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A SURVEY ON SHOOTING ARGUMENTS FOR BOUNDARY VALUE PROBLEMS

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In this survey, we explain shooting arguments for proving existence of solutions to boundary value problems posed for nonlinear ordinary differential equations. All the examples (except one) involve shooting on infinite intervals.

Key Words: Nonlinear ordinary differential equations; shooting arguments; existence of solutions; boundary value problems

1. INTRODUCTION

In this expository article, we explain the shooting arguments for proving the existence of solutions for boundary value problems. In general, shooting arguments may be applied in two ways. In one of the approaches, the continuous dependence of the parameters is made use of and in the other approach, the existence of upper and lower solutions is utilized. Some interesting comments may be seen in Kaper *et al*. [8].

We concentrate on the former approach here and explain a shooting argument with an example (see Mawhin [12]). Consider the boundary value problem

$$y'' + f(x, y, y') = 0, x \in [a, b], y(a) = A, y(b) = B. \tag{1.1}$$

In shooting arguments, we replace the boundary conditions with suitable initial conditions and thus a suitable initial value problem is posed. Here the initial value problem

$$y'' + f(x, y, y') = 0, \ \ y(a) = A, \ \ y'(a) = c \tag{1.2}$$

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is considered. Then look for the existence, uniqueness of the solutions of (1.2) and continuous dependence of the solutions of (1.2) on the parameter c. Now suppose that a unique solution of (1.2) exists and is denoted by $y(x, c)$; further $y(x, c)$ depends on c continuously. Our problem now is to show the existence of a real number $\hat{c}$ such that $y(b, \hat{c}) = B$. If we can show that there exist two constants c_1 and c_2 such that $y(b, c_1) > B$ and $y(b, c_2) < B$, then by the intermediate value theorem and the continuous dependence on the parameter c, the given boundary value problem (1.1) would have a solution.

Mawhin [12] pointed out that shooting argument was introduced by Severini (see [20, 21]). Severini considered the boundary value problem (1.1) and assumed f to be continuous. Severini proved that there exists a solution to (1.1) if $b-a$ and $\frac{B-A}{b-a}$ are sufficiently small. We refer the reader to an interesting survey on boundary value problems for nonlinear ordinary differential equations by Mawhin [12].

A shooting argument may involve the following aspects: (i) choice of shooting parameter(s) (ii) 'shooting site' (iii) local existence and uniqueness of solutions (iv) asymptotic behaviour of solutions at $\pm\infty$ (v) characterization of solutions and (vi) openness of some sets. In a recent article, Kwong [9] puts forward the importance of shooting arguments in the following way: " in the special case of second-order equations, the shooting method can be an effective tool, sometimes yielding better results than those obtainable via fixed point techniques". We may point out that the analysis with small/large values of shooting parameter(s) is relatively easy when compared to moderate values of shooting parameter(s). Moderate values of shooting parameters may be handled by numerical solutions. Thus we may observe that a judicial mix of analysis and numerics of a differential equation gives a better insight to the understanding of solutions of differential equations. We quote the comments of McLeod [13] on the shooting arguments: "A shooting argument thus breaks naturally into two parts. One of which should be accessible to an analyst and other to the computer. It becomes therefore a very powerful tool for the proof of existence of solutions of two-point boundary value problems".

For the readers, who are interested in an introduction to numerical aspects of shooting arguments for boundary value problems, we refer to Brian Bradie [3].

The plan of this article is as follows. Section 2 deals with a solution of the heat equation with absorption. Section 3 deals with the self-similar solutions of the modified burgers equation. In section 4, we give a review of some other related work on shooting arguments.

2. Existence of a Solution of the Heat Equation with Absorption

In this section, we explain the concepts of shooting arguments with an example due to Brezis, Peletier and Terman [4]. They considered the ordinary differential equation

$$u'' + \left(\frac{N-1}{x} + \frac{x}{2}\right) u' + \frac{\alpha}{2} u - u^p = 0, \quad 0 < x < \infty, \tag{2.1}$$

subject to the conditions

$$u > 0, \quad 0 \leq x < \infty, \tag{2.2}$$

$$u'(0) = 0, \quad u(\infty) = 0, \tag{2.3}$$

where $\alpha > 0, p > 1$ and $\alpha > N$. Equation (2.1) is a generalization of an ordinary differential equation obtained from the heat equation with absorption, namely

$$u_t - \triangle u + u^p = 0, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \quad N \geq 1.$$

Brezis *et al*. [4] proved the existence and uniqueness of a solution of (2.1)–(2.3) satisfying the condition

$$\lim_{x\to\infty} x^\alpha u(x) = 0, \tag{2.4}$$

by applying shooting arguments; the analysis was done in phase plane (u, u').

Now we explain their argument. Recasting (2.1) as a system of first order equations, we get

$$u' = v, \tag{2.5}$$
$$v' = -\left(\frac{N-1}{x} + \frac{x}{2}\right) v - f(u); \tag{2.6}$$

here $f(u) = \frac{\alpha}{2}u - u^p$. Suppose that $(u(x,\nu), v(x,\nu))$ be the solution of (2.5)–(2.6) satisfying

$$(u(0,\nu), v(0,\nu)) = (\nu, 0). \tag{2.7}$$

Note that $\nu > 0$ is the shooting parameter here and $0 < \nu < \left(\frac{\alpha}{2}\right)^{1/(p-1)} \equiv A$. Local existence and uniqueness of solutions for (2.5)–(2.7) are assured (see [10, 11, 15]). Peletier and Serrin [15] showed that $\lim\limits_{x\to 0} \dfrac{u'(x,\nu)}{x} = u''(0,\nu) = -\dfrac{1}{N} f(\nu)$. $u''(0,\nu) < 0$ implies that for sufficiently small x, u' is negative and hence initially u is decreasing. Suppose that $u' = 0$ for some $\tilde{x} > 0$ and $u(\tilde{x}) > 0$. Then $u(\tilde{x}) < A$ and $u''(\tilde{x}) = -f(u(\tilde{x})) < 0$. Hence u can't have a positive local minimum. Thus u decreases $(u' < 0)$ as long as $u > 0$. The task now is to prove the existence of a positive real number ν such that $u(x,\nu)$, the solution of the initial value problem (2.1)–$(2.3)_1$ with $u(0,\nu) = \nu$, has the required behaviour $(2.3)_2$ at infinity.

Define

$L_\lambda = \{(u,v) : 0 \leq u \leq A, v \leq 0, v \geq -\lambda u\}$, where $A = \left(\frac{\alpha}{2}\right)^{1/(p-1)}$ and $\lambda > 0$.

Brezis *et al*. [4] proved that, if $(u(x,\nu), v(x,\nu))$ enters L_λ for sufficiently large x_0, then $(u(x,\nu), v(x,\nu)) \in L_\lambda$ for all $x \geq x_0$. In otherwords, the solution gets trapped in L_λ. Precisely, they proved the following lemma.

Lemma 2.1. — Let $\lambda > 0$ and $(u(x,\nu), v(x,\nu))$ be the solution of (2.5)–(2.6) and $(u(x_0,\nu), v(x_0,\nu)) \in L_\lambda$ for some $x_0 \geq \xi_\lambda \equiv 2\lambda + \frac{\alpha}{\lambda}$. Then $(u(x,\nu), v(x,\nu)) \in L_\lambda$ for all $x \geq x_0$.

PROOF: The solution $(u(x,\nu), v(x,\nu))$ of (2.5)–(2.7) can neither pass through the point $(A, 0)$ nor (0, 0) as they are the critical points. Along the line segment $v = 0, 0 < u < A$, by (2.6), $v' = -f(u) < 0$. i.e., v decreases and hence the solution can't leave L_λ via $v = 0, 0 < u < A$. Again along the line $u = A, v < 0$, we observe that $u' = v < 0$ and hence u decreases and the solution is pointed inward L_λ. On $v = -\lambda u$, we have

$$\begin{aligned} \frac{v'}{u'} &= -\left(\frac{N-1}{x} + \frac{x}{2}\right) - \frac{f(u)}{v} \\ &= -\left(\frac{N-1}{x} + \frac{x}{2}\right) + \frac{\alpha}{2\lambda} - \frac{u^{p-1}}{\lambda} \\ &\quad < -\left(\frac{N-1}{x} + \frac{x}{2}\right) + \frac{\alpha}{2\lambda} < -\lambda, \text{ since } x > \xi_\lambda. \end{aligned}$$

Thus $v' + \lambda u' > 0$. This in turn gives $(v + \lambda u)' > 0$. i.e., $v + \lambda u$ is increasing. Thus we have $\frac{v}{u} > -\lambda$. Hence $(u, v) \in L_\lambda$ for all $x \geq x_0$. Thus the solution $(u(x, \nu), v(x, \nu))$ of (2.5)–(2.7) can't exit the region L_λ through the segments $u = A, v \leq 0$ or $v = -\lambda u$ or $v = 0, 0 \leq u \leq A$.

Remark: Brezis *et al*. [4] proved that the positive solutions of (2.5)–(2.7) with $0 < \nu < A$ vanish as $x \to \infty$. Further they showed that if $u(x, \nu)$ is positive on $[0, \infty)$, then $\lim_{x\to\infty} \dfrac{v(x)}{u(x)}$ exists and is either $-\infty$ or 0 (see Brezis *et al*. [4] for detailed proofs). Thus, positive solutions are characterized by the value of the limit $\lim_{x\to\infty} \dfrac{v(x)}{u(x)}$. They proved that

(*i*) $\lim_{x\to\infty} \dfrac{u'}{u} = 0$ corresponds to the solutions with asymptotic behaviour $u = O(x^{-\alpha})$ as $x \to \infty$ [referred to as slow orbits].

(*ii*) $\lim_{x\to\infty} \dfrac{u'}{u} = -\infty$ corresponds to the solutions with asymptotic behaviour $u = O\left(\mathrm{e}^{-x^2/4} x^{\alpha - N}\right)$ as $x \to \infty$ [referred to as fast orbits].

Let $S = \left\{\nu \in (0, A) : u(x, \nu) > 0 \quad \forall x > 0 \ \text{ and } \ \lim_{x\to\infty} \dfrac{v(x, \nu)}{u(x, \nu)} = 0\right\}$.

Proposition 2.2. — S is nonempty and open.

PROOF: It is known that $v' < 0$ on the line segment $v = 0$, $0 < u < A$ (see (2.6)) and $(u(x, A), v(x, A)) = (A, 0)$ for all $x \geq 0$. So for a given λ, making use of the continuous dependence on the initial data, we can choose ν sufficiently close to *A* such that $(u(x, \nu), v(x, \nu)) \in L_\lambda$ for $0 \leq x \leq \xi_\lambda$. And now by Lemma 2.1, $(u(x, \nu), v(x, \nu)) \in L_\lambda$ for all $x \geq 0$. Thus $\dfrac{v(x, \nu)}{u(x, \nu)} \geq -\lambda$ for all $x \geq 0$, which in turn gives $\lim_{x\to\infty} \dfrac{v(x, \nu)}{u(x, \nu)} \neq -\infty$. Hence $\lim_{x\to\infty} \dfrac{v(x, \nu)}{u(x, \nu)} = 0$ (see the Remark above Proposition 2.2.) and hence S is nonempty. To prove the openness of S, let $\nu^* \in S$. By the definition of S, $\lim_{x\to\infty} \dfrac{v(x, \nu^*)}{u(x, \nu^*)} = 0$. This implies that for any $\lambda > 0$ there exists a positive x_λ such that $\dfrac{v(x, \nu^*)}{u(x, \nu^*)} > -\lambda$ for all $x > x_\lambda$. Let $x^* = 2 \ \max\{\xi_{2\lambda}, x_\lambda\}$ and $B_\rho = \{(u, v) : \|(u, v) - (u(x^*, \nu^*), v(x^*, \nu^*))\| < \rho\}$; ρ is chosen so that $B_\rho \subseteq L_{2\lambda}$. The continuous dependence on the initial data leads to existence of an $\epsilon > 0$ such that $(u(x^*, \nu), v(x^*, \nu))$ $\in B_\rho$ when $|\nu - \nu^*| < \epsilon$. Since $u(x, \nu)$ is a decreasing function, $u(x, \nu) \geq u(x^*, \nu) > 0$ for $0 \leq x \leq x^*$ and by Lemma 2.1, $(u(x, \nu), v(x, \nu)) \in L_{2\lambda}$ for $x \geq x^*$ (see the definition of

x^*). Thus $\lim_{x\to\infty} \dfrac{v(x,\nu)}{u(x,\nu)} \geq -2\lambda > -\infty$. Therefore $\lim_{x\to\infty} \dfrac{v(x,\nu)}{u(x,\nu)} = 0$ (see the Remark) and hence $(\nu^* - \epsilon, \nu^* + \epsilon) \subset S$.

In the above lemma, it was shown that when ν is sufficiently close to A, the solutions of (2.5)–(2.7) are with algebraic decay as $\eta \to \infty$. Brezis *et al.* [4] showed that the solutions of (2.5)-(2.7) with sufficiently small ν have a finite zero and this result is presented in the following lemma.

Lemma 2.3 — Suppose that $\alpha > N$ and $\nu_1 = (\frac{\alpha-N}{2})^{\frac{1}{p-1}}$. Then for $0 < \nu < \nu_1$, $u(x,\nu)$ satisfying (2.5)-(2.7) vanishes at a finite x_0.

PROOF: Suppose for contradiction that $u(x,\nu) > 0$ for all $x \geq 0$. Multiplying (2.1) with x^{N-1} and integrating between 0 and x, we arrive at

$$x^{N-1}u'(x,\nu) + \frac{1}{2}x^N u(x,\nu) = \frac{N-\alpha}{2}\int_0^x s^{N-1}u(s,\nu)ds + \int_0^x s^{N-1}u^p(s,\nu)ds. \tag{2.8}$$

Note that integration by parts was used for the integral containing $x^N u'(x,\nu)$ to arrive at (2.8). As u is decreasing on $[0,\infty)$, (2.8) can be written as

$$x^{N-1}u'(x,\nu) + \frac{1}{2}x^N u(x,\nu) < -\left(\nu_1^{p-1} - \nu^{p-1}\right)\int_0^x s^{N-1}u(s,\nu)ds. \tag{2.9}$$

Using the fact that $\nu_1^{p-1} - \nu^{p-1} > 0$, we have $u'(x,\nu) + \frac{1}{2}xu(x,\nu) < 0$. This in turn implies that

$$u(x,\nu) < \nu \exp\left(-\frac{x^2}{4}\right) \quad \text{for all } x > 0. \tag{2.10}$$

Let $\left(\nu_1^{p-1} - \nu^{p-1}\right)\int_0^\infty s^{N-1}u(s,\nu)ds = 2\delta$ [integral exists because of (2.10)]. Making use of (2.10), we get from (2.9) that

$$\limsup_{x\to\infty} x^{N-1}u'(x,\nu) \leq -2\delta.$$

Then, there exists a $R_0 > 0$ such that $x^{N-1}u' < -\delta$ for $x > R_0$. Dividing with x^{N-1} and integrating from x to $R, x > R_0$, we have

$$u(x,\nu) \quad > \quad u(R,\nu) + \frac{\delta}{2-N}\left(R^{2-N} - x^{2-N}\right), \quad \text{if } N \neq 2, \tag{2.11}$$

$$u(x,\nu) \quad > \quad u(R,\nu) + \delta\log\left(\frac{R}{x}\right), \quad \text{if } N = 2. \tag{2.12}$$

For $N < 2$, RHS of (2.11) becomes unbounded as $R \to \infty$ which is impossible. When $N > 2$, as $R \to \infty$ in (2.11), we arrive at $u(x,\nu) \geq \frac{\delta}{N-2}x^{2-N}$ for $x > R_0$ which contradicts (2.10). Again for $N = 2$, taking the limit $R \to \infty, u$ becomes unbounded, a contradiction. Thus $u(x,\nu)$ has to vanish for some finite $x > 0$.

Let

$$\tilde{S} = \{0 < \nu < A : \; u(x, \nu) \text{ has a finite zero}\}.$$

The above lemma gives the nonemptyness of $\tilde{S}$ and it can be seen easily that $\tilde{S}$ is open. Since S, $\tilde{S}$ are disjoint, nonempty and open, $S \cup \tilde{S} \neq (0, A)$ and hence there exists a ν_0 such that $\nu_0 \in (0, A) \setminus \{S \cup \tilde{S}\}$. By the Remark, $u(x, \nu_0)$ is a solution of (2.1)–(2.4) with exponential decay as $x \to \infty$.

Further the solution of (2.1) subject to (2.2)–(2.4) is unique. They also gave the expressions for asymptotic behaviors of u as $x \to \infty$ when u is positive using an elementary principle - L'Hôspital's rule (see Brezis *et al*. [4]).

3. Self-Similar Solutions of the Modified Burgers Equation

Here we explain a shooting argument with an example due to Srinivasa Rao *et al*. [23]. Srinivasa Rao *et al*. [23] studied the boundary value problem

$$f'' + 2\eta f' + \frac{2}{\alpha} f - 2^{\frac{3}{2}} f^{\alpha} f' = 0, \tag{3.1}$$

$$f'(0) = 0, f \to 0 \;\text{ as }\; \eta \to \pm\infty \text{ and } f > 0 \text{ on } (-\infty, \infty). \tag{3.2}$$

Eq. (3.1) is obtained from the modified Burgers equation

$$u_t + u^{\alpha} u_x = \frac{\delta}{2} u_{xx}, \;\; \alpha > 0, \;\; \delta > 0, \tag{3.3}$$

using a similarity transformation. To prove the existence of a solution of (3.1)-(3.2), Srinivasa Rao *et al*. [23] considered (3.1) with initial conditions

$$f(0) = \nu \quad \text{and} \quad f'(0) = 0, \tag{3.4}$$

where $\nu > 0$ is a constant. Here ν is the shooting parameter. Local existence and uniqueness of solutions for (3.1), (3.4) are assured by the Picard's theorem. Let $f(\eta, \nu)$ be the solution of (3.1) satisfying (3.4) on its maximal interval of existence. We use $f(\eta)$ instead of $f(\eta, \nu)$ for the sake of simplicity. We explain the results for $\alpha > 1$ and $\eta \geq 0$ only. However, Srinivasa Rao *et al*. [23] analyzed (3.1) with (3.4) on $(-\infty, 0]$ also. The following are their results for the interval $[0, \infty)$.

Theorem — *Assume that $\alpha > 1$. Let f be a solution of* (3.1) *satisfying* (3.4). *Then there exists $\nu^* > 0$ such that*

(*i*) *if $\nu > \nu^*$, f has a zero at a finite point on $(0, \infty)$,*

(*ii*) *if $\nu = \nu^*$, f is positive on $[0, \infty)$ and decays exponentially to zero as $\eta \to \infty$,*

(*iii*) *if $\nu < \nu^*$, f is positive on $[0, \infty)$ and decays algebraically to zero as $\eta \to \infty$.*

Now we explain their proof of the above theorem. Srinivasa Rao *et al.* [23] proved that if f is a positive solution of (3.1) satisfying (3.4) on $[0,\infty)$, then $f,\, f' \to 0$ as $\eta \to \infty$ and f is decreasing on $[0,\infty)$. Following the work of Brezis *et al.* [4], they arrived at the following

(a) If $f > 0$ on $[0,\infty)$, then $\lim\limits_{\eta\to\infty} \dfrac{f'}{f}$ exists and is either 0 or $-\infty$.

(b) [Exponentially decaying solutions] If $f > 0$ on $[0,\infty)$ and $\lim\limits_{\eta\to\infty} \dfrac{f'}{f} = -\infty$, then $f = O(\mathrm{e}^{-\eta^2}\eta^{-(\alpha-1)/\alpha})$ as $\eta \to \infty$.

(c) [Algebraically decaying solutions] If $f > 0$ on $[0,\infty)$ and $\lim\limits_{\eta\to\infty} \dfrac{f'}{f} = 0$, then $f = O\left(\eta^{-1/\alpha}\right)$ as $\eta \to \infty$.

Results (a)–(c) stated above characterize the positive solutions of (3.1) satisfying (3.4).

Srinivasa Rao *et al.* [23] gave another characterization for the positive solutions of (3.1) subject to (3.4) in terms of the function $f\mathrm{e}^{\eta^2}$

(i) f decays algebraically to zero as $\eta \to \infty$ iff $f\mathrm{e}^{\eta^2}$ increases for η sufficiently large.

(ii) f decays exponentially to zero as $\eta \to \infty$ iff $f\mathrm{e}^{\eta^2}$ decreases for η sufficiently large.

Define

$S_1 = \{\nu > 0 \mid f(\eta,\nu)$ is positive on $[0,\infty)$ and decreases algebraically to zero as $\eta \to \infty\}$,

$S_2 = \{\nu > 0 \mid f(\eta,\nu)$ is positive on $[0,\infty)$ and decreases exponentially to zero as $\eta \to \infty\}$ and

$S_3 = \{\nu > 0 \mid f(\eta,\nu)$ has a finite zero on $(0,\infty)\}$.

Clearly, S_1, S_2 and S_3 are disjoint.

Lemma 3.1 — S_1 is nonempty.

PROOF: Srinivasa Rao *et al.* [23] showed that for $\nu < \left[\frac{(\alpha-1)^3}{4\alpha^2}\right]^{1/2\alpha}$, the solution $f(\eta)$ of (3.1) satisfying (3.4) is positive on $[0,\infty)$ and decays algebraically to zero as $\eta \to \infty$. For proving this, define

$$h(s) \equiv cf^{1-\alpha}(s) + f'(s), \quad c = \frac{\alpha-1}{\sqrt{2}\alpha} > 0, \tag{3.5}$$

$$h_1(\eta) = 2^{\frac{3}{2}} f^{\alpha}(\eta) h(\eta). \tag{3.6}$$

Observe that $h(0) = c\nu^{1-\alpha} > 0$ and $h_1(0) = 2^{\frac{3}{2}}\nu c > 0$. Thus h and h_1 are initially positive. We explain the proof of this lemma in two steps.

Step 1 — We claim that if η_1 and η_2 are the first zeros of h_1 and f respectively, then $\eta_2 > \eta_1$. Suppose for contradiction that $\eta_2 \le \eta_1$. Then $h_1(\eta) \ge 0$ on $[0,\eta_2]$, $f(\eta) \ge 0$ on $[0,\eta_2]$, $f'(\eta_2) \le 0$. An integration of (3.1) with respect to η from 0 to η gives

$$f'(\eta) + 2\eta f = \int_0^\eta h_1(s)ds. \tag{3.7}$$

Inserting $\eta = \eta_2$ in (3.7), we get $f'(\eta_2) > 0$, a contradiction. Hence $\eta_2 > \eta_1$ and thus f can't vanish before the zero of h_1.

Step 2 — We prove here that for $\nu < \left[\frac{(\alpha-1)^3}{4\alpha^2}\right]^{\frac{1}{2\alpha}}$, $h_1 > 0$ on $(0,\infty)$. Suppose for contradiction that η_1 is the first zero of h_1. Then by step 1, $f > 0$ on $[0,\eta_1]$. By the definition of h, $h > 0$ on $[0,\eta_1)$ and $h(\eta_1) = 0$, $h'(\eta_1) \leq 0$ and $f'(\eta_1) = -cf^{1-\alpha}(\eta_1)$. It can be shown that

$$h'(\eta_1) > f(\eta_1)\left[\frac{(\alpha-1)^3}{2\alpha^2}\nu^{-2\alpha} - 2\right] > 0,$$

since $\nu < \left[\frac{(\alpha-1)^3}{4\alpha^2}\right]^{\frac{1}{2\alpha}}$, a contradiction. Thus h can't have a zero on $[0,\infty)$ and hence h_1 is positive on $[0,\infty)$, which in turn gives $f > 0$ on $(0,\infty)$. Further, the equation (3.7) gives that

$$(f\mathrm{e}^{\eta^2})' > 0 \text{ for } \eta \in (0,\infty).$$

Using the characterization of algebraically decaying solutions as $\eta \to \infty$ (see the comments above Lemma 3.1), we conclude that f decays algebraically to zero as $\eta \to \infty$.

Lemma 3.2 — S_3 is nonempty.

PROOF: Srinivasa Rao *et al*. [23] showed that for sufficiently large ν, $f(\eta,\nu)$ has a finite zero. Suppose that

$$f(\eta_0(\nu)) = \left(\frac{\eta_0(\nu)}{\sqrt{2}}\right)^{1/\alpha}.$$

Observe that

$$f'' = -f'[2\eta - 2^{\frac{3}{2}}f^\alpha] - \frac{2}{\alpha}f < 0$$

on $[0,\eta_0(\nu))$ as $f > \left(\frac{\eta}{\sqrt{2}}\right)^{\frac{1}{\alpha}}$ and $f' < 0$. This implies that

$$f(\eta) \geq \nu + \eta\left[\frac{f(\eta_0(\nu)) - \nu}{\eta_o(\nu)}\right], \quad \eta \in [0,\eta_0(\nu)]. \tag{3.8}$$

Now onwards, η_0 will be used instead of $\eta_0(\nu)$ for the sake of simplicity. They proved this lemma in four steps.

Step 1 — $f'(\eta_0)$ satisfies the following bounds.

$$\begin{aligned} &-\frac{\alpha}{\alpha+1}2^{(2\alpha-1)/2\alpha}\eta_0^{(\alpha+1)/\alpha} - \frac{2^{3/2}}{\alpha+1}\nu^{\alpha+1} + \frac{2(\alpha-1)}{\alpha}\nu\eta_0 \geq f'(\eta_0) \\ &\geq \frac{2^{-1/2\alpha}}{\alpha(\alpha+1)}\left(-\alpha^2 - 1\right)\eta_0^{(\alpha+1)/\alpha} - \frac{2^{3/2}}{\alpha+1}\nu^{\alpha+1} + \frac{(\alpha-1)}{\alpha}\nu\eta_0. \end{aligned} \tag{3.9}$$

An integration of (3.1) from 0 to η_0 gives

$$f'(\eta_0) = -\frac{\alpha}{\alpha+1}2^{(2\alpha-1)/2\alpha}\eta_0^{(\alpha+1)/\alpha} - \frac{2^{3/2}}{\alpha+1}\nu^{\alpha+1} + \frac{2\alpha-2}{\alpha}\int_0^{\eta_0} f ds. \tag{3.10}$$

The bounds given above are the consequence of $f(\eta) \leq \nu$, (3.8) and (3.10).

Step 2 — Srinivasa Rao *et al*. [23] next proved that $\eta_0(\nu)$ is bounded. Rewriting (3.1),

$$f'' + \frac{2}{\alpha}f = f'\left[2^{3/2}f^\alpha - 2\eta\right].$$

Since $f' < 0$ and $f(\eta) \geq (\eta/\sqrt{2})^{1/\alpha}$ on $[0, \eta_0]$,

$$f'' + \frac{2}{\alpha}f \leq 0$$

on $[0, \eta_0]$. Integrating with respect to η from 0 to η , we get

$$f'(\eta) \leq -\frac{2}{\alpha}\int_0^\eta f ds. \tag{3.11}$$

Using (3.8) in (3.11), we arrive at

$$f'(\eta) \leq -\frac{2}{\alpha}\left[\nu\eta + \frac{\eta^2}{2}\left(\frac{f(\eta_0)-\nu}{\eta_0}\right)\right].$$

Integrating again w.r.t. η from 0 to η_0, we arrive at

$$\nu\left(1 - \frac{2}{3\alpha}\eta_0^2\right) - f(\eta_0)\left(\frac{1}{3\alpha}\eta_0^2 + 1\right) \geq 0.$$

This in turn gives that

$$\eta_0 \leq \left(\frac{3\alpha}{2}\right)^{1/2}.$$

Step 3 — $f'(\eta_0) \sim -\frac{2^{3/2}}{\alpha+1}\nu^{\alpha+1}$ as $\nu \to \infty$.

Using the fact that η_0 is bounded from *step 2* in the bounds for f' (see (3.9)), we have the required result.

Step 4 — We now show that for ν sufficiently large, f has a finite zero.

Suppose that $f > 0$ on $[0, \infty)$ and hence $f' < 0$ on $(0, \infty)$. Eq. (3.1) gives

$$f'' + 2\eta f' + \frac{2}{\alpha}f \leq 0.$$

An Integration of the above equation from η_0 to η and using $f(\eta) \leq f(\eta_0)$ on $[\eta_0, \infty)$, we get after some simplification

$$f'(\eta) - f'(\eta_0) - 2\eta_0 f(\eta_0) \leq \frac{2(\alpha-1)}{\alpha}f(\eta_0)(\eta - \eta_0). \tag{3.12}$$

Again an integration of (3.12) on (η_0, η) gives

$$f(\eta) \leq f(\eta_0) + \Big(f'(\eta_0) + 2\eta_0 f(\eta_0) \Big) (\eta - \eta_0) + \frac{(\alpha - 1)}{\alpha} f(\eta_0)(\eta - \eta_0)^2. \tag{3.13}$$

Fix η. Let $\nu \to \infty$ in (3.13), then $f(\eta) < 0$, because η_0 is bounded and $f'(\eta_0) \sim \frac{-2^{3/2}}{\alpha+1}\nu^{\alpha+1}$ as $\nu \to \infty$. Hence a contradiction. Thus f can't be positive on $(0, \infty)$ for ν sufficiently large. As in the previous section, openness of S_1 and S_3 can be shown. Since S_1 and S_3 are disjoint and $S_1 \cup S_3 \neq (0, \infty)$, there exists ν_0 such that $f(\eta, \nu_0)$ is positive and decays exponentially to zero as $\eta \to \infty$. In fact, such ν_0 is unique (see Srinivasa Rao *et al*. [23] for more details).

4. Some More Examples

We give here an account of some of the related work, wherein the shooting arguments were used for proving the existence of solutions (satisfying certain conditions at $-\infty$ and / or at $+\infty$) for boundary value problems involving nonlinear ordinary differential equations.

Example 1 — Cazenave and Escobedo [5] studied the boundary value problem

$$u'' + p|u|^{p-1}u' + \frac{t}{2}u' + \frac{1}{2(p-1)}u = 0, \tag{4.1}$$

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to-\infty} (-t)^{-1/(p-1)}u(t) = p^{-1/(p-1)} \text{ and } u \geq 0; \tag{4.2}$$

here $1 < p < 2$ and $t \in \mathbb{R}$. Grundy *et al*. [7] arrived at the boundary value problem (4.1)–(4.2) while constructing the large time asymptotic solutions for the initial value problem

$$\frac{\partial}{\partial t}[u + u^p] + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{4.3}$$

$$u(x, 0) = u_0(x) \geq 0, \ \ u_0 + u_0^p \in L^1(\mathbb{R}); \ \ p > 0. \tag{4.4}$$

For the parametric range $1 < p < 2$, Cazenave and Escobedo [5] proved the existence of positive, monotonically decreasing solutions u^λ on $(-\infty, \infty)$ satisfying $\lim\limits_{t\to-\infty}(-t)^{-1/(p-1)}u^\lambda(t)$ $= p^{-1/(p-1)}$ and $\lim\limits_{t\to\infty} t^{1/(p-1)}u^\lambda(t) = c(\lambda)$; here $c(\lambda)$ is a continuous, increasing function defined on $[0, \infty)$ to $[0, \infty)$ with $c(0) = 0$ and $c(\lambda) \to \infty$ as $\lambda \to \infty$. Further they proved that any nontrivial, non-negative solution of (4.1) on $\mathbb{R}$ should be a member of the above family $\{u^\lambda\}$. Cazenave and Escobedo [5] analyzed the problem on the intervals $(-\infty, 0]$ and $[0, \infty)$ by posing two initial value problems. They fixed $u(0) = u_0$ and showed that there exists $u'(0) = u_0'$ such that the corresponding solution u of (4.1) decays exponentially to zero as $t \to +\infty$. They also proved the existence of $u'(0) = u_1'$ for which the solution is global and is of $O\left((-t)^{1/(p-1)}\right)$ as $t \to -\infty$. Further they showed the existence of $\underline{x}$ such that for $u_0 = \underline{x}$, u_0' becomes equal to u_1'; the solution of (4.1) with $u(0) = \underline{x}$, $u'(0) = u_1' = u_0'$ has the required exponential decay at $+\infty$ and behaves like

$(-t)^{1/(p-1)}$ at $-\infty$. When $u_0 > \underline{x}, u'(0) = u_1'$, the corresponding solution of (4.1) is $O(t^{-1/(p-1)})$ at $+\infty$ and $O((-t)^{1/(p-1)})$ at $-\infty$. Cazenave and Escobedo [5] referred to this problem as two parameter shooting problem. An interesting feature of this work is that both $u(0)$ and $u'(0)$ are the shooting parameters.

Having given a synopsis of their work, we shall explain now the steps involved in some detail.

Cazenave and Escobedo [5] considered (4.1) with initial conditions

$$u(0) = u_0, \quad u'(0) = u_0' \tag{4.5}$$

for $t \geq 0$. Here u_0 and u_0' are the shooting parameters. Local existence and uniqueness of solutions are a consequence of Picard's theorem. It is easy to see that the solutions are global. Cazenave and Escobedo [5] showed that three types of solutions are possible: (i) solutions with finite zero (ii) positive solutions with algebraic decay at $t = +\infty$ (iii) positive solutions with exponential decay at $t = +\infty$.

They considered the sets

$$\begin{aligned}
N^+ &= \{(u_0, u_0') \in (0,\infty) \times \mathbb{R}; u \text{ has a finite zero}\},\\
P^+ &= \{(u_0, u_0') \in (0,\infty) \times \mathbb{R}; u > 0 \text{ and } \liminf_{t\to\infty}\ t^{1/(p-1)}u(t) > 0\} \text{ and}\\
E^+ &= \{(u_0, u_0') \in (0,\infty) \times \mathbb{R}; u > 0 \text{ and } \liminf_{t\to\infty}\ t^{1/(p-1)}u(t) = 0\}.
\end{aligned}$$

These sets are mutually disjoint and $N^+ \cup P^+ \cup E^+ = (0,\infty)\times\mathbb{R}$. Assuming that u is a nonnegative solution of (4.1) on $[0,1]$ satisfying (4.5) with $u_0 > 0$ and $u_0' \leq 0$, Cazenave and Escobedo [5] proved that u_0 and u_0' satisfy the relation $u_0' \geq -\dfrac{u_0 C_0}{1-\mathrm{e}^{-C_0}},\ C_0 = p|u_0|^{p-1} + 1/2$. This in turn implies that when $u_0' \leq -M$, with $M > \dfrac{u_0 C_0}{1-\mathrm{e}^{-C_0}}$, solutions of (4.1) satisfying (4.5) cannot be nonnegative on $[0,1]$. Thus N^+ is non-empty. Openness of N^+ follows from the continuous dependence of u on the initial data (u_0, u_0'). They also proved that $u, u' \to 0$ as $t \to \infty$ for any solution u of (4.1) with $u_0, u_0' \in \mathbb{R}$. They characterized the positive solutions of (4.1) in terms of $\dfrac{u'}{u}$ as in Brezis *et al.* [4]: (i) If u is a positive solution of (4.1), then $\dfrac{u'}{u} \to l$ as $t \to \infty$; $l = 0$ or $-\infty$. (ii) $l = 0$ corresponds to the positive solutions with algebraic decay $(u = O(t^{-1/(p-1)})$ as $t \to \infty)$ and $l = -\infty$ corresponds to the positive solutions with exponential decay $\left(u = O(t^{(2-p)/(p-1)}\mathrm{e}^{-t^2/4}) \text{ as } t \to \infty\right)$. They also arrived at a trapping region similar to that of Brezis *et al.* [4]. The result reads as follows.

Let $\nu > 0$ and $D_\nu = \{(u,v) \in \mathbb{R}^2 : u > 0,\ v < 0,\ v + \nu u > 0\}$. If u is a solution of (4.1) with $(u(\tau), u'(\tau)) \in D_\nu$ for $\tau > 2\nu + \dfrac{1}{(p-1)\nu}$, then $(u(t), u'(t)) \in D_\nu$ for all $t \geq \tau$.

For proving the existence of algebraically decaying positive solutions as $t \to \infty$, they used the fact that the solution that falls eventually in the set D_1 has algebraic decay as $t \to \infty$. Now we shall explain the idea of the proof of the existence of algebraically decaying positive solutions as $t \to \infty$.

Let $0 < \delta < 1$ and $M > 0$ be the constants satisfying the conditions $\dfrac{1-\delta}{\delta} > 2 + \dfrac{1}{p-1}$ and $\delta p(p-1)M^{p-1} = 1$. Suppose that u is a solution of (4.1) with $u(t_0) \geq M$ and $u'(t_0) \geq -\delta u(t_0)$ for some $t_0 \geq 0$. It is easy to see that there exists $\tau \geq t_0$ such that $u(\tau) \geq M$ and $0 \geq u'(\tau) \geq -\delta u(\tau)$. Thus, $(u(\tau), u'(\tau)) \in D_1$ and Cazenave and Escobedo [5] proved that $(u(t),\ u'(t))$ can't exit D_1 for $t \geq \tau$ and hence u has the required algebraic decay at $+\infty$. Thus P^+ is nonempty. They also proved that for every $u_0 > 0$ and $u_0' > 0$ sufficiently large, the solution of (4.1) satisfying (4.5) is positive for all $t \geq 0$ and has algebraic decay as $t \to \infty$. By making use of trapping region for solutions of (4.1) in the phase plane and continuous dependence on initial data, it is proved that P^+ is open. The solutions with the initial data $(u_0, 0)$ for u_0 sufficiently small are with finite zero. This result was proved using scaling arguments. Finally, Cazenave and Escobedo [5] proved the following

(*i*) there exists a unique $x_0 > 0$ such that $(x_0, 0) \in E^+$.

(*ii*) for every $u_0 > x_0$, there exists a unique u_0' such that $(u_0, u_0') \in E^+$.

For proving (i)-(ii), openness of $P^+,\ N^+$ and a comparison theorem were made use of. Define $a(u_0) = u_0'$ for $u_0 \geq x_0$. Then $a(x_0) = 0$ and $(u_0, a(u_0)) \in E^+$. This function a is proved to be continuous and decreasing on $[x_0, \infty)$.

To study the equation (4.1) on $(-\infty, 0]$, they transformed it to

$$v'' - p|v|^{p-1}v' + \frac{t}{2}v' + \frac{1}{2(p-1)}v = 0 \tag{4.6}$$

by using $v(t) = u(-t)$. The initial conditions become $v(0) = u(0),\ v'(0) = -u'(0)$. They studied (4.6) with initial conditions

$$v(0) = v_0, \quad v'(0) = v_0'. \tag{4.7}$$

Here v_0 and v_0' are the shooting parameters. Local existence and uniqueness are obtained by the standard theorems. $v(v_0, v_0', t)$ represents the solution of the IVP (4.6)–(4.7). When there is no ambiguity, we simply write $v(t)$. Further suppose that v is defined on $[0, T^*(v_0, v_0'))$. Consider

$$\begin{aligned}
N^- &= \{(v_0, v_0') \in (0,\infty) \times \mathbb{R};\ \exists t_0 \in [0, T^*), v > 0 \text{ on } [0, t_0] \text{ and } v'(t_0) \leq 0\},\\
B^- &= \{(v_0, v_0') \in (0,\infty) \times \mathbb{R};\ v' > 0 \text{ on } [0, T^*) \text{ and } T^* < \infty\} \text{ and}\\
E^- &= \{(v_0, v_0') \in (0,\infty) \times \mathbb{R}; v \text{ is defined on } [0,\infty) \text{ and } v' > 0 \text{ on } [0,\infty)\}.
\end{aligned}$$

Clearly, the above sets are mutually disjoint and $N^- \cup B^- \cup E^- = (0,\infty) \times \mathbb{R}$. A choice of $v_0' = 0$ gives the nonemptyness of N^- and openness can be proved using continuous dependence of solutions on initial conditions. To prove that B^- is nonempty, they showed that if v is a solution of (4.6) satisfying

$$v(t_0)^{p-1} > Kt_0,\ v'(t_0) > K(v(t_0)^{2-p} + 1) \text{ for some } t_0 \in [0, T^*) \text{and some } K > 0, \tag{4.8}$$

then $T^* < \infty$ and v increases on $[t_0, T^*)$. The properties given in (4.8) characterize the solutions that blow up in finite time. Choosing $t_0 = 0$, we arrive at the nonemptyness of B^-. If $(v_0, v_0') \in B^-$, then the solution v of (4.6)–(4.7) satisfies $v(t_2)^{p-1} > Kt_2$, $v'(t_2) > K(v(t_2)^{2-p} + 1)$ for some $t_2 \in [0, T^*)$. By continuous dependence of solutions on (v_0, v_0'), there exists a neighbourhood of (v_0, v_0') such that any solution of (4.6)–(4.7) with initial data in this neighbourhood satisfies the above conditions (4.8) at $t = t_2$. This in turn leads to the blow up of solutions. Hence B^- is open. For given $v_0 > 0$, consider $A = \{w_0' : (v_0, w_0') \in B^-\}, B = \{w_0' : (v_0, w_0') \in N^-\}$ and $C = \{w_0' : (v_0, w_0') \in E^-\}$. Since A, B are disjoint, nonempty open subsets of $\mathbb{R}$ and can't cover $\mathbb{R}$, C is nonempty. By comparison results, C becomes a singleton set and let that be $\{(v_0, v_0')\}$. If v is a solution of (4.6)–(4.7) with $(v_0, v_0') \in C$, then Cazenave and Escobedo [5] proved that $t^{-1/(p-1)}v(t)$ is decreasing on $(0, \infty)$ and decreases to $p^{-1/(p-1)}$ as $t \to \infty$. Define for all $v_0 > 0, \alpha(v_0) = v_0'$ so that $(v_0, \alpha(v_0)) \in E^-$. Then α is continuous due to openness of B^- and N^-.

We give here an outline of the construction of the family of solutions $\{u^\lambda\}$ of (4.1), which are positive, monotonic, $O((-t)^{1/(p-1)})$ as $t \to -\infty$ and decays to zero either exponentially or algebraically as $t \to \infty$. Cazenave and Escobedo [5] showed that there exists $\underline{x} > x_0$ such that

$$\alpha(x) + a(x) = \begin{cases} \text{positive,} & \text{if } x_0 \leq x < \underline{x}, \\ \text{zero,} & \text{if } x = \underline{x}, \\ \text{negative,} & \text{if } x > \underline{x}. \end{cases} \tag{4.9}$$

They defined for all $\lambda \geq 0$, $u^\lambda(t)$ is a solution of (4.1) with $u^\lambda(0) = \underline{x} + \lambda$ and $u^{\lambda'}(0) = -\alpha(u^\lambda(0))$ and showed that the solution $u^\lambda(t)$ of (4.1) is monotone decreasing and $u^\lambda(t) = O((-t)^{1/(p-1)})$ as $t \to -\infty$. It is shown that for $x \geq \underline{x}$, the solution u^λ of (4.1), (4.5) is positive on $\mathbb{R}$ and decreases to zero exponentially for $x = \underline{x}$ and algebraically for $x > \underline{x}$ as $t \to \infty$.

Example 2 — Peletier and Serafini [14] studied the boundary value problem

$$u'' + \frac{x}{2}u' + \frac{\alpha}{2}u + p|u|^{p-1}u' = 0, \quad 0 < x < \infty, \tag{4.10}$$

$$u'(0) = 0, \quad x^\alpha u(x) \to 0 \text{ as } x \to \infty, \tag{4.11}$$

$$u(x) > 0, \quad 0 \leq x < \infty; \tag{4.12}$$

here $p > 1, \alpha > 0$. A special case of the ordinary differential equation (4.10) with $\alpha = 1/(p-1)$ is obtained from the heat equation with convection by a similarity transformation. Peletier and Serafini [14] used a shooting argument to prove the existence of a solution for (4.10)–(4.12). They posed an initial value problem for (4.10) with initial conditions

$$u(0) = \gamma, \quad u'(0) = 0, \tag{4.13}$$

where $\gamma > 0$ is the shooting parameter. There exists a unique solution for the initial value problem (4.10), (4.13) locally and is denoted by $u(x, \gamma)$. They proved that the solution $u(x, \gamma)$ can be

continued for all $x > 0$. Further, if u is positive for all $x > 0, (u(x,\gamma), u'(x,\gamma)) \to (0,0)$ as $x \to \infty$. They pointed out that it is possible to prove that $\dfrac{u'(x,\gamma)}{u(x,\gamma)} \to 0$ or $-\infty$ as $x \to \infty$ when $u > 0$ on $(0,\infty)$ following Brezis *et al.* [4]. They defined the sets:
$S_1 = \{\gamma > 0 : \ u(x,\gamma) \text{ has a finite zero on } (0,\infty)\}$,
$S_2 = \{\gamma > 0 : \ u(x,\gamma) \text{ is positive for all } x > 0 \text{ and decays algebraically to } 0 \text{ as } x \to \infty\}$,
$S_3 = \{\gamma > 0 : \ u(x,\gamma) \text{ is positive for all } x > 0 \text{ and decays exponentially to } 0 \text{ as } x \to \infty\}$.
They used 'scaling arguments' to prove that S_1 (when $\alpha > 1$) and S_2 (when $\alpha > 0$) are nonempty. The transformation $\gamma = \epsilon$, and $\gamma^{-1}u(x,\gamma) = w_\epsilon(x)$ transforms (4.10), (4.13) to

$$w_\epsilon'' + \frac{x}{2}w_\epsilon' + \frac{\alpha}{2}w_\epsilon + \epsilon^{p-1}p|w_\epsilon|^{p-1}w_\epsilon' \quad = \quad 0, \tag{4.14}$$

$$w_\epsilon(0) = 1 \text{ and } w_\epsilon'(0) \quad = \quad 0. \tag{4.15}$$

It was shown that as $\epsilon \to 0, w_\epsilon \to w_0$ in $C^2([0,X])$ for any $X > 0$, where w_0 is the solution of the initial value problem

$$w'' + \frac{x}{2}w' + \frac{\alpha}{2}w \quad = \quad 0, \tag{4.16}$$

$$w(0) = 1 \text{ and } w'(0) \quad = \quad 0. \tag{4.17}$$

Further w_0, the solution of (4.16)–(4.17), has a finite zero for $\alpha > 1$. Thus w_ϵ vanishes at a finite x_0 for sufficiently small ϵ. Hence small solutions of (4.10), (4.13) are with finite zero and S_1 is nonempty for $\alpha > 1$. They also proved that for the parametric range $\alpha \leq 1$, there is no solution for (4.10), (4.13) with finite zero. Peletier and Serafini [14] transformed (4.10), (4.13) to

$$\epsilon y_\epsilon'' + (y_\epsilon^p)' + \frac{t}{2}y_\epsilon' + \frac{\alpha}{2}y_\epsilon \quad = \quad 0, \tag{4.18}$$

$$y_\epsilon(0) = 1 \text{ and } y_\epsilon'(0) \quad = \quad 0 \tag{4.19}$$

using the transformation $t = \gamma^{-(p-1)}x, \gamma^{-1}u(x,\gamma) = y_\epsilon(t)$ and $\epsilon = \gamma^{-2(p-1)}$. As $\gamma \to \infty$ $(\epsilon \to 0)$, the solution y_ϵ of (4.18)–(4.19) converges to the solution y_0 of the initial value problem

$$(y^p)' + \frac{t}{2}y' + \frac{\alpha}{2}y \quad = \quad 0, \tag{4.20}$$

$$y(0) = 1 \text{ and } y'(0) \quad = \quad 0, \tag{4.21}$$

which has a unique (positive) solution decaying algebraically to zero as $t \to \infty$ and therefore for large γ, the solutions of (4.10), (4.13) are with algebraic decay as $x \to \infty$. Thus they proved that S_2 is nonempty for $\alpha > 0$. It is easy to see that S_1 and S_2 are open and disjoint. As $S_1 \cup S_2$

$\neq (0, \infty)$, S_3 is nonempty and hence a positive solution of (4.10) satisfying (4.11) exists for $\alpha > 1$. Peletier and Serafini [14] gave explicit asymptotic behaviours (as $x \to \infty$) for positive solutions following the work of Brezis *et al*. [4]. They also proved the uniqueness of the fast decaying solution for $\alpha > 1$. For $0 < \alpha \leq 1$, all the solutions of (4.10) satisfying (4.13) are positive on $(0, \infty)$ with algebraic decay at ∞.

Example 3 — Srinivasa Rao *et al*. [23] studied the self-similar solutions of the generalized Burgers equation

$$u_t + u^\alpha u_x + \frac{ju}{2t} = \frac{\delta}{2} u_{xx}; \tag{4.22}$$

here $\alpha > 0, j \geq 0$ and $\delta > 0$ (small) are constants. Inspired by the work of Sachdev and Nair [19] and Peletier and Serafini [14], Srinivasa Rao *et al*. [23] posed the following two problems:
(I) [IVP]

$$f'' + 2\eta f' + \frac{2(1-\alpha j)}{\alpha} f - 2^{\frac{3}{2}} f^\alpha f' = 0, \tag{4.23}$$

$$f(0) = \nu, \quad f'(0) = 0, \ \ \nu > 0 \text{ is a constant.} \tag{4.24}$$

(II) [Connection Problem]

$$f'' + 2\eta f' + \frac{2(1-\alpha j)}{\alpha} f - 2^{\frac{3}{2}} f^\alpha f' = 0, \tag{4.25}$$

$$f \sim A\mathrm{e}^{-\eta^2} \eta^{-[\alpha(j+1)-1]/\alpha} \quad \text{as } \eta \to \infty, \tag{4.26}$$

$$f \to 0 \text{ as } \eta \to -\infty, \tag{4.27}$$

$$|f| < \infty, \ \ A > 0 \text{ is an arbitrary constant.} \tag{4.28}$$

Equation (4.23) is obtained from (4.22) by a similarity transformation. We discuss here their results for the initial value problem posed in (4.23)–(4.24). Their interest was to prove the existence of positive solutions vanishing at $\pm\infty$ and thus the initial value problem (4.23)-(4.24) was posed. Srinivasa Rao *et al*. [23] carried out the analysis on the intervals $[0, \infty)$ and $(-\infty, 0]$ respectively. They showed that positive and bounded solutions exist for the parametric range $\frac{1}{j+1} < \alpha \leq \frac{1}{j}$ only. If $\alpha j = 1, f \equiv \nu$. We first discuss their results for the parametric range $\frac{1}{j+1} < \alpha < \frac{1}{j}$ on $[0, \infty)$. The local existence and uniqueness of solutions of (4.23)–(4.24) are a consequence of the Picard's existence theorem. The solution of (4.23)–(4.24) on it's maximal interval of existence is denoted by $f(\eta, \nu)$. If f is a positive solution of (4.23)–(4.24) on $(0, \infty)$, then $f, f' \to 0$ as $\eta \to \infty$. Following the techniques of Brezis *et al*. [4], they proved that if $f(\eta, \nu) > 0$ on $(0, \infty)$, then $\lim\limits_{\eta \to \infty} \dfrac{f'}{f} = 0$ or $-\infty$. If $f(\eta, \nu) > 0$ for all $\eta \geq 0$, then the solutions of (4.23) with $\lim\limits_{\eta \to \infty} \dfrac{f'}{f} = 0$ are algebraically decaying at $+\infty$ and the solutions of (4.23) with $\lim_{\eta\to\infty}\frac{f'}{f}=-\infty$ are with exponential decay at $+\infty$. They also gave another type of asymptotic characterization for the positive solutions of (4.23) as $\eta\to\infty$ for the parametric range $\frac{1}{j+1}<\alpha<\frac{1}{j}$. A positive solution f of (4.23) decays exponentially to zero as $\eta\to\infty$ iff $e^{\eta^2}f$ decreases for η sufficiently large. Similarly, a positive solution f of (4.23) decays algebraically to zero as $\eta\to\infty$ iff $e^{\eta^2}f$ increases for η sufficiently large.

They considered the following sets.

$S_1=\{\nu>0: f(\eta,\nu)>0 \text{ on } [0,\eta_1) \text{ and } f(\eta_1)=0 \text{ for some } \eta_1\}$,

$S_2=\{\nu>0: f(\eta,\nu)>0 \text{ on } [0,\infty) \text{ and } f\eta^{(1-\alpha j)/\alpha}\to c\neq 0 \text{ as } \eta\to\infty\}$ and

$S_3=\{\nu>0: f(\eta,\nu)>0 \text{ on } [0,\infty) \text{ and } f \text{decays exponentially to zero as } \eta\to\infty\}$.

Clearly, S_1, S_2 and S_3 are disjoint and $S_1\cup S_2\cup S_3=(0,\infty)$. As expected,"small solutions" are solutions with algebraic decay as $\eta\to\infty$. They proved that if $\nu<\left(\frac{\alpha-1}{4\alpha^2}[\alpha j+\alpha-1]^2\right)^{1/2\alpha}$ and $\alpha>1$, then $f(\eta,\nu)$ is positive on $[0,\infty)$ and decays algebraically to zero as $\eta\to\infty$. That is, S_2 is nonempty. Using the "trapping region argument" as in Brezis *et al*. [4], Srinivasa Rao *et al*. [23] proved that S_2 is open. Further they proved that "large solutions" are the solutions with finite zero. That is, when ν is sufficiently large, $f(\eta,\nu)$ can't remain positive on the interval $[0,\infty)$. This is the place where nonlinear term plays an important role. For proving this, they defined $\eta_0(\nu)$ as the first point of intersection of f and $(\frac{\eta}{\sqrt{2}})^{\frac{1}{\alpha}}$, i.e., $f(\eta_0(\nu),\nu)=\left(\eta_0(\nu)/\sqrt{2}\right)^{1/\alpha}$. They proved that (i) $f(\eta,\ \nu)\geq\nu+\eta\left(\frac{f(\eta_0(\nu),\nu)-\nu}{\eta_0(\nu)}\right)$ for all $0\leq\eta\leq\eta_0(\nu)$, (ii) $\eta_0(\nu)$ is bounded and (iii) $f'(\eta_0(\nu),\nu)\sim-\frac{2^{\frac{3}{2}}}{\alpha+1}\nu^{\alpha+1}$ as $\nu\to\infty$. Using (i)-(iii), it was proved that for ν sufficiently large, $f(\eta,\nu)$ can't remain positive on $(0,\infty)$. Hence S_1 is nonempty. It is straightforward to show that S_1 is open. Since $S_1,\ S_2$ are nonempty, open and disjoint, $S_1\cup S_2\neq(0,\infty)$. Hence S_3 is nonempty. As a result, there exists $\nu_0>0$ such that $f(\eta,\nu_0)$ is a positive solution of (4.23)–(4.24) with exponential decay at $+\infty$. It was also proved that if $f(\eta,\nu_1), f(\eta,\nu_2)$ are solutions with exponential decay at ∞, then $f(\eta,\nu_1)\equiv f(\eta,\nu_2)$. Thus the parametric range $(0,\infty)$ for 'ν' is partitioned into three sets $S_1=(\nu_0,\infty), S_2=(0,\nu_0)$ and $S_3=\{\nu_0\}$. Further they proved that $f(\eta,\nu)$ is positive on $(-\infty,0)$ for all $\nu>0$ and decays algebraically to zero as $\eta\to-\infty$. Thus they arrived at the following theorem.

Theorem — *Assume that* $\frac{1}{j+1}<\alpha<\frac{1}{j}$ *and* $\alpha>1$. *Then there exixts a* ν_0 *such that the following holds*

(i) If $\nu>\nu_0$, *every solution* $f(\eta,\nu)$ *of* (4.23)–(4.24) *is positive on* $(-\infty,0)$ *and decays algebraically to zero as* $\eta\to-\infty$; f *has a zero at a finite point in* $(0,\infty)$.

(ii) If $\nu=\nu_0$, *solution* f *of* (4.23)–(4.24) *is positive on* $(-\infty,\infty)$ *and decays algebraically to*

zero as $\eta \to -\infty$; f *decays exponentially to zero as* $\eta \to \infty$.

(iii) If $\nu < \nu_0$, *every solution* f *of* (4.23)–(4.24) *is positive on* $(-\infty, \infty)$ *and decays algebraically to zero as* $\eta \to \pm\infty$.

For the connection problem (II), Srinivasa Rao *et al*. [23] proved that when $\dfrac{1}{j+1} < \alpha < \dfrac{1}{j}$, all solutions of (4.25) satisfying (4.26)–(4.28) are positive and decay algebraically to zero $\eta \to -\infty$. An explicit solution of (4.25) having exponential decay at $\pm\infty$ for $\alpha = 1/(j+1)$ was given by Sachdev and Nair [19] and Cazenave and Escobedo [5]. Note that in section 3, we explained the analysis for special case of (4.23) with $j = 0$ in a detailed manner.

Example 4 — Srinivasa Rao *et al*. [24] studied the ordinary differential equation

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g|g|^{(\alpha-3)/2} g' - 4\lambda g|g|^{\alpha-1} = 0 \tag{4.29}$$

obtained from the similarity reduction of

$$u_t + u^{(\alpha-1)/2} u_x + \lambda u^\alpha = \frac{\delta}{2} u_{xx}, \;\; x \in \mathbb{R}, \;\; t > 0; \tag{4.30}$$

here $\alpha > 1,\ \lambda \in \mathbb{R}$ and $\delta > 0$ (small) are constants. They studied the connection problem for (4.29) with the boundary conditions

$$\begin{aligned} & g(\eta) \sim A\mathrm{e}^{-\eta^2} \eta^{(3-\alpha)/(\alpha-1)} \text{ as } \eta \to \infty, \\ & g \to 0 \text{ as } \eta \to -\infty \;\text{ or } g \to [\lambda(\alpha-1)]^{-1/(\alpha-1)} \text{ as } \eta \to -\infty, \\ & g > 0,\ |g| < \infty \text{ on } (-\infty, \infty). \end{aligned} \tag{4.31}$$

This was initially studied by Sachdev *et al*. [18] while studying the intermediate asymptotic nature of the solutions of ordinary differential equation (4.29) for the initial value problems of (4.30). Sachdev *et al*. [18] observed numerically that for $1 < \alpha < 3$ and $\lambda > 0$, self-similar solutions are the intermediate asymptotics for solutions of (4.30) with initial conditions that have rapid decay at $\pm\infty$.

We explain here the results of Srinivasa Rao *et al*. [24] for the parametric range $\alpha > 3,\ \lambda \in \mathbb{R}$ only. They proved the local existence and uniqueness of solution as $\eta \to \infty$ using a contraction principle by analyzing an integral equation obtained from (4.29) and (4.31). Here A is the shooting parameter. If the solution g of (4.29) is positive on $(-\infty, \infty)$ and $g \to 0$ as $\eta \to -\infty$, then $g' \to 0$ as $\eta \to -\infty$. They showed that if the positive solution g of (4.29) satisfying (4.31) is tending to zero as $\eta \to -\infty$, then

$$\begin{aligned} g &= O(\mathrm{e}^{-\eta^2} |\eta|^{(3-\alpha)/(\alpha-1)}) \;\; \text{or} \\ g &= O(|\eta|^{-2/(\alpha-1)}) \;\; as \;\; \eta \to -\infty. \end{aligned}$$

These expressions for asymptotic behaviors and also characterization of solutions decaying to 0 (as $\eta \to -\infty$) were given in terms of g'/g as in Brezis *et al*. [4].

For all $A > 0$, all the solutions of (4.29) satisfying (4.31) are positive on $\mathbb{R}$ and have algebraic decay as $\eta \to -\infty$ in the parametric range $\alpha > 3,\ \lambda = 0$.

For the parametric range $\alpha > 3,\ \lambda > 0$, three types of solutions of (4.29), (4.31) are possible.

(*i*) $g > 0$ exists on $(-\infty, \infty)$ and $g \to 0$ algebraically as $\eta \to -\infty$.

(*ii*) $g > 0$ exists on $(-\infty, \infty)$ and $g \to \gamma_0$ as $\eta \to -\infty$; here $\gamma_0 = (\lambda(\alpha-1))^{-1/(\alpha-1)}$.

(*iii*) $g > 0$ exists on $[a_0, \infty)$ such that $g(a_0) = \gamma_0$ for some $a_0 \in \mathbb{R}$.

Srinivasa Rao *et al*. [24] characterized these solutions of (4.29) satisfying (4.31) using the function

$$E(\eta) = \frac{g'^2}{2} + \frac{2}{(\alpha-1)}g^2 - \frac{4\lambda}{(\alpha+1)}|g|^{\alpha+1}$$

in the following manner.

(*i*) If $g,\ g' \to 0$, then $E \to 0$ as $\eta \to -\infty$.

(*ii*) If $g \to \gamma_0,\ g' \to 0$ as $\eta \to -\infty$, then $E \to \dfrac{2}{\alpha+1}\gamma_0^2$ as $\eta \to -\infty$.

(*iii*) If $g(\eta_1) = \gamma_0$ for some η_1, then $E(\eta_1) \geq \dfrac{2}{\alpha+1}\gamma_0^2$.

For $A > 0$ sufficiently small, the solutions of (4.29) satisfying (4.31) are positive on $\mathbb{R}$ and decay algebraically to zero as $\eta \to -\infty$ when $\alpha > 3,\ \lambda > 0$. Existence of a solution of (4.29), (4.31) assuming the value $(\lambda(\alpha-1))^{-1/(\alpha-1)}$ at some η_0 was proved using upper and lower solution methods for the same parametric range.

Define

$S_1 = \{A > 0 | g_A(\eta) > 0$ is the solution of (4.29) and (4.31) and $g_A(\eta_1) = \gamma_0$ for some $\eta_1\}$,
$S_2 = \{A > 0 | g_A(\eta) > 0$ is the solution of (4.29) and (4.31) and $g_A(\eta) \to \gamma_0$ as $\eta \to -\infty\}$ and
$S_3 = \{A > 0 | g_A(\eta) > 0$ is the solution of (4.29) and (4.31) and $g_A(\eta) \to 0$ as $\eta \to -\infty\}$.
S_1 and S_3 are shown to be non-empty open sets and $S_1 \cup S_3 \neq (0, \infty)$. This proves the existence of a monotonic solution of (4.29), (4.31) tending to γ_0 as $\eta \to -\infty$.

For $\alpha > 3,\ \lambda < 0$, when A is sufficiently small, solution of (4.29) satisfying (4.31) is positive on $(-\infty, \infty)$ and decays to zero as $\eta \to -\infty$. Their numerical study suggested that for A sufficiently large, solutions are with finite zero. This may lead to the existence of positive solutions with exponential decay at both $+\infty$ and $-\infty$.

We refer to Srinivasa Rao *et al*. [22] for a study of an initial value problem for (4.29) numerically and analytically. A reference may be made to the work of Gmira *et al*. [6], wherein they studied the initial value problem

$$(|u'|^{p-2}u')' + \beta u' + \alpha u - \gamma|u|^{q-1}u|u'|^{p-2}u' = 0,\ \ r > 0, \tag{4.32}$$

$$u(0) = A \text{ and } u'(0) = 0, \tag{4.33}$$

where $A > 0, p > 2, q > 1, \alpha > 0, \beta > 0$ and $\gamma \in \mathbb{R}$. Eq. (4.32) is a generalization of the ordinary differential equation derived from the nonlinear parabolic equation.

$$\begin{aligned} U_t &- (|U_x|^{p-2}U_x)_x = -\frac{k}{t}U + |U|^{q-1}U|U_x|^{p-2}U_x, \\ &- \infty < x < \infty,\ 0 < t < \infty, \end{aligned} \tag{4.34}$$

where $k > 0,\ p > 2$ and $q > 1$. They discussed three aspects-existence and uniqueness of solutions, asymptotic behaviour of positive solutions and classification of solutions. The problem (4.32)–(4.33) is an initial value problem but the analysis fits very well here.

Example 5 — Peletier *et al.* [16] studied the boundary value problem

$$u'' - |u'|^q + \lambda u^p = 0 \quad \text{on } \mathbb{R},\ \ \lambda > 0, \tag{4.35}$$

$$u(x) \geq 0 \text{ on } \mathbb{R},\ u \text{ is non } - \text{ trivial}, \tag{4.36}$$

$$u(x) \to 0 \ \text{ as } \ x \to \pm\infty, \tag{4.37}$$

where p, q are positive constants. This problem arises when dealing with nonnegative radially symmetric equilibrium solutions [called ground states if $u \to 0$ as $x \to \infty$] of the partial differential equation

$$u_t = \Delta u - |\nabla u|^q + \lambda |u|^{p-1}u, \quad \lambda > 0. \tag{4.38}$$

To show the existence and nonexistence of solutions by shooting arguments for the boundary value problem (4.35)-(4.37) in the parametric range $0 < p < \infty$ and $q < \dfrac{2p}{p+1}$, they studied the initial value problem

$$u'' - |u'|^q + u^p = 0 \quad \text{on } \mathbb{R}, \tag{4.39}$$

$$u(0) = \gamma \quad \text{and} \quad u'(0) = 0,\ \ \gamma > 0, \tag{4.40}$$

with the condition

$$xu'(x) < 0 \ \text{ whenever } u \text{ is positive and } x \neq 0. \tag{4.41}$$

In (4.39), λ is taken to be 1 as it can be scaled out when $q \neq \dfrac{2p}{p+1}$. Since the equation is invariant under the transformation $x \to -x$, it suffices to consider the boundary value problem on $[0, \infty)$ only. The solutions of (4.35)–(4.37) satisfying (4.40)–(4.41) are referred to as 'restricted ground states'. Note that γ is the shooting parameter. Further the equation (4.35) is an autonomous, hence it is possible to express the ground states in terms of restricted ground states.

Peletier *et al.* [16] analyzed the initial value problem (4.39)–(4.41) in the phase plane by considering the relevant system of first order differential equations

$$\begin{aligned} u' &= -v, \\ v' &= u^p - v^q \end{aligned}$$

in the region $Q = \{u > 0,\ v \geq 0\}$. Local existence and uniqueness for the above system were proved. They also proved the invariance of the region $\Omega_\lambda = \{0 < u < u_\lambda,\ \ u^\alpha < v < \lambda u^\alpha\}$ for every $\lambda > 1$; this λ is different from that appearing in (4.35), $\alpha = p/q$. They defined ξ to be the supremum of values of x along the orbit (u, v) in Q. Peletier *et al*. [16] showed that

$$(u(x), v(x)) \to (0, v_0) \text{ as } x \to \xi \text{ for some } v_0 \geq 0.$$

When $v_0 = 0$, they characterized the positive solutions of (4.39)–(4.40) using the values of the limit $\lim\limits_{x\to\xi} v(x)/u^\alpha(x) = 1$ or ∞, where $\alpha = p/q$. The latter case corresponds to

$$\lim_{x\to\xi} \frac{v(x)}{u^\beta(x)} = \beta^{-\beta}, \quad \beta = 1/(2-q) \tag{4.42}$$

and corresponding orbit is unique (if such orbit exists). They also gave the explicit asymptotic expressions for v in terms of u when u is small. They considered the sets

$$\begin{aligned} I_0 &= \{\gamma \in (0,\infty) : (u_\gamma, v_\gamma) \text{ ultimately enters one of the invariant domains } \Omega_\lambda\}, \\ I_1 &= \{\gamma \in (0,\infty) : (u_\gamma, v_\gamma) \text{ ultimately enters the domain } v > (u/\beta)^\beta\} \end{aligned}$$

and showed that small values of γ fall in I_0 and large values of γ in I_1. Further I_0 are I_1 are open and disjoint. Since $I_0 \cup I_1 \neq (0,\infty)$, there exists a γ_0 such that the corresponding orbit has the property (4.42). This γ_0 turns out to be unique. They proved that

(*i*) if $u(0) < \gamma_0, u$ has compact support when $0 < p < q$ and u is positive on $(0,\infty)$ for $p \geq q$

(*ii*) if $u(0) = \gamma_0, u$ has a compact support when $q < 1$ and u is positive on $(0,\infty)$ for $q \geq 1$

(*iii*) if $u(0) > \gamma_0, u$ changes the sign.

Finally, they showed that (i) no ground state exists for $q > \dfrac{2p}{p+1}, p > 0$; (ii) when $q = \dfrac{2p}{p+1}$, ground state exists for the boundary value problem (4.35)-(4.37) if and only if $\lambda \leq \dfrac{(2p)^p}{(p+1)^{2p+1}}, p >$ 0.

Example 6 — Kaper, Knaap and Kwong [8] considered the differential equation

$$(|u'|^{p-2}u')' + f(t,u) = 0, \quad t_0 < t < t_1,\ \ p > 1, \tag{4.43}$$

where p is a constant and f is continuous, need not be Lipschitz. While working with p-Laplace equation,

$$\nabla.(|\nabla u|^{p-2}\nabla u) + f(|x|, u) = 0, \quad x \in \Omega \subset \mathbb{R}^N, \quad N > p, \tag{4.44}$$

we come across (4.43). Assuming that (i) f is continuous on $[t_0, t_1] \times [0,\infty)$,

(*ii*) $\lim\limits_{u\to\infty} f(t,u)/u^{p-1} = \infty$, uniformly in t on $[t_0, t_1]$ and (iii) $\lim\limits_{u\to 0^+} f(t,u)/u^{p-1} = l \leq 0$, uniformly in t on $[t_0, t_1]$, Kaper *et al*. [8] showed that the differential equation (4.43) subject to the conditions

$$u'(t_0) = 0, \quad u(t_1) = 0, \tag{4.45}$$

$$u(t) > 0, \;\; t_0 \le t < t_1 \tag{4.46}$$

has a solution. Since f is not assumed to be Lipschitz, uniqueness may not be there. So they used a perturbation technique. They replaced the given problem with a sequence of boundary value problems, which are amenable to analysis using shooting arguments. The existence of the given problem is obtained in the limit.

We also refer the reader to Sachdev [17], Bailey *et al.* [1] and Bernfeld and Lakshmikantham [2] for some more work on shooting arguments.

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