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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF PERTURBED NONLINEAR DIFFERENCE EQUATIONS

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In this paper, by employing the generalized Riccati technique, we establish some new oscillation criteria for the second-order perturbed nonlinear difference equation

$$\Delta(a_{n-1}\,(\Delta x_{n-1})^{\beta}) + F(n, x_n) = G(n, x_n, \Delta x_n), \quad n \geq n_0,$$

where $\beta > 0$ is quotient of odd positive integers. Our results improve some well-known results in the literature. Some comparison between our theorems and those previously known results are indicated. Some examples are included to illustrate the relevance of the main results.

Key Words: Oscillation; second-order difference equations; Riccati techniques

1. INTRODUCTION

The theory of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 2, 5, 6, 10]. In this paper, we are concerned with the oscillation of the solutions of second-order perturbed nonlinear difference equation

$$\Delta(a_{n-1}\,(\Delta x_{n-1})^{\beta}) + F(n, x_n) = G(n, x_n, \Delta x_n), \quad n \geq n_0, \tag{1.1}$$

where β is quotient of odd positive integers, Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, $\{a_n\}_{=n_0}^{\infty}$ is a real positive sequence. We consider the two cases:

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{\frac{1}{\beta}} = \infty, \tag{1.2}$$

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{\frac{1}{\beta}} < \infty. \tag{1.3}$$

By a solution of (1.1), we mean a nontrivial sequence $\{x_n\}$ satisfying equation (1.1) for $n \geq n_0$. A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 \geq n_0$ there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The problem of determining oscillation criteria for second order difference equations has received a fair amount of attention, see for example the papers [8, 9, 12, 13, 14, 15, 16, 20] and the references cited therein. For equations with perturbation terms such as eq.(1.1) relatively few oscillation criteria are known, for example, we refer to the results of Saker [11], Szmanda [17], Thandapani [18] and Wong and Agarwal [19].

In [19], the authors considered eq.(1.1) when $a_{n-1} = 1$, $\beta = 1$ and assumed that: There exist real sequences $\{q_n\}$, $\{p_n\}$ and a function $f : \mathbf{R} \to \mathbf{R}$ such that

$$uf(u) > 0 \quad \text{for} \quad u \neq 0, \tag{1.4}$$

$$f(u) - f(v) = g(u,v)(u-v), \text{for } u, v \neq 0 \text{ where } g \text{ is nonnegative function}, \tag{1.5}$$

$$\frac{F(n,u)}{f(u)} \geq q_n, \qquad \frac{G(n,u,v)}{f(u)} \leq p_n \text{ for } u \neq 0, \tag{1.6}$$

and used the summation averaging technique similar to that of the technique used in differential equations and offered new necessary conditions for existence of nonoscillatory solutions.

In [18], the author considered eq.(1.1), $\beta = 1$ and assumed that (1.4)-(1.6) hold, and established several sufficient conditions for oscillation. Most of the oscillation criteria in [18] required the conditions (1.2), and

$$\sum_{n=n_0}^{\infty} (q_n - p_n) = \infty. \tag{1.7}$$

But one can easily see that the results in [17, 18, 19] can not applied for the equation

$$\Delta^2 x_{n-1} + \frac{\mu}{n^2} x_n(1+x_n^2) = \frac{\alpha}{n^2}\frac{(\Delta x_n)^2 x_n}{1+(\Delta x_n)^2}, \quad n = 1, 2, ..., \tag{1.8}$$

since the condition (1.7) is not satisfied.

In [11], the author considered the perturbed equation

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) + F(n, x_n) = G(n, x_n, \Delta x_n), \quad n \geq 1, \tag{1.9}$$

where $\gamma > 1$ is quotient of odd positive integers and there exist two positive sequences $\{q_n\}$, and $\{p_n\}$ such that

$$\frac{F(n,u)}{u^{\beta}} \geq q_n, \quad \frac{G(n,u,v)}{u^{\beta}} \leq p_n \text{ for } u \neq 0, \tag{1.10}$$

where β is quotient of odd positive integers and by using the Riccati substitution some sufficient conditions for oscillation are established, which improve the results investigated in [17, 18, 19].

In this paper, by employing Riccati and generalized Riccati technique we establish some new oscillation criteria for eq.(1.1) bypassing the condition (1.5) and improving the condition (1.7). When (1.2) holds, we will establish some sufficient conditions for oscillation of eq.(1.1), and when (1.3) holds the sufficient conditions guarantee that every solution oscillates or converges to zero. Our results improve the results in [11, 17, 18, 19] when (1.2) holds and when (1.3) holds our results are essentially new. Examples are included to illustrate the relevance of our results.

2. Main Results

In the following, we will use the generalized Riccati technique to establish some sufficient conditions for oscillation of eq. (1.1). Throughout this Section, we will assume that there exist two real sequences $\{q_n\}_{n=n_0}^{\infty}$ and $\{p_n\}_{n=n_0}^{\infty}$ such that

$$q_n - p_n \geq 0 \text{ and not identically zero for large } n, \tag{2.1}$$

$$\begin{cases} F : \mathbf{N}\times\mathbf{R} \to \mathbf{R} = (-\infty,\infty), \mathbf{G} : \mathbb{N}\times\mathbf{R^2} \to \mathbf{R}, \\ uF(n,u) > 0 \text{ and } uG(n,u,v) > 0 \text{ for } u \neq 0, \\ \dfrac{F(n,u)}{u^{\beta}} \geq q_n, \quad \dfrac{G(n,u,v)}{u^{\beta}} \leq p_n \text{ for } u \neq 0, \end{cases} \tag{2.2}$$

where $\beta \geq 1$ is quotient of odd positive integers.

First, we consider the case when (1.2) holds.

***Theorem* 2.1** — *Assume that (1.2), (2.1) and (2.2) hold. Furthermore, assume that there exists a positive constant M and a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that* $\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^{\beta} \leq M$ *and*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\psi_s = \infty, \tag{2.3}$$

for some $n_0 \geq 0$, *where*

$$\psi_n = \rho_n\left[(q_n - p_n) - \Delta\left(a_{n-1}\alpha_{n-1}\right) + a_n\left(\alpha_n\right)^{1+\frac{1}{\beta}}\right], and \quad (\alpha_n)^{\frac{1}{\beta}} = -\frac{\Delta\rho_n}{(\beta+1)\rho_n}.$$

Then every solution of eq. (1.1) oscillates.

PROOF: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1), say $x_n > 0$ for all $n \geq n_0$. We shall consider only this case, because the proof when $x_n < 0$ is similar. From eq. (1.1) and (2.2), we have

$$\Delta(a_{n-1}\left(\Delta x_{n-1}\right)^{\beta}) \leq -[q_n - p_n]x_n^{\beta} \leq 0, \quad n \geq n_0. \tag{2.4}$$

From which we have that $\{a_{n-1}\left(\Delta x_{n-1}\right)^{\beta}\}$ is an eventually nonincreasing sequence. We first show that $a_n\Delta x_n \geq 0$ for $n \geq n_0$. In fact, if there exists an integer $n_1 \geq n_0$ such that $a_{n_1}\left(\Delta x_{n_1}\right)^{\beta} = c < 0$, then $a_n\left(\Delta x_n\right)^{\beta} \leq a_{n_1}\left(\Delta x_{n_1}\right)^{\beta} = c$ for $n \geq n_1$ that is

$$\Delta x_n \leq c^{\frac{1}{\beta}}\left(\frac{1}{a_n}\right)^{\frac{1}{\beta}}.$$

Upon summing the above inequality from n_1 to $n-1$, we get

$$x_n \leq x_{n_1} + c^{\frac{1}{\beta}}\sum_{s=n_1}^{n-1}\left(\frac{1}{a_s}\right)^{\frac{1}{\beta}} \to -\infty \text{ as } n \to \infty,$$

which contradicts the fact that $x_n > 0$ for $n \geq n_0$. Therefore, we have

$$x_n > 0, \quad \Delta x_n \geq 0 \quad \text{and,} \quad \Delta(a_{n-1}\left(\Delta x_{n-1}\right)^{\beta}) \leq 0 \text{ for } n \geq n_0. \tag{2.5}$$

Define the sequence $\{w_n\}$ by the generalized Riccati substitution

$$w_n = \rho_n\left[\frac{a_{n-1}\left(\Delta x_{n-1}\right)^{\beta}}{x_n^{\beta}} + a_{n-1}\alpha_{n-1}\right]. \tag{2.6}$$

Then

$$\Delta w_n = a_n\left(\Delta x_n\right)^{\beta}\Delta\left[\frac{\rho_n}{x_n^{\beta}}\right] + \frac{\rho_n\Delta(a_{n-1}\left(\Delta x_{n-1}\right)^{\beta})}{x_n^{\beta}} + \rho_n\Delta\left(a_{n-1}\alpha_{n-1}\right) + a_n\alpha_n\Delta\rho_n.$$

Thus, (2.4) and (2.6), imply that

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \rho_n \Delta\left(a_{n-1}\alpha_{n-1}\right) + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n a_n \left(\Delta x_n\right)^{\beta} \Delta(x_n^{\beta})}{x_n^{\beta} x_{n+1}^{\beta}}. \tag{2.7}$$

Now, by using the inequality (cf. [7, p. 39])

$$x^{\beta} - y^{\beta} \geqslant \beta y^{\beta-1}(x-y) \text{ for all } x \neq y > 0 \text{ and } \beta \geqslant 1,$$

we have

$$\Delta(x_n^{\beta}) = x_{n+1}^{\beta} - x_n^{\beta} \geqslant \beta x^{\beta-1}(x_{n+1} - x_n) = \beta x_n^{\beta-1}\left(\Delta x_n\right), \quad \beta \geqslant 1. \tag{2.8}$$

But, in view of (2.5) we have $x_{n+1} \geq x_n$, and hence from (2.7) and (2.8), we obtain

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \rho_n \Delta\left(a_{n-1}\alpha_{n-1}\right) + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta \rho_n a_n \left(\Delta x_n\right)^{\beta+1}}{x_{n+1}^{\beta+1}}. \tag{2.9}$$

From (2.6), we have

$$w_{n+1} = \rho_{n+1}\left[\frac{a_n \left(\Delta x_n\right)^{\beta}}{x_{n+1}^{\beta}} + a_n \alpha_n\right],$$

and this implies that

$$\left(\frac{\Delta x_n}{x_{n+1}}\right)^{\beta+1} = \left[\frac{w_{n+1}}{a_n \rho_{n+1}} - \alpha_n\right]^{1+\frac{1}{\beta}}.$$

Now, by using the inequality, [cf. [9]).

$$(v-u)^{1+\frac{1}{\beta}} \geq v^{1+\frac{1}{\beta}} + \frac{1}{\beta} u^{1+\frac{1}{\beta}} - \left(1+\frac{1}{\beta}\right) u^{\frac{1}{\beta}} v, \quad \beta = odd/odd \geq 1,$$

we have,

$$\left(\frac{\Delta x_n}{x_{n+1}}\right)^{1+\frac{1}{\beta}} = \left[\frac{w_{n+1}}{a_n \rho_{n+1}} - \alpha_n\right]^{1+\frac{1}{\beta}}$$

$$\geq \left(\frac{w_{n+1}}{a_n \rho_{n+1}}\right)^{1+\frac{1}{\beta}} + \frac{1}{\beta}\left(\alpha_n\right)^{1+\frac{1}{\beta}} - \frac{\left(1+\frac{1}{\beta}\right)\left(\alpha_n\right)^{\frac{1}{\beta}}}{a_n \rho_{n+1}} w_{n+1}. \tag{2.10}$$

Substitute (2.10) into (2.9) gives

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \rho_n \Delta\left(a_{n-1}\alpha_{n-1}\right) - a_n\rho_n\left(\alpha_n\right)^{1+\frac{1}{\beta}}$$
$$+\left[\frac{\Delta\rho_n}{\rho_{n+1}} + \beta\rho_n \frac{\left(1+\frac{1}{\beta}\right)\left(\alpha_n\right)^{\frac{1}{\beta}}}{\rho_{n+1}}\right] w_{n+1} - \frac{\beta\rho_n}{\left(a_n\right)^{\frac{1}{\beta}}\left(\rho_{n+1}\right)^{1+\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}}. \quad (2.11)$$

Using the fact that the maximum of the function $f(u) = Bw - Aw^{1+\frac{1}{\beta}}$ for $A > 0$ is given by

$$\frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{B^{\beta+1}}{A^\beta}, \quad (2.12)$$

we get from (2.11) that

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \rho_n \Delta\left(a_{n-1}\alpha_{n-1}\right) - a_n\rho_n\left(\alpha_n\right)^{1+\frac{1}{\beta}}$$
$$+\frac{a_n(\rho_{n+1})^{\beta+1}}{(\beta+1)^{\beta+1}(\rho_n)^\beta}\left[\frac{\Delta\rho_n}{\rho_{n+1}} + \beta\rho_n \frac{\left(1+\frac{1}{\beta}\right)\left(\alpha_n\right)^{\frac{1}{\beta}}}{\rho_{n+1}}\right]^{\beta+1}. \quad (2.13)$$

Now, since $(\alpha_n)^{\dfrac{1}{\beta}} = -\dfrac{\Delta\rho_n}{(\beta+1)\rho_n}$, we have

$$\frac{\Delta\rho_n}{\rho_{n+1}} + \beta\rho_n \frac{\left(1+\frac{1}{\beta}\right)\left(\alpha_n\right)^{\frac{1}{\beta}}}{\rho_{n+1}} = 0,$$

and this implies from (2.13) that

$$\Delta w_n \leq -\psi_n.$$

Summing the last inequality from n_1 to n, we obtain

$$\sum_{s=n_1}^{n} [\psi_s] < w_{n_1} - w_{n+1}. \quad (2.14)$$

Now by (2.6), we have

$$w_{n+1} \geq -\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{(\beta+1)\rho_n}\right)^\beta.$$

Using the fact that $\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^\beta \leq M$, we have

$$w_{n+1} > -\frac{1}{(\beta+1)^\beta}\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^\beta \geq -\frac{1}{(\beta+1)^\beta}M, \quad (2.15)$$

and therefore, it follows that the right side of (2.14) is bounded above. This is contrary to (2.3). The proof is complete.

Remark 2.1 : From Theorem 2.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. For examples one can let $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant, or $\rho_n = R(n, n_0) = \sum_{s=n_0}^{n-1} \left(\frac{1}{a_s}\right)^{\frac{1}{\beta}}$. The details are left to the reader.

Remark 2.2 : Note that, when $\beta = 1$ and $p_n = 0$, eq. (1.1) reduces to the linear difference equation

$$\Delta(a_{n-1}\Delta x_{n-1}) + q_n x_n = 0, \quad n = 1, 2..., \tag{2.16}$$

and the condition (2.3) reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \rho_l \left[q_l - \Delta\left(a_{l-1}\alpha_{l-1}\right) + a_l \alpha_l^2\right] = \infty, \tag{2.17}$$

for some $\mathrm{n}_0 \geq 0$, where

$$\alpha_n = -\frac{\Delta \rho_n}{2\rho_n}. \tag{2.18}$$

The following examples are illustrative.

Example 2.1 — As special case of eq. (1.1), we consider the discrete Euler equation

$$\Delta^2 x_{n-1} + \frac{\mu}{n^2} x_n = 0, \quad n \geq 1, \tag{2.19}$$

where $\mu > \frac{1}{4}$. Here $\beta = 1$, $a_n = 1$, $p_n = 0$ and $q_n = \frac{\mu}{(n+1)^2}$. If we take $\rho_n = n + 1$, we have

$$\alpha_n = -\frac{\Delta \rho_n}{2\rho_n} = -\frac{1}{2(n+1)},$$

and

$$\begin{aligned} \sum_{s=n_0}^{n} (s+1)\left[q_s - \Delta\alpha_{s-1} + \alpha_s^2\right] &= \sum_{s=1}^{n} (s+1)\left[\frac{\mu}{s^2} + \frac{1}{2}\Delta\left(\frac{1}{s}\right) + \frac{1}{4(s+1)^2}\right] \\ \geq \quad & \sum_{s=1}^{n} (s+1)\left[\frac{\mu}{(s+1)^2} - \frac{1}{2s(s+1)} + \frac{1}{4(s+1)^2}\right] \\ = \quad & \frac{1}{4}\sum_{s=1}^{n}\left[\frac{4\mu - 1}{(s+1)} - \frac{2}{s(s+1)}\right] \to \infty, \end{aligned}$$

as $n \to \infty$ provided that $\mu > \frac{1}{4}$. Then by (2.17) and Theorem 2.1, every solution of (2.19) oscillates if $\mu > \frac{1}{4}$, and this is compatible with the oscillation results of Euler equation (see [19]). Hence, Theorem 2.1 is able to conclude the oscillation of (2.19). Note that the results in [17, 18] can not be applied to eq. (2.19).

Example 2.2 — Consider the second-order perturbed difference equation

$$\Delta\left(\frac{1}{n-1}\Delta x_{n-1}\right)+\frac{\gamma}{n^3}x_n=0,\ \text{for}\quad n\geq 2. \tag{2.20}$$

Here $\beta=1$, $a_{n-1}=\dfrac{1}{n-1}$ and $q_n-p_n=\dfrac{\gamma}{n^3}$. Note that a_n satisfies the condition (1.2). To apply Theorem 2.1, it remains to prove that (2.3) holds. By choosing $\rho_n=n^2$, then $\rho_{n+1}a_n\left(\dfrac{\Delta\rho_n}{\rho_n}\right)^\beta\leq 5$, $\alpha_n=-\dfrac{2n+1}{2n^2}$ and

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}[\psi_s]=\lim_{s\to\infty}\sup\sum_{s=2}^{n}\rho_s\left[q_s-p_s-\Delta\left(a_{s-1}\alpha_{s-1}\right)+a_s\alpha_s^2\right]$$

$$=\lim_{n\to\infty}\sup\sum_{s=2}^{n}s^2\left[\frac{\gamma}{s^3}+\frac{1}{s}\left(\frac{s+s+1}{2s^2}\right)^2+\Delta\left(\frac{s-1+s}{2(s-1)^3}\right)\right]$$

$$=\lim_{n\to\infty}\sup\sum_{s=2}^{n}s^2\left[\frac{\gamma}{s^3}+\frac{1}{s}\left(\frac{2s+1}{2s^2}\right)^2+\Delta\left(\frac{2s-1}{2(s-1)^3}\right)\right]$$

$$=\lim_{n\to\infty}\sup\sum_{s=2}^{n}s^2\left[\frac{\gamma}{s^3}+\frac{1}{s}\left(\frac{2s+1}{2s^2}\right)^2+\frac{2s+1}{2s^3}-\frac{2s-1}{2(s-1)^3}\right]$$

$$=\lim_{n\to\infty}\sup\sum_{s=1}^{n}s^2\left[\frac{\gamma}{s^3}+\frac{1}{4s^5}\left(2s+1\right)^2+\frac{1}{2s^3}+\frac{1}{s^2}-\frac{1}{2\left(s-1\right)^3}-\frac{1}{\left(s-1\right)^2}\right]$$

$$\geq\lim_{n\to\infty}\sup\sum_{s=2}^{n}s^2\left[\frac{\gamma}{s^3}+\frac{1}{s^3}+\frac{1}{2s^3}+\frac{1}{s^2}-\frac{1}{2\left(s-1\right)^3}-\frac{1}{\left(s-1\right)^2}\right]$$

$$=\lim_{n\to\infty}\sup\sum_{s=2}^{n}\left[\frac{\gamma}{s}+\frac{1}{s}+\frac{1}{2s}+1-\frac{s^2}{2\left(s-1\right)^3}-\frac{s^2}{\left(s-1\right)^2}\right]$$

$$= \sum_{s=2}^{\infty}\left[\frac{1}{2}\frac{2\gamma s^3-6\gamma s^2+6\gamma s-2\gamma-2s^3-3s^2+7s-3}{s\left(s-1\right)^3}\right]$$

$$= \operatorname{signum}(\gamma\infty)-\frac{5}{2}-\frac{1}{2}\zeta\left(3\right)-\frac{1}{3}\pi^2+\frac{1}{2}\left(2\gamma-2\right)\left(\text{gamma}-1\right)$$

$$= \infty,$$

provided that $\gamma > 1$, where

$$\text{signum} = \begin{cases} \dfrac{z}{|z|} & \text{if} z \neq 0, \\ 0 & \text{if} z = 0, \end{cases} ,$$

ζ is Zeta function and

$$\text{gamma} = \lim_{n\to\infty}\sum_{m=1}^{n}\left[\frac{1}{m}-\ln n\right] = \infty.$$

Then, by Theorem 2.1 every solution of difference equation (2.20) oscillates if $\gamma > 1$.

Also, one can see that, all the results established in [11, 17, 18, 19] cannot be applied to eq.(2.20), since the condition (1.7) is not satisfied.

In the following, we derive new oscillation criteria, which can be considered the generalized of the Kamenev-type oscillation criteria for second order differential equation.

***Theorem* 2.2** — *Assume that (1.2), (2.1) and (2.2) hold. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence such that $\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^{\beta} \leq M$. Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that (i) $H_{m,m} = 0$ for $m \geq 0$, (ii) $H_{m,n} > 0$ for $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$. If*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\psi_n - \frac{a_n\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(\beta+1)^{\beta+1}\rho_n\left(H_{m,n}\right)^{\beta}}\right] = \infty, \tag{2.21}$$

where

$$\zeta_n = \Delta\rho_n + \beta\rho_n\left(1+\frac{1}{\beta}\right)\left(\alpha_n\right)^{\frac{1}{\beta}} \text{ and } h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ m > n \geq 0.$$

Then every solution of (1.1) oscillates.

PROOF: Proceeding as in the proof of Theorem 2.1, we assume that eq. (1.1) has a nonoscillatory solution, say $x_n > 0$ for all $n \geq n_0$. From the proof of Theorem 2.1 by (1.2) we saw that $\Delta x_n > 0$ for all $n \geq n_0$. By defining again w_n by (2.6) and proceeding as in Theorem 2.1, we get (2.12). From (2.12), we have for $n \geq n_1$

$$\psi_n \leq -\Delta w_n + \frac{\zeta_n}{\rho_{n+1}}w_{n+1} - \frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^{1+\frac{1}{\beta}}}w_{n+1}^{1+\frac{1}{\beta}},\ \text{where } \bar{\rho}_n = \frac{\beta\rho_n}{\left(a_n\right)^{\frac{1}{\beta}}}. \tag{2.22}$$

Therefore, we have

$$\sum_{n=n_1}^{m-1} H_{m,n}\psi_n \leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\zeta_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_1}^{m-1} \frac{H_{m,n}\bar{\rho}_n}{(\rho_{n+1})^{1+\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}} \tag{2.23}$$

which yields, after summing by parts

$$\begin{aligned}
&\sum_{n=n_1}^{m-1} H_{m,n}\psi_n \leq H_{m,n_1} w_{n_1} \\
&+ \sum_{n=n_1}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\zeta_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_1}^{m-1} \frac{H_{m,n}\bar{\rho}_n}{(\rho_{n+1})^{1+\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}} \\
= &\ H_{m,n_1} w_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}} w_{n+1} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\zeta_n}{\rho_{n+1}} w_{n+1} \\
&- \sum_{n=n_1}^{m-1} \frac{H_{m,n}\bar{\rho}_n}{(\rho_{n+1})^{1+\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}} \\
= &\ H_{m,n_1} w_{n_1} + \sum_{n=n_1}^{m-1} \left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}} \right] w_{n+1} - \sum_{n=n_1}^{m-1} \frac{H_{m,n}\bar{\rho}_n}{(\rho_{n+1})^{1+\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}}.
\end{aligned}$$

Then, as in the proof of Theorem 2.1, we have

$$\sum_{n=n_1}^{m-1} H_{m,n}\psi_n \leq H_{m,n_1} w_{n_1} + \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \sum_{n=n_1}^{m-1} \frac{(\rho_{n+1})^{\beta+1}\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(H_{m,n}\bar{\rho}_n)^\beta}.$$

Then

$$\sum_{n=n_1}^{m-1} \left[H_{m,n}\psi_n - \frac{a_n\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(\beta+1)^{\beta+1}\rho_n\,(H_{m,n})^\beta} \right] < H_{m,n_1} w_{n_1} \leq H_{m,0} w_{n_1},$$

which implies that

$$\sum_{n=0}^{m-1} \left[H_{m,n}\psi_n - \frac{a_n\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(\beta+1)^{\beta+1}\rho_n\,(H_{m,n})^\beta} \right] < H_{m,0}\left(w_{n_1} + \sum_{n=0}^{n_1-1}\psi_n \right).$$

Hence

$$\begin{aligned}
&\lim_{m\to\infty}\sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left[H_{m,n}\psi_n - \frac{a_n\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}} - h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(\beta+1)^{\beta+1}\rho_n\,(H_{m,n})^\beta} \right] \\
< &\ \left(w_{n_1} + \sum_{n=0}^{n_1-1}\psi_n \right) < \infty,
\end{aligned}$$

and this contradicts (2.21). The proof is complete.

As an immediate consequence of Theorem 2.2, we get the following:

Corollary 2.1 — Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.21) is replaced by

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{n=\mathrm{n}_0}^{m-1}H_{m,n}\psi_n=\infty, \tag{2.24}$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\frac{a_n\left[H_{m,n}\frac{\zeta_n}{\rho_{n+1}}-h_{m,n}\sqrt{H_{m,n}}\right]^{\beta+1}}{(\beta+1)^{\beta+1}\rho_n\left(H_{m,n}\right)^{\beta}}<\infty.$$

Then every solution of eq. (1.1) oscillates.

Remark 2.3 : By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, one can derive several oscillation criteria for (1.1). For instance, one can use the double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n}=(m-n)^{\lambda},\ \text{ or } H_{m,n}=\left(\log\frac{m+1}{n+1}\right)^{\lambda}\ \lambda\geq 1, m\geq n\geq 0, \tag{2.25}$$

or

$$H_{m,n}=(m-n)^{(\lambda)}\quad \lambda>2, m\geq n\geq 0. \tag{2.26}$$

where $(m-n)^{(\lambda)}=(m-n)(m-n+1)...(m-n+\lambda-1)$ and

$$\Delta_2(m-n)^{(\lambda)}=(m-n-1)^{(\lambda)}-(m-n)^{(\lambda)}=-\lambda(m-n)^{(\lambda-1)}. \tag{2.27}$$

We observe that $H_{m,m}=0$ for $m\geq 0$ and $H_{m,n}>0$ and $\Delta_2 H_{m,n}\leq 0$ for $m>n\geq 0$. The details are left to the interesting reader.

Next, we will consider the case when (1.3) holds.

***Theorem* 2.3** — *Assume that (1.3), (2.1) and (2.2) hold. Furthermore, we assume that there exist positive sequences* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that* $\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^{\beta}\leq M$ and *(2.3) holds, and*

$$\sum_{n=n_0}^{\infty}[q_n-p_n]=\infty\ \ and \sum_{n=n_0}^{\infty}\left(\frac{1}{a_{n-1}}\sum_{i=n_0}^{n-1}[q_i-p_i]\right)^{\frac{1}{\beta}}=\infty, \tag{2.28}$$

for some $n_0>0$. *Then every solution of eq. (1.1) oscillates or* $\lim_{n\to\infty}x_n=0$.

PROOF: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n>0$ for all $n\geq n_0$. We shall consider only this case, since the proof when $x_n<0$ is similar. By (2.1) and Theorem 2 in [19] we see that $\{\Delta x_n\}$ does not oscillate and there exist two possible cases. If $\{\Delta x_n\}$ is eventually positive, we are then back to the case where (2.5) holds. Thus the proof of Theorem 2.1 goes through, and we may conclude that $\{x_n\}$ cannot be eventually positive, which is not possible. Suppose that $\Delta x_n<0$ for $n\geq n_1\geq n_0$. It follows that $\lim_{n\to\infty}x_n=b\geq 0$. Now, we claim that $b = 0$. If not, we assume that $\lim_{n\to\infty} x_n = b > 0$ which implies that $x_n^\beta \to b^\beta > 0$ as $n \to \infty$, and hence there exists $n_2 \geq n_1$ such that $x_n^\beta \geq b^\beta$. Therefore from (1.1) and (2.2), we have

$$\Delta(a_{n-1}\left(\Delta x_{n-1}\right)^\beta) \leq -[q_n - p_n]b^\beta.$$

Define the sequence $u_n = a_{n-1}\left(\Delta x_{n-1}\right)^\beta$ for $n \geq n_2$. Then we have

$$\Delta u_n \leq -b^\beta[q_n - p_n].$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \leq u_{n_2} - b^\beta \sum_{s=n_2}^{n-1}[q_s - p_s].$$

In view of (2.28), since $\sum\limits_{n=n_0}^{\infty}[q_n - p_n] = \infty$ it is possible to choose integer n_3 sufficiently large such that for all $n \geq n_3$

$$u_n \leq -\frac{b^\beta}{2}\sum_{n=n_2}^{n-1}[q_s - p_s].$$

Summing the last inequality from n_3 to n we obtain

$$x_{n+1} \leq x_{n_3} - \left(\frac{b^\beta}{2}\right)\sum_{s=n_3}^{n}\left(\frac{1}{a_{s-1}}\sum_{i=n_2}^{s-1}[q_i - p_i]\right)^{\frac{1}{\beta}}.$$

Condition (2.28) implies that $\{x_n\}$ is eventually negative, which is a contradiction. Thus the proof is complete.

Remark 2.4 : From Theorem 2.3, one can obtain different conditions for oscillation of all solutions of eq.(1.1) when (1.3) holds by different choices of $\{\rho_n\}$. The details are left to the reader.

The following example is illustrative.

Example 2.3 — Consider the linear difference equation

$$\Delta((n-1)^2\Delta x_{n-1}) + \mu x_n = 0, \quad n \geq 1, \tag{2.29}$$

where $\mu > \frac{1}{4}$. Here $a_{n-1} = (n-1)^2$, $\beta = 1$, $p_n = 0$ and $q_n = \mu$. If we take $\rho_n = n$, then one can easily see that (2.3) and (2.28) hold. Thus, Theorem 2.3 asserts that every solution of (2.29) oscillates or $x_n \to \infty$ as $n \to \infty$. Note that none of the above mentioned papers can be applied to eq.(2.29).

***Theorem* 2.4** — *Assume that (1.3), (2.1), (2.2) and (2.28) hold. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence such that $\rho_{n+1}a_n\left(\frac{\Delta\rho_n}{\rho_n}\right)^\beta \leq M$. Furthermore, we assume that there exists a double* *sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *as defined in Theorem 2.2 and (2.21) holds. Then every solution of eq.(1.1) oscillates or* $\lim_{n\to\infty} x_n = 0$.

PROOF : Suppose that $\{x_n\}$ is an eventually positive solution of (1.1). Then as seen in the proof of Theorem 2.3, either $\{\Delta x_n\}$ is eventually positive or is eventually negative. In the case, when $\{\Delta x_n\}$ is eventually positive we may follow the proof of Theorem 2.2 and obtain a contradiction. If $\{\Delta x_n\}$ is eventually negative, then we may follow the proof of Theorem 2.3 to show that $\{x_n\}$ converges to zero.

By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ be defined as in Remark (2.5). Hence we have the following conditions which are sufficient for oscillation or imply that $\lim_{n\to\infty} x_n = 0$ when (1.3) holds.

Corollary 2.2 — Assume that all the assumptions of Theorem 2.4 hold, except that the condition (2.21) is replaced by (2.25) or (2.26) or (2.27). Then every solution of eq.(1.1) oscillates or $\lim_{n\to\infty} x_n = 0$.

In the following, we will use the Riccati substitution to establish some oscillation criteria for the equation

$$\Delta(a_{n-1}(\Delta x_{n-1})) + F(n, x_n) = G(n, x_n, \Delta x_n), \quad n \geq n_0, \tag{2.30}$$

when $0 < \beta < 1$.

First, we consider the case when $0 < \beta < 1$, $\sum\limits_{i=n_0}^{\infty} \left(\frac{1}{a_i}\right) = \infty$ and $\Delta a_n \geq 0$.

***Theorem* 2.5** — *Let* $0 < \beta < 1$. *Assume that (2.1) and (2.2) hold,* $\sum\limits_{i=n_0}^{\infty} \left(\frac{1}{a_i}\right) = \infty$ *and* $\Delta a_n \geq 0$. *Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that for every* $b \geq 1$

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l[q_l - p_l] - \frac{(a_l)\, b^{1-\beta}(l+1)^{1-\beta}(\Delta\rho_n)^2}{4\beta\rho_l}\right] = \infty. \tag{2.31}$$

Then every solution of eq.(2.31) oscillates.

PROOF: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (2.30), say $x_n > 0$ for all $n \geq n_0$. We shall consider only this case, because the proof when $x_n < 0$ is similar. From eq.(2.30) and (2.2), we have

$$\Delta(a_{n-1}(\Delta x_{n-1})) \leq -[q_n - p_n]x_n^{\beta} \leq 0, \quad n \geq n_0, \tag{2.32}$$

and so $\{a_{n-1}(\Delta x_{n-1})\}$ is an eventually nonincreasing sequence, and as in the proof of Theorem 2.1 we can easily show that $a_n \Delta x_n \geq 0$ for $n \geq n_0$. Define w_n by the Riccati substitution

$$w_n = \rho_n \frac{a_{n-1}(\Delta x_{n-1})}{x_n^{\beta}}. \tag{2.33}$$

Then $w_n > 0$ and as in the proof of Theorem 2.1 by using (2.32), we obtain

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n a_n\left(\Delta x_n\right)\Delta(x_n^\beta)}{\left(x_{n+1}^\beta\right)^2}. \tag{2.34}$$

Now we claim $\Delta^2 x_n \leq 0$. If not there exists $n_1 \geq n_0$ such that $\Delta^2 x_n > 0$ and this implies that $\Delta x_n > \Delta x_{n-1}$, so that since $\Delta a_n \geq 0$, $a_n\left(\Delta x_n\right) > a_n\left(\Delta x_{n-1}\right) \geq a_{n-1}\left(\Delta x_{n-1}\right)$ and this contradicts the fact that $\{a_{n-1}\left(\Delta x_{n-1}\right)\}$ is nonincreasing sequence, then $\Delta^2 x_n \leq 0$, and therefore we have

$$x_n > 0, \quad \Delta x_n \geq 0 \quad \text{and} \quad \Delta^2 x_n \leq 0 \text{ for } n \geq n_0. \tag{2.35}$$

Now, by using the inequality (cf. [7, p. 39])

$$x^\beta - y^\beta \geqslant \beta x^{\beta-1}(x-y) \text{ for all } x \neq y > 0 \text{ and } 0 < \beta \leq 1,$$

we have

$$\Delta(x_n^\beta) = x_{n+1}^\beta - x_n^\beta \geqslant \beta\left(x_{n+1}\right)^{\beta-1}\left(x_{n+1} - x_n\right) = \beta\left(x_{n+1}\right)^{\beta-1}\left(\Delta x_n\right). \tag{2.36}$$

Substitute (2.36) into (2.34) gives

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \rho_n a_n \frac{\beta\left(x_{n+1}\right)^{\beta-1}\left(\Delta x_n\right)\left(\Delta x_n\right)}{\left(x_{n+1}^\beta\right)^2}, \tag{2.37}$$

or

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\beta\rho_n}{a_n\left(\rho_{n+1}\right)^2\left(x_{n+1}\right)^{1-\beta}}\frac{\left(\rho_{n+1}a_n\right)^2\left(\Delta x_n\right)^2}{\left(x_{n+1}^\beta\right)^2}.$$

Hence,

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\beta\rho_n}{a_n\left(\rho_{n+1}\right)^2\left(x_{n+1}\right)^{1-\beta}}w_{n+1}^2. \tag{2.38}$$

From (2.35), we conclude that

$$x_n \leq x_{n_0} + \Delta x_{n_0}(n - n_0), \quad n \geq n_0.$$

Consequently there exists a $n_1 \geq n_0$ and appropriate constant $b \geq 1$ such that

$$x_n \leq bn \text{ for } n \geq n_1,$$

and this implies that

$$x_{n+1} \leq b(n+1) \text{ for } n \geq n_2 = n_1 - 1,$$

and, hence

$$\frac{1}{\left(x_{n+1}\right)^{1-\beta}} \geq \frac{1}{b^{1-\beta}(n+1)^{1-\beta}}. \tag{2.39}$$

Then from (2.38) and (2.39), we obtain

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta\rho_n}{(\rho_{n+1})^2\,(a_n)\,b^{1-\beta}(n+1)^{1-\beta}} w_{n+1}^2. \tag{2.40}$$

The remainder of the proof is similar to that of the proof of Theorem 2.1 and hence is omitted.

Remark 2.5 : Note that from Theorem 2.5, we can obtain different conditions for oscillation of all solutions of eq.(1.1) when (1.2) holds by different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant, or $\rho_n = R(n, n_0) = \sum_{s=n_0}^{n-1} \frac{1}{a_s}$. Hence we have the following results.

Corollary 2.3 — Assume that all the assumptions of Theorem 2.5 hold, except that the condition (2.31) is replaced by

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[s^\lambda[q_s - p_s] - \frac{(a_s)\,b^{1-\beta}(s+1)^{1-\beta}((s+1)^\lambda - s^\lambda)^2}{4\beta s^\lambda} \right] = \infty, \tag{2.41}$$

or

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[R(s,n_0)[q_s - p_s] - \frac{(a_l)\,b^{1-\beta}(l+1)^{1-\beta}\,(\Delta R(s,n_0))^2}{4\beta R(s,n_0)} \right] = \infty. \tag{2.42}$$

Then, every solution of eq.(2.30) oscillates.

Remark 2.8: When $p_n = 0$, eq.(2.30) reduces to the linear difference equation

$$\Delta(a_{n-1}\Delta x_{n-1}) + q_n x_n^\beta = 0, \quad n = 0, 1, 2..., \tag{2.43}$$

and condition (2.31) reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(a_l)\,b^{1-\beta}(l+1)^{1-\beta}(\Delta\rho_n)^2}{4\beta\rho_l} \right] = \infty,$$

which improves Theorem 4.3 in [8] in the sublinear case.

Theorem 2.6 — *Let $0 < \beta < 1$. Assume that (2.1) and (2.2) hold,* $\sum_{i=n_0}^{\infty} \left(\frac{1}{a_i}\right) = \infty$ *and* $\Delta a_n \geq 0$. *Let $\{\rho_n\}_{n=1}^\infty$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that (i) $H_{m,m} = 0$ for $m \geq 0$, (ii) $H_{m,n} > 0$ for $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$. If*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=n_0}^{m-1} \left[H_{m,n}\rho_n[q_n - p_n] - \frac{\rho_{n+1}^2}{4P_n} \left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] = \infty, \tag{2.44}$$

where

$$P_n = \frac{\beta \rho_n}{b^{1-\beta}(a_n)(n+1)^{1-\beta}},$$

and $h_{m,n}$ be defined in Theorem 2.2. Then every solution of eq.(2.30) oscillates.

PROOF: Proceeding as in the proof of Theorem 2.5, we assume that eq.(2.30) has a nonoscillatory solution, say $x_n > 0$ and for all $n \geq n_0$. From the proof of Theorem 2.5 we obtain (2.40) for all $n \geq n_1$. From (2.40), we have for $n \geq n_1$

$$\Delta w_n \leq -\rho_n[q_n - p_n] + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{P_n}{(\rho_{n+1})^2} w_{n+1}^2. \tag{2.45}$$

The remainder of the proof is similar to that of Theorem 2.2 and hence is omitted.

Corollary 2.4 — Assume that all the assumptions of Theorem 2.6 hold, except the condition (2.44) is replaced by

$$\lim_{m\to\infty} \sup \frac{1}{m^\lambda} \sum_{n=0}^{m-1} \left[(m-n)^\lambda \rho_n [q_n - p_n] - \frac{\rho_{n+1}^2}{4P_n} A_{m,n}^2 \right] = \infty, \tag{2.46}$$

where

$$A_{m,n} = \left(\lambda (m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n \sqrt{(m-n)^\lambda}}{\rho_{n+1}} \right),$$

or

$$\lim_{m\to\infty} \sup \frac{1}{(\log(m+1))^\lambda} \sum_{n=0}^{m-1} \left[\left(\log \frac{m+1}{n+1} \right)^\lambda \rho_n [q_n - p_n] - \frac{\rho_{n+1}^2}{4P_n} B_{m,n} \right] = \infty, \tag{2.47}$$

where

$$B_{m,n} = \left(\frac{\lambda}{n+1} \left(\log \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1} \right)^\lambda} \right)^2$$

or

$$\lim_{m\to\infty} \sup \frac{1}{m^{(\lambda)}} \sum_{n=0}^{m-1} (m-n)^{(\lambda)} \left[\rho_n [q_n - p_n] - \frac{\rho_{n+1}^2}{4P_n} \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta \rho_n}{\rho_{n+1}} \right)^2 \right] = \infty. \tag{2.48}$$

Then, every solution of eq. (2.30) oscillates.

Next, we consider eq. (2.30) when $0 < \beta < 1$, $\sum\limits_{i=n_0}^{\infty} \left(\frac{1}{a_i}\right) < \infty$ and $\Delta a_n \geq 0$.

***Theorem* 2.7** — *Let $0 < \beta < 1$. Assume that (2.1) and (2.2) hold, $\sum\limits_{i=n_0}^{\infty} \left(\frac{1}{a_i}\right) < \infty$ and $\Delta a_n \geq 0$. Furthermore, assume that there exists a positive sequences $\{\rho_n\}_{n=n_0}^{\infty}$ such that (2.31) hold and*

$$\sum_{n=n_0}^{\infty} [q_n - p_n] = \infty \quad and \quad \sum_{n=n_0}^{\infty} \left(\frac{1}{a_{n-1}} \sum_{i=n_0}^{n-1} [q_i - p_i] \right) = \infty. \tag{2.49}$$

Then every solution of eq.(2.30) oscillates or $\lim_{n\to\infty} x_n = 0$.

PROOF : The proof is similar to that of Theorem 2.3 and hence is omitted.

***Theorem* 2.8** — *Let* $0 < \beta < 1$. *Assume that (2.1)-(2.2) hold, and let* $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequences such that (2.49) holds. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *as defined in Theorem 2.6 and (2.44) holds. Then every solution of eq. (2.30) oscillates or* $\lim_{n\to\infty} x_n = 0$.

PROOF : The proof is similar to that of Theorem 2.4 by using the inequality (2.40) and hence is omitted.

By choosing $\{\rho_n\}$ and $\{H_{m,n}\}$ in appropriate manners, one can derive several oscillation criteria for (2.30) when $\sum\limits_{i=n_1}^{\infty} \left(\frac{1}{a_i}\right) < \infty$ holds and due to the limited space the details are left to the reader.

3. CONCLUSION

By employing Riccati and generalized Riccati technique, we have developed some new oscillation criteria for eq.(1.1) which do not require the conditions (1.5) and (1.7). When (1.2) holds, we have established some sufficient conditions for the oscillation of eq.(1.1), and when (1.3) holds the sufficient conditions guarantee that every solution oscillates or converges to zero. Our results improve the results in [11, 17, 18, 19] when (1.2) holds, and when (1.3) holds our results are essentially new. We have also given some examples to illustrate the usefulness of our results. Readers who are interested in other work which uses a different approach may refer to [3, 4].

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