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MATRIX PARTIAL ORDERS AND REVERSAL LAW

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Let A, B be any $m \times n$ matrices such that A is below B under any of the three matrix order relations, namely space pre-order and therefore minus, star and also when matrices are square under the sharp order. We give conditions on matrices that ensure the validity of the reverse order law for various g- inverses.

Key Words: Reversal law; g-inverse; group inverse; Moore-Penrose inverse; range-hermitian matrix; space pre-order; minus order; star order; sharp order

1. INTRODUCTION AND PRELIMINARIES

One of the basic problems in linear algebra is to know when a g- inverse of a product of matrices is the product of g- inverses of the matrices. It is immediate that if for a g- inverse A^- of A and a g- inverse B^- of B, the matrices A^-A and BB^- commute then B^-A^- is a g- inverse of AB. Similarly, if A, B and AB are matrices each of index 1 such that $BB^\#$ and $A^\#A$ commute, then $(AB)^\# = B^\#A^\#$. However, for the Moore-Penrose inverse the problem is well studied and several conditions [2], [3] that imply $B^\dagger A^\dagger = (AB)^\dagger$ are known. In this short note, we show that under very mild conditions on either the matrix or its g- inverse the reverse order law holds for

(i) g-inverse of a product AB of matrices A and B such that $A <^s B$,

(ii) the group inverse of the product AB of matrices A and B such that $A <^\# B$, and

(iii) the Moore-Penrose inverse of the product AB of matrices A and B such that $A <^* B$.

For the sake of completeness, we include the definitions of the various g- inverses and order relations on matrices used here in this note. For details on g- inverses, we refer the reader to [1] and [7], and for properties of order relations to [4], [5] and [6].

Definition 1.1 — (g- inverse [1], [7]). For an $m \times n$ matrix A over a field F, a g- inverse is an $n \times m$ matrix G such that $AGA = A$.

A g- inverse of A that also satisfies $GAG = G$ is called a reflexive g- inverse of it. Moreover, a g- inverse that satisfies $GA = AG$ is called a commuting g- inverse.

Definition 1.2 — ((Group inverse) [1], [7]). For an $n \times n$ matrix A over a field F with index 1, its group inverse $A^{\#}$ is the unique matrix G satisfying (i) $AGA = A$, (ii) $GAG = G$ and (iii) $AG = GA$.

Definition 1.3 — ((Moore-Penrose inverse) [1], [7]). For an $m \times n$ matrix over the field of complex numbers $\mathbb{C}$, its Moore-Penrose inverse $A^{\dagger}$ is the unique $n \times m$ matrix G satisfying (i) $AGA = A$ (ii) $GAG = G$ (iii) $(AG)^* = AG$, and (iv) $(GA)^* = GA$; where X^* denotes the conjugate transpose of a matrix X.

A g-inverse of A that satisfies (iii) is called least square g- inverse of A and a g- inverse that satisfies (iv) is called a minimum norm g-inverse of A.

Definition 1.4 — (Space pre-order [4]). Let A, B be any $m \times n$ matrices over a field F (not necessarily complex). We say A is under B in space pre-order if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(A') \subseteq \mathcal{C}(B')$, where X' denotes the transpose of a matrix X. When this holds we say $A <^s B$.

Note that $A <^s B \Leftrightarrow A = AB^-B = BB^-A$ for some g-inverse B^- of B and therefore, for all g-inverses B^- of B.

Definition 1.5 — (Minus order [4]). Let A, B be any $m \times n$ matrices over a field F (not necessarily complex). We say A is under B in minus order if we can find a g- inverses A^- and $A^\sim$ of the matrix A such that $A^-A = A^-B$, $AA^\sim = BA^\sim$. When this happens, we say $A <^- B$.

We note that $A <^- B \Leftrightarrow A = AB^-B = BB^-A = AB^-A$ for some g- inverse B^- of B

$\Leftrightarrow \mathrm{Rank}(B - A) = \mathrm{Rank}\, B - \mathrm{Rank}\, A$.

Definition 1.6 — (Sharp order [5]). Let A, B be any $n \times n$ matrices over a field F (not necessarily complex). Let A be of index 1. We say A is under B in sharp order if $A^{\#}A = A^{\#}B$, $AA^{\#} = BA^{\#}$. When this holds we say $A <^{\#} B$.

One can easily show that $A <^{\#} B \Leftrightarrow A^2 = AB = BA$.

Definition 1.7 — (Star order [5], [6]). Let A, B be any $m \times n$ matrices over $\mathbb{C}$, the field of complex numbers. We say A is under B in star order if $A^*A = A^*B = B^*A$, $AA^* = BA^* = AB^*$. When this happens we say $A <^* B$.

In fact, $A <^* B \Leftrightarrow A^{\dagger}A = A^{\dagger}B = B^{\dagger}A$, $AA^{\dagger} = BA^{\dagger} = AB^{\dagger}$.

Clearly, (i) minus order implies space pre-order, (ii) sharp order implies space pre-order and (iii) star order implies space pre-order.

Any other unexplained notations and terminology is as in [1], [4], [5]; [6] and [7].

2. Order Relations and Reversal Law for Matrices over a General Field

***Theorem* 2.1** — *Let A, B be any $m \times n$ matrices over an arbitrary field such that $A <^s B$. Then we have the following*:

(i) Let A^- be a commuting g- inverse of A. Then

(a) for any g- inverse B^- of B, B^-A^- is a g- inverse of AB and

(b) for all g- inverses G of A of the form $G = A^- + U(I - AA^-)$ of A with U arbitrary and all g- inverses B^- of B, B^-G is a g- inverse of AB.

(ii) Let B^- be a commuting g- inverse of B. Then

(a) for any g- inverse A^- of A, B^-A^- is a g- inverse of AB and

(b) for all g- inverses H of B of the form $H = B^- + (I - B^-B)V$ with V arbitrary and for all g- inverses A^- of A, HA^- is a g- inverse of AB.

In particular, (i) and (ii) hold when $A <^- B$.

Proof: (i) Let A^- be any commuting g- inverse of A and consider any g- inverse B^- of B. Then,

$$\begin{aligned}(AB)(B^-A^-)(AB) &= (AB)B^-(A^-AB) \\ &= (AB)(B^-AA^-B), \\ &\qquad \text{since } A^- \text{ is a commuting } g\text{- inverse of } A \\ &= A(BB^-A)A^-B.\end{aligned}$$

Since $A <^s B$, $BB^-A = A$, so,

$$(AB)(B^-A^-)(AB) = AAA^-B = (AA^-A)B = AB\,.$$

Now, given any commuting g- inverse of A^- of A, every g- inverse of A is of the form $G = A^- + U(I - AA^-) + (I - A^-A)V$ with matrices U and V of suitable sizes. Therefore, for any g- inverse B^- of B, we have

$$\begin{aligned}AB(B^-G)AB &= AB(B^-A^-)AB + ABB^-U(I - AA^-)AB \\ &\quad + ABB^-(I - A^-A)VAB \\ &= AB + (ABB^- - A)VAB.\end{aligned}$$

Therefore, $AB(B^-G)AB = AB \Leftrightarrow (ABB^- - A)VAB = 0$. Thus, for all g- inverses G of the form $G = A^- + U(I - AA^-)$ of A and all g- inverses B^- of B and, B^-G is a g- inverse of AB.

(ii) Proof is similar to the proof of (i). □

***Theorem* 2.2** — *Let A, B be any $n \times n$ matrices over an arbitrary field such that $A <^s B$. Suppose further that A and B are each of index 1 and $A^\# A$ and B commute. Then*

(i) $B^\# A^\#$ is the group inverse of AB.

(ii) For any $B^- \in \{B^-_{\text{com}}\}$ and $A^- \in \{A^-_{\text{com}}\}, B^-A^- \in \{(AB)^-_{\text{com}}\}$. In other words, $\{B^-_{\text{com}}\}\{A^-_{\text{com}}\} \subseteq \{(AB)^-_{\text{com}}\}$.

PROOF: (i) Now, $AB(B^\# A^\#)AB = (ABB^\#)A^\# AB = AA^\# AB = AB$ and,

$$B^\# A^\# (AB) B^\# A^\# = B^\# A^\# A A^\# = B^\# A^\#.$$

Moreover,

$$AB(B^\# A^\#) = AA^\#$$

and

$$\begin{aligned}(B^\# A^\#)(AB) &= B^\#(A^\# AB) = B^\#(BA^\# A) \\ &= B^\#(BAA^\#) = (B^\# BA)A^\# \\ &= AA^\#.\end{aligned}$$

Thus, $B^\# A^\#$ is the group inverse of AB.

(ii) Any $A^- \in \{A^-_{\text{com}}\}$ is of the form $A^\# + (I - A^\# A)V(I - AA^\#)$ for a suitable matrix V. Similarly, any $B^- \in \{B^-_{\text{com}}\}$ is of the form $B^\# + (I - B^\# B)U(I - BB^\#)$ for a suitable matrix U.

Clearly, $B^- A^-$ is a g- inverse of AB by Theorem 2.1. To complete it is enough to show that $B^\# A^-$ and AB commute. Now,

$$\begin{aligned}B^\# A^- AB &= B^\#[A^\# + (I - A^\# A)V(I - AA^\#)]AB \\ &= B^\#(A^\# AB) \\ &= AA^\#, \text{ by } (i) \quad \text{and} \\ ABB^\# A^- &= (AB)B^\#[A^\# + (I - A^\# A)V(I - AA^\#)] \\ &= A[A^\# + (I - A^\# A)V(I - AA^\#)] \\ &= AA^\#.\end{aligned}$$

***Theorem* 2.3** — *Let A, B be any $n \times n$ matrices over an arbitrary field. If A and B are each of index* 1 *and $A <^\# B$, then*

$$(AB)^\# = B^\# A^\# = A^\# B^\# = (A^2)^\# = (A^\#)^2.$$

PROOF : As $A <^{\#} B$, we have $A^{\#}A = A^{\#}B$, $AA^{\#} = BA^{\#}$ equivalently,

$$AB = BA = A^2,$$

Clearly, $(AB)^2 = A^4$. Since index of A is 1, rank A^n = rank A for each positive integer n. Therefore, rank $(AB)^2$ = rank A^4 = rank A and rank(AB) = rank A^2 = rank A. This implies rank $(AB)^2$ = rank (AB), so, index AB is 1. In view of Theorem 2.2, we only need to show that $A^{\#}A$ and B commute. Now,

$$A^{\#}AB = AA^{\#}B = AA^{\#}A = A$$

and

$$BA^{\#}A = AA^{\#}A = A, \text{so } A^{\#}A \text{ and } B \text{ commute.}$$

As $AB = A^2$ we have $(AB)^{\#} = (A^2)^{\#}$. Taking $B = A$, we obtain $(A^2)^{\#} = (A^{\#})^2$. Since, $A <^{\#} B$, and $A^{\#}$ and $B^{\#}$ are polynomials in A and B respectively, therefore $A^{\#}, B^{\#}$ commute [Theorem 2.5(b), 4].

3. ORDER RELATIONS AND REVERSAL LAW FOR MATRICES OVER COMPLEX FIELD

Before we give our next theorem we recall that a matrix is range-Hermitian [1] (also known as EPr or EP matrix) if $\mathcal{C}(A) = \mathcal{C}(A^*)$; equivalently $A^\dagger A = AA^\dagger$.

***Theorem* 3.1** — *Let A, B be any complex $n \times n$ matrices such that $A <^s B$ and let B be range-Hermitian. Then*

(i) $B^\dagger A^\dagger$ is a reflexive least square g- inverse of AB.

(ii) Every matrix of the form $B^\dagger A^\dagger + (I - B^\dagger A^\dagger AB)V$ with V arbitrary and of suitable size is a least square g- inverse of AB.

(iii) For any $B^- \in \{B_\ell^-\}$ and $A^- \in \{A_\ell^-\}$, $B^-A^- \in \{(AB)_\ell^-\}$.

Thus, $\{B_\ell^-\}\{A_\ell^-\} \subset \{(AB)_\ell^-\}$.

(iv) $A^\dagger B^\dagger$ is a reflexive minimum norm g- inverse of BA.

(v) Every matrix of the form $G = A^\dagger B^\dagger + U(I - BAA^\dagger B^\dagger)$ with U arbitrary and of suitable size is a minimum norm g- inverse of BA.

(vi) For any $B^- \in \{B_m^-\}$ and *$A^- \in \{A_m^-\}$, $A^-B^- \in \{(BA)^-m\}$. In other words, $\{A_m^-\}\{B_m^-\} \subset \{(BA^-)_m\}$.*

PROOF: (i) Consider

$$\begin{aligned}(AB)(B^\dagger A^\dagger)(AB) &= A(BB^\dagger)A^\dagger(AB) = (AB^\dagger B)(A^\dagger AB) \\ &= A(A^\dagger AB) = AB\end{aligned}$$

implying $B^\dagger A^\dagger$ is a g- inverse of AB.

Also,

$$\begin{aligned}(B^\dagger A^\dagger)(AB)(B^\dagger A^\dagger) &= B^\dagger(A^\dagger ABB^\dagger)A^\dagger = B^\dagger(A^\dagger AB^\dagger B)A^\dagger \\ &= B^\dagger(A^\dagger A)A^\dagger = B^\dagger A^\dagger.\end{aligned}$$

Moreover, $(AB)(B^\dagger A^\dagger) = (AA^\dagger)$ is Hermitian. Therefore, $B^\dagger A^\dagger$ is a reflexive least square g-inverse of AB.

(ii) If G is of the form $G = B^\dagger A^\dagger + (I - B^\dagger A^\dagger AB)V$, where V is a matrix of suitable size, then,

$$\begin{aligned}(AB)^*(AB)G &= (AB)^*ABB^\dagger A^\dagger + (AB)^*(I - B^\dagger A^\dagger AB)V \\ &= (AB)^*.\end{aligned}$$

Therefore, the result follows by (i).

(iii) Any $A^- \in \{A_\ell^-\}$ is of the form $A^\dagger + (I - A^\dagger A)V$ for a suitable V and any $B^- \in \{B_\ell^-\}$ is of the form $B^\dagger + (I - B^\dagger B)W$ for a suitable choice of W. Now,

$$\begin{aligned}B^-A^- &= (B^\dagger + (I - B^\dagger B)W)(A^\dagger + (I - A^\dagger A)V) \\ &= B^\dagger A^\dagger + B^\dagger(I - A^\dagger A)V + (I - B^\dagger B)WA^\dagger \\ &\quad + (I - B^\dagger B)W(I - A^\dagger A)V.\end{aligned}$$

Therefore,

$$\begin{aligned}(AB)^*(AB)(B^-A^-) &= (AB)^*(AB)(B^\dagger A^\dagger) + (AB)^*(AB)(B^\dagger(I - A^\dagger A)V) \\ &\quad + (AB)^*(AB)((I - B^\dagger B)W)A^\dagger) \\ &\quad + (AB)^*(AB)((I - B^\dagger B)W(I - A^\dagger A)V) \\ &= (AB)^*.\end{aligned}$$

(iv) Since,

$$BA(A^\dagger B^\dagger)BA = (B(AA^\dagger(B^\dagger BA)) = BAA^\dagger A = BA,$$
$$(A^\dagger B^\dagger)(BA)(A^\dagger B^\dagger) = A^\dagger A(A^\dagger B^\dagger) = A^\dagger B^\dagger$$

and

$$(A^\dagger B^\dagger)BA = A^\dagger A, \text{ a Hermitian matrix.}$$

So, $A^\dagger B^\dagger$ is a minimum norm g- inverse of BA.

(v) Similar to the proof of (ii).

(vi) Note that any $X^- \in \{X_m^-\}$ is of the form $X^\dagger + W(I - XX^\dagger)$ with W of suitable size, rest of the proof follows on the lines of (iii).

***Theorem* 3.2** — *Let A, B be any complex $n \times n$ matrices such that $A <^* B$ and let B be Hermitian. Then $B^\dagger A^\dagger = (AB)^\dagger$.*

PROOF: As $A <^* B$, we have $A <^s B$. Moreover, B is Hermitian implies B is range-Hermitian. Therefore, by Theorem 3.1, $B^\dagger A^\dagger$ is a reflexive least square g- inverse of AB. Moreover, $(B^\dagger A^\dagger)(AB) = B^\dagger(A^\dagger AB) = (B^\dagger A^*) = (AB^\dagger)^*$. Since, $AB^\dagger$ is Hermitian, $(B^\dagger A^\dagger)(AB)$ is Hermitian. So, $B^\dagger A^\dagger$ is the Moore-Penrose inverse of AB.

Remark 1 : It is known that [1, p. 160] for complex $n \times n$ matrices A and B $B^\dagger A^\dagger = (AB)^\dagger \Leftrightarrow \mathcal{C}(A^*AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(BB^*A^*) \subseteq \mathcal{C}(A^*) \Leftrightarrow A^*ABB^*$ is range-Hermitian. It is easy to see that the conditions of Theorem 2.4, imply $\mathcal{C}(A^*AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(BB^*A^*) \subseteq \mathcal{C}(A^*)$. However, if we take the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $A \not<^* B$, B is Hermitian and $\mathcal{C}(A^*AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(BB^*A^*) \subseteq \mathcal{C}(A^*)$.

2. The condition that $A <^* B$ can not be replaced by $A <^- B$, even when A, B are Hermitian as the following example shows:

Take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^\dagger = (1/4)A \text{ and } B^\dagger = B^{-1}.$$

It is easy to check that $A <^- B$, A and B are Hermitian still $(AB)^\dagger \neq B^\dagger A^\dagger$.

3. In Theorem 3.2 the condition "B is Hermitian" can not be replaced with "B is range-Hermitian".

For take

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Clearly, $A <^* B$, B is range Hermitian. Simple computations show that

$$A^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix},$$

$$B^\dagger A^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, (AB)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

So, $(AB)^\dagger \neq B^\dagger A^\dagger$.

It looks like that the range-Hermitian property of the upper matrix is crucial in clinching the results of Theorem 3.1. We give an example in its support.

Example 3.3 — Take A and B as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly, $A^\dagger = A$ and $B^\dagger$ is given by

$$B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, B^\dagger A^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now $A <^s B$, A is range-Hermitian but B is not, $AB(B^\dagger A^\dagger)AB = AB$

and

$B^\dagger A^\dagger AB = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, a non-Hermitian matrix, therefore $B^\dagger A^\dagger$ can not be a minimum norm g- inverse of AB.

***Theorem* 3.4** — *Let $A <^s B$, $B <^s A$ and B, A be both range-Hermitian. Then $B^\dagger A^\dagger$ is the Moore-Penrose inverse of AB.*

PROOF: By Theorems 3.1, $B^\dagger A^\dagger$ is a reflexive least square g- inverse of AB. Moreover,

$$\begin{aligned} (B^\dagger A^\dagger)(AB) &= B^\dagger(A^\dagger AB) \\ &= B^\dagger(AA^\dagger B), \text{ since } A \text{ is range} - \text{Hermitian} \\ &= B^\dagger B\,. \end{aligned}$$

As $B^\dagger B$ is Hermitian, we have $B^\dagger A^\dagger$ is a minimum norm g- inverse of AB. Therefore, $B^\dagger A^\dagger$ is the Moore-Penrose inverse of AB.

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