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A CHARACTERIZATION OF TARSKI MONSTERS

RAMJI LAL AND R. P. SHUKLA

Department of Mathematics, University of Allahabad, Allahabad 211 002, *India*
E-mail: ramjilal@mri.ernet.in, rps@mri.ernet.in

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Tarski monsters have been characterized in terms of transversals. It is also proved that an infinite group all of whose proper right transversals are isomorphic as right quasigroups is isomorphic to the p-component of $\mathbf{Q} / \mathbf{Z}$ for some prime p.

Key Words: Transversal; Tarski Monster.

1. INTRODUCTION

Transversals in groups are important as they determine the embeddings of subgroups in groups. Whenever we make our study of extension of groups independent of transversals, we enter into obstructions and indeterminacies. This prompts us to study transversals individually and abstractly (see [2], [4], [9]). Transversals in topological groups have been discussed in [5].

Let H be a subgroup of a group G and S a right transversal of H in G. Then S is a right quasigroup with identity with respect to the operation o on S defined by $\{x \circ y\} = Hxy \cap S$. Conversely, every right quasigroup with identity can be embedded as a right transversal in a group with some universal property [2, Th. 3.4]. Let $\langle S \rangle$ denote the subgroup of G generated by S and let H_S denote the subgroup $\langle S \rangle \cap H$. Then $H_S = \langle \{xy\,(x \circ y)^{-1} : x, y \in S\} \rangle$ and $H_S S = \langle S \rangle$, where o is denned as above. Identifying S with the set $G/^r H$ of right cosets of H in G, we obtain a permutation representation $\phi_S : G \to Sym\,(S)$ ($\{\phi_S(g)(x)\} = S \cap Hxg, g \in G, x \in S$ and we adopt the convention that $(r.s)(x) = s(r(x))$, $r, s \in Sym\,(S)$ and $x \in S$). Let $G_S = \phi_S(H_S)$. Then G_S depends only on the right quasigroup structure o on S and not on the subgroup H (see [2]). This group is called the group torsion of the right quasigroup $(S, \circ)$ [2, Def. 3.1]. It is easy to observe that ϕ_S is injective on S and if we identify S with $\phi_S(S)$, then $\phi_S(\langle S \rangle) = G_S S$ which also depends only on the right quasigroup S and in which S is a right transversal of G_S. Further, one observes that $Ker\,(\phi_S \mid_{H_S S} : H_S S \to G_S S) = Ker\,(\phi_S \mid_{H_S} : H_S \to G_S)$ (for details see [2, Prop. 3.10]). Thus if $\langle S \rangle = G$ and if $Core_G(H)$ is trivial, then $G_S S \cong G$ and $G_S \cong H$. Also $(S, \circ)$ is a group if and only if G_S is trivial and H is normal if and only if all right transversals of H in G are groups. Thus if H is normal in G, then the group torsions of all right transversals of H in G are isomorphic. The converse of this statement is not true (for example if H is a subgroup of order 2 of a non-abelian finite simple group G, then the group torsion of every right transversal of H in G is isomorphic to H). This prompts to have the concept of stable subgroups [3, Def. 1.1]: A subgroup H of a group G is called a stable subgroup if for any pair of right transversals S_1 and S_2 of H in G, G_{S_1} is isomorphic to G_{S_2}. Description of finite groups all of whose proper subgroups are stable are as follows :

***Theorem* [3, Th. 2.4]** — *A finite solvable group all of whose subgoups are stable is a Dedekind group.*

***Theorem* [3, Cor. 2.6]** — *A non-Dedekind finite group all of whose proper subgroups are stable and all of whose proper quotients are Dedekind, is a simple group.*

***Theorem* [3, Th. 3.9]** — *A finite simple group has all its subgroups stable if and only if it is non-factorisable.*

Dedekind groups have the property that the group torsions of all right transversals of subgroups are isomorphic (being trivial). This prompts to have the following definition : A non-Dedekind group G will be called a *-group if given any pair S_1 and S_2 of right transversals of nontrivial proper subgroups H_1 and H_2 respectively, G_{S_1} is isomorphic to G_{S_2}.

A group G is usually called Tarski monster if either it is an infinite simple group all of whose proper subgroups are of same prime order p or it is an infinite simple group all of whose proper subgroups are infinite cyclic (for existence see [6] and [7]). In this paper we shall give a characterization of Tarski monsters (see Theorems 2.3 and 2.4).

Again, if a subgroup H of a group G is normal, then all right transversals of H in G are isomorphic (to G/H). This prompted to have the following definition [4, p.643]: A subgroup H of a group G is called perfectly stable if all right transversals of H in G are isomorphic. We do not have any example of a non-normal perfectly stable subgroup. Indeed in [4] it is proved that every perfectly stable subgroup of a finite group is normal.

For a fixed prime p, let P denote the p-primary component of $\mathbb{Q}/\mathbb{Z}$. Then P has the property that all proper right transversals are isomorphic. In this paper we also prove (Theorem 3.1) that these are the only infinite groups with this property:

2. *-GROUPS

Definition 2.1 — A non-Dedekind group G will be called a *-group if given any pair S_1 and S_2 of right transversals of nontrivial proper subgroups H_1 and H_2 respectively, G_{S_1} is isomorphic to G_{S_2}.

Proposition 2.2 — Let G be a *-group. Then either G is an infinite simple torsion group all of whose elements are of same prime order or it is an infinite torsion free simple group.

PROOF : If G has a proper normal subgroup, then the group torsions of all right transversals of all subgroups of G will be trivial (as G is supposed to be a *-group). But, then every subgroup of G will be normal [2, Cor. 3.3] and G will be Dedekind, a contradiction. Thus G is simple. Since the group torsion of every right transversal of a subgroup H of G is a subquotient of H [2, Prop. 3.10], G can not have subgroups of co-prime orders. Further, G cannot be finite, for otherwise it will be a p-group for some prime p and so it can not be non-Dedekind and simple. Suppose that G has an element of a prime order p. Then no torsion element of G can be of order co-prime to p. We show that every non-identity element of G is of same prime order p. Let b be a non-identity element of G and $H = \langle b \rangle$, the subgroup of G generated by b. There is an element $x \in G \setminus H$ such that $x^2 \notin H$, for otherwise H would be normal in G. Choose a right transversal S of H in G containing $\{e, x, bx^2\}$, where e is the identity in G. Then $\langle S \rangle = G$ and $H_S = \langle b \rangle = H$. Since G is simple, $G_S \cong H_S = \langle b \rangle$. Again, since G is a *-group, G_S is of order p and so b is of order p. $\square$

***Theorem* 2.3** — *A group G is a torsion Tarski monster if and only if it is a. torsion *-group.*

PROOF : Let G be a torsion Tarski monster. Then G is an infinite simple group generated by two elements all of whose proper subgroups are of same prime order p [6]. This implies that the group torsion of every right transversal of a subgroup is of order p.

Conversely, suppose that G is a torsion *-group. By Proposition 2.2, G is an infinite simple group all of whose non-identity elements are of order p for same prime p. Since G is non-abelian, $p \neq 2$. Suppose that $p = 3$. If G contains a proper subgroup which is not cyclic, then it contains a proper non-cyclic subgroup H generated by two elements a and b. Let $x \in G \setminus H$. Since x is of order 3, $x^2 \notin H$. Consider the subgroup K of G generated by H and x. Since a finitely generated group of exponent 3 is finite (see [8] for a proof), $K \neq G$. Let $y \in G \setminus K$. Then $y^2 \notin K$ (for y is of order 3). Hence H, Hx, Hx^2, Hy and Hy^2 are distinct right cosets of H in G. Let S be a right transversal of H in G containing $\{e, x, ax^2, y, by^2\}$, where e is the identity element of G. Then clearly, $G = \langle S \rangle = HS$. Since G is simple, $G_S \cong H$. This contradicts our assumption that G is a *-group. This also proves that G is generated by two elements. Since a finitely generated group of exponent 3 is finite, we get the contradiction that G is finite. This proves that $p \neq 3$. Hence $p > 3$.

Suppose that G has a proper subgroup which is not cyclic. As before, let H be a non-cyclic subgroup of G generated by two elements a and b. Let $x \in G \setminus H$. Since the order of x is p, a prime, $x^i \notin H$ for $0 < j < p$. Take a right transversal of H in G containing $\{e, x, ax^2, b^{-1}ax^3, x^4, \ldots, x^{p-1}\}$. Then $G = \langle S \rangle = HS$ and $G_S \cong H$. This is again a contradiction to our assumption that G is a *-group. This proves that every proper subgroup of G is cyclic of order p. □

For torsion free *-groups, we have the following characterization:

***Theorem* 2.4** — *A group G is a torsion free *-group if and only if it is torsion free Tarski monster.*

PROOF : Let G be a torsion free *-group. Then by Proposition 2.2, G is simple. Since the group torsion of a right transversal of a subgroup H in G is a subquotient of H, it follows that the group torsion of any proper right transversal is a nontrivial cyclic group. Suppose that there is a proper subgroup of G which is non-cyclic. Then there is a proper non-cyclic subgroup H of G generated by two elements a and b. We can find an element x in G such that $x^6 \notin H$. For, if $x^6 \in H$ for all $x \in G$, then the subgroup $\langle \{x^6 \mid x \in G\} \rangle$ generated by the sixth powers of elements of G would be normal in G and contained in H. Since H is proper and G is simple, this is a contradiction. Let $x \in G$ such that $x^6 \notin H$. Then x, x^2 and x^3 will lie in different right cosets of H in G. Let S be a right transversal of H in G containing $\{e, x, ax^2, b^{-1}ax^3\}$. Clearly, $G = \langle S \rangle = HS$. Since G is simple, $G_S \cong H$. This is a contradiction for H is not cyclic. Thus all proper subgroups of G are cyclic.

Conversely, suppose that G is a torsion free Tarski monster. Let H be a nontrivial proper subgroup of G and S be a right transversal of H in G. If the subgroup $\langle S \rangle$ of G generated by S is proper, then it is abelian and $G = HS = H\langle S \rangle$ is a product of its abelian subgroups H and $\langle S \rangle$. By a Theorem of Ito [1], the commutator subgroup of G is abelian. But this is a contradiction. Thus $\langle S \rangle = G$ for all right transversals of H in G. Hence $H_S = H \cap \langle S \rangle = H$. Since G is simple, $G_S \cong H$. This proves that G is a torsion free *-group. □

3. Groups all of whose Proper Transversals are Isomorphic

From [4], it follows that if G is a finite group all of whose proper right transversals are isomorphic, then G is a finite Dedekind group all of whose proper quotients are isomorphic and so G will be a group of order p or p^2 for some prime p. As mentioned in the introduction, for any prime p, the p-component of the group $\mathbb{Q}/\mathbb{Z}$ has also the property that all proper right transversals are isomorphic. The following theorem says that these are the only infinite groups with this property.

***Theorem* 3.1** — *Let G be an infinite group all of whose proper right transversals are isomorphic. Then G is isomorphic to the p-component of $\mathbb{Q}/\mathbb{Z}$ for some prime p.*

Proof : First, we shall prove that G is abelian. From the structure theorem of Dedekind groups [8, Th. 5.3.7, p. 143], it follows that a non-abelian Dedekind group can not have all proper right transversals isomorphic. Assume that G is non-abelian. Then G is non-Dedekind. A non-Dedekind group all of whose proper right transversals are isomorphic is clearly *a* *-group. It follows from Theorems 2.3 and 2.4 that G is a Tarski monster. Since all proper right transversals of G are isomorphic, all subgroups of G are perfectly stable. Since a finite perfectly stable subgroup of a finitely generated group is normal [4, Prop. 3.6], G can not be a torsion Tarski monster. Hence G is a torsion free Tarski monster and all of whose proper right transversals are isomorphic. Since G is generated by two elements, $Aut(G)$ is by atmost countable. Next, let H be a nontrivial proper subgroup of G. Let S be a right transversal of H in G. Then $\langle S \rangle = G$ and $G_S \cong H$. (see the proof of Theorem 2.4). For any right transversal T, fixing an isomorphism θ between S and T induces an isomorphism ψ_T from $G_S S$ to $G_T T$ [2] and so gives rise an automorphism $\alpha_S = \phi_T^{-1} \circ \psi_T \circ \phi_S$ of G whose restriction to S is θ, where ϕ_S and ϕ_T denote permutation representations of G on S and T respectively as described in the introduction. Since there are uncountably many right transversals of H in G, $Aut(G)$ would be uncountable. This is a contradiction. Hence G is abelian.

Next, since G is abelian and all proper transversals (and so all proper quotients) of G are isomorphic, it is easy to observe that G is a p-group for some prime p. Next, we prove that G is divisible. If $n \in \mathbb{Z}$ is co-prime to p, then the nth power map on G is an isomorphism. Further, if the subgroup $G^p = \{x^p \mid x \in G\}$ of G is proper, then the quotient group G/G^p is an elementary abelian p-group. Now, the property of G that all proper transversal are isomorphic forces that all proper quotients of G are isomorphic to the cyclic group $\mathbb{Z}_p$ of order p. But, this is a clearly a contradiction. Hence $G^p = G$ and so G is a divisible group. Now the structure theorem for torsion abelian divisible group [8, Th. 4.1.5] together with the property of G that all proper quotients of G are isomorphic implies that G is isomorphic to the p-component of $\mathbb{Q}/\mathbb{Z}$. □

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