[
(1 - \beta) \left( \frac{zf'(z)}{f(z)} \right) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) - p < az \left( p + \frac{\beta}{1 + az} \right). \quad \ldots \ (2.25)
\]

Since the function \( h(z) = az \left( p + \frac{\beta}{1 + az} \right) \) is univalent in \( U \) and

\[
| h(z) | \geq a \text{Re} \left( p + \frac{\beta}{1 + az} \right) \geq a \left( p + \frac{\beta}{1 + az} \right)
\]

for \( |z| = 1 \left( z \neq -\frac{1}{a} \right) \), it follows from (2.23) that the subordination (2.25) holds. Hence an application of Corollary 9 yields the inequality (2.24).

**Remark 11:** Putting \( p = a = 1 \) in Corollary 10, we get the result obtained earlier by Mocanu [4, Theorem 3] using a different method.

**Corollary 11** — Let \( 0 < \gamma < 1 \) and \( \beta > 0 \). If \( f(z) \in A_p \) satisfies \( f(z)f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
(1 - \beta) \left( \frac{zf'(z)}{f(z)} \right) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) < p \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \gamma)}, \quad \ldots \ (2.26)
\]

where

\[
\alpha(\beta, \gamma) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\gamma \pi}{2} + \frac{\beta \gamma}{p(1 + \gamma)} \frac{1 + \gamma}{(1 - \gamma) \frac{1 - \gamma}{2} \cos \frac{\gamma \pi}{2}}}{1} \right), \quad \ldots \ (2.27)
\]
then \( f(z) \in \mathcal{S}^*_p (\gamma) \) and the bound \( \alpha(\beta, \gamma) \) in (2.26) is sharp.

**Proof**: For \( a = 1, \ b = -1, \ 0 < \gamma < 1 \) and \( \beta > 0 \), (2.22) in Corollary 9 becomes

\[
(1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec h(z),
\]

where

\[
h(z) = p \left( \frac{1+z}{1-z} \right)^\gamma \frac{2 \beta \gamma z}{1-z^2}.
\]

Letting \( 0 < \theta < \pi \) and \( x = \cot \frac{\theta}{2} > 0 \), then

\[
\arg h(e^{i \theta}) = \arg \left\{ px^{\frac{i \gamma \pi}{2}} + \frac{\beta \gamma i}{2} \left( x + \frac{1}{x} \right) \right\} = \arctan g(x),
\]

where

\[
g(x) = \tan \frac{\gamma \pi}{2} + \frac{\beta \gamma}{2p \cos \frac{\gamma \pi}{2}} \left( x^{1 - \gamma} + \frac{1}{x^{1 + \gamma}} \right).
\]

Further, we have

\[
\inf_{|z| = 1 (z \neq \pm 1)} |\arg h(z)| = \frac{\pi}{2} \arctan \left( g \left( \sqrt{\frac{1-\gamma}{1+\gamma}} \right) \right)
\]

\[
= \frac{\alpha(\beta, \gamma) \pi}{2},
\]

where \( \alpha(\beta, \gamma) \) is given by (2.27).

Now, if \( f(z) \) satisfies (2.26), then it follows from (2.29) that the subordination (2.28) holds.

Therefore an application of Corollary 9 yields \( f(z) \in \mathcal{S}^*_p (\gamma) \).

Next we consider the function \( f(z) \) given by (2.13). Then \( f(z) \) satisfies

\[
(1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) = p \left( \frac{1+z}{1-z} \right)^\gamma \frac{2 \beta \gamma z}{1-z^2} = h(z)
\]
and it follows from (2.29) that the bound $\alpha(\beta, \gamma)$ in (2.26) is sharp.

Setting $m = 1$, $\lambda = \mu = 0$ and $p(z) = \frac{zf''(z)}{pf'(z)}$ in Theorem 3, we have

**Corollary 12** — Let $-1 \leq b < a \leq 1$ and $0 < \gamma \leq 1$. If $f(z) \in A_p$ satisfies $f(z)f'(z) \neq 0$ ($0 \leq |z| < 1$) and

$$
1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} < \frac{\gamma(a-b)z}{(1+az)(1+bz)^*},
$$

then $f(z) \in S_p^* (\gamma, a, b)$.

**REFERENCES**