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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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In this paper, the existence of nonoscillatory solutions of the second-order nonlinear neutral differential equations is studied. Some new sufficient conditions are given and four examples are constructed to show that the results here are almost sharp.

Key Words : Neutral differential equation, positive and negative term, Banach's contraction mapping principle, Krasnoselskii's fixed point theorem

1. INTRODUCTION

In this paper, we consider the second-order nonlinear neutral differential equation with positive and negative terms

$$(x(t) - p(t)x(t-\tau))'' + f_1(t, x(\sigma_1(t))) - f_2(t, x(\sigma_2(t))) = 0, \tag{1.1}$$

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and its corresponding perturbed equation

$$(x(t) - p(t)x(t-\tau))'' + f_1(t, x(\sigma_1(t))) - f_2(t, x(\sigma_2(t))) = g(t) \tag{1.2}$$

as well as the neutral equation with oscillatory coefficients:

$$(x(t) - p(t)x(t-\tau))'' + \sum_{i=1}^{n} A_i(t)x(\rho_i(t)) = 0, \ \ i = 1, 2, \cdots, n. \tag{1.3}$$

We have the following assumptions for eq. (1.1) and eq. (1.2)

(*i*) $p, \sigma_i \in C([t_0, \infty), R)$, $\tau > 0$ is a constant, and $\lim_{t\to\infty} \sigma_i(t) = \infty$, $i = 1, 2$;

(*ii*) $f_i(t, u) \in C([t_0, \infty) \times R, R)$ is nondecreasing on u, and $uf_i(t, u) > 0$, for $u \neq 0$, $i = 1, 2$. Furthermore, f_i satisfies the following Lipschitz condition:

$$|f_i(t, u) - f_i(t, v)| \leq q_i(t)|u - v|, \ \ for\ t \in [t_0, \infty),\ u, v \in [a, b], \tag{1.4}$$

where

$$q_i \in C([t_0, \infty), R^+), \text{ and } \int_t^\infty (s - t)q_i(s)ds < \infty; \tag{1.5}$$

(*iii*) $g \in C([t_0, \infty), R)$, and

$$\int_t^\infty (s - t)|g(s)|ds < \infty, \ \ for\ t \in [t_0, \infty). \tag{1.6}$$

And for eq.(1.3), we assume:

(*iv*) $p,\ \rho_i \in C([t_0, \infty), R)$, $\tau > 0$ is a constant, and $\lim_{t\to\infty} \rho_i(t) = \infty$, $i = 1, 2, \cdots, n$;

(*v*) $A_i \in C([t_0, \infty), R)$ is oscillatory around the zero, and

$$\int_t^\infty (s - t)|A_i(s)|ds < \infty, \ \ for\ t \in [t_0, \infty),\ i = 1, 2, \cdots, n. \tag{1.7}$$

The oscillatory behavior of solutions for second-order neutral and non-neutral differential equations have been studied in many papers [1-11]. However, the second-order neutral equations (1.1) and (1.2) received much less attention due mainly to the technical difficulties arising in its analysis.

The purpose of this paper is to present some new criteria for the existence of non-oscillatory solutions of eq. (1.1) and to derive sufficient conditions for the existence of non-oscillatory solution of eq. (1.2) and eq. (1.3).

Let $m = \min\{\mathrm{t} - \tau, \sigma_1(\mathrm{t}), \sigma_2(\mathrm{t})\}$(or $min\{t - \tau, \rho_i(t),\ i = 1, 2, \cdots, n\}$). By a solution of eq. (1.1), ((1.2) or (1.3)), we mean a function $y \in C([m,\infty), R)$ with $(y(t)-p(t)y(t-\tau)) \in C^2[t_0,\infty)$ and satisfies eq. (1.1), ((1.2) or (1.3)) for $t \geq t_0$.

Let $\phi \in C([m, t_0], R)$ be a given initial function. One can easily see by the method of steps that eq. (1.1), ((1.2) or (1.3)) has a unique solution $y \in C([m,\infty), R)$ such that

$$y(t) = \phi(t), \quad for\ \ m \leq t \leq t_0.$$

As is customary, a nontrivial solution of eq. (1.1), ((1.2) or (1.3)) is said to oscillate if it has arbitrarily large zeros. Otherwise it is non-oscillatory.

Throughout we assume that BC is the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm.

2. Main Results

***Theorem* 2.1** — *Assume that (i) and (ii) hold, if one of the following conditions holds:*

$$(a)\ 0 \leq p(t) \leq p < 1, \qquad (b)\ 1 < p_1 \leq p(t) \leq p_2 < \infty,$$

$$(c)\ -1 < -p \leq p(t) < 0, \quad (d)\ -\infty < -p_1 \leq p(t) \leq -p_2 < -1.$$

Then eq. (1.1) has a unique bounded non-oscillatory solution.

PROOF : The proof of this theorem will be divided into four cases according to the above four different conditions of $p(t)$.

Case 1. $0 \leq p(t) \leq p < 1.$

Set

$$\Omega = \{x \in BC : M_1 \leq x(t) \leq M_2,\, t \geq t_0\},$$

where M_1 and M_2 are positive constants with $M_1 < (1-p)M_2$, $M_2 \leq b$.

It is obvious that Ω is a bounded, closed and convex subset of BC. Choose $\alpha, C > 0$ such that

$$M_1 < \alpha < (1-p)M_2,$$

$$C = min\{\frac{1-p}{3}, \frac{\alpha - M_1}{M_2}, \frac{(1-p)M_2 - \alpha}{M_2}\}. \tag{2.1}$$

In view of (1.5), there exists a $T \geq t_0$ sufficiently large such that

$$\int_t^\infty (s-t)q_i(s)ds \leq C, \quad for\, t \geq T. \tag{2.2}$$

Define a mapping $\Gamma : \Omega \to BC$ as

$$(\Gamma x)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \int_t^{\infty}(s-t)[f_2(s,x(\sigma_2(s))) - f_1(s,x(\sigma_1(s)))]ds, & t \geq T, \\ (\Gamma x)(T), & t_0 \leq t < T. \end{cases}$$

It is easy to see that Γx is continuous. For every $x \in \Omega$ and $t \geq T$, we have $(\Gamma x)(t) \geq \alpha - CM_2 \geq M_1$ and $(\Gamma x)(t) \leq \alpha + pM_2 + CM_2 \leq M_2$. Thus $\Gamma\Omega \subset \Omega$.

For any $x, y \in \Omega$ and $t \geq T$, we have

$$\begin{aligned} |(\Gamma x)(t) - (\Gamma y)(t)| &\leq p|x(t-\tau) - y(t-\tau)| \\ &\quad + \sum_{i=1}^{2} \int_t^{\infty} (s-t)|f_i(s,x(\sigma_i(s)) - f_i(s,y(\sigma_i(s))|ds \\ &\leq (p + \sum_{i=1}^{2} \int_t^{\infty} (s-t)q_i(s)ds)\|x-y\| \\ &\leq (p+2C)\|x-y\|. \end{aligned}$$

Hence $\|\Gamma x - \Gamma y\| \leq (p+2C)\|x-y\|$, where $\|\cdot\|$ denotes the sup norm. In view of $C \leq \dfrac{1-p}{3} < \dfrac{1-p}{2}$, we have $0 < p + 2C < 1$, which implies that Γ is a contraction mapping. By Banach's Contraction Mapping Principle, there exists a unique $x \in \Omega$ such that $\Gamma x = x$. It is easy to see that $x(t)$ is a bounded positive solution of (1.1).

Case 2. $1 < p_1 \leq p(t) \leq p_2 < \infty$.
Set

$$\Omega = \{x \in BC : M_3 \leq x(t) \leq M_4,\, t \geq t_0\},$$

where M_3 and M_4 are positive constants with $(p_2 - 1)M_3 < (p_1 - 1)M_4$, $M_4 \leq b$.

It is obvious that Ω is a bounded, closed and convex subset of BC. Choose $\alpha, C > 0$ such that

$$(p_2 - 1)M_3 < \alpha < (p_1 - 1)M_4,$$

$$C = min\{\frac{p_1 - 1}{3},\ \ \frac{p_1}{p_2} \cdot \frac{\alpha - (p_2-1)M_3}{M_4},\ \ \frac{(p_1-1)M_4 - \alpha}{M_4}\}. \tag{2.3}$$

In view of (1.5), there exists a $T \geq t_0$ sufficiently large such that

$$\int_t^{\infty} (s-t)q_i(s)ds \leq C, \quad for\ t \geq T. \tag{2.4}$$

Define a mapping $\Gamma : \Omega \to BC$ as

$$(\Gamma x)(t) = \begin{cases} \frac{\alpha}{p(t+\tau)} + \frac{x(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int_{t+\tau}^{\infty}(s-t-\tau)[f_1(s,x(\sigma_1(s))) - f_2(s,x(\sigma_2(s)))]ds, & t \geq T, \\ (\Gamma x)(T), & t_0 \leq t < T. \end{cases}$$

It is easy to see that Γx is continuous. For every $x \in \Omega$ and $t \geq T$, we have $(\Gamma x)(t) \geq \frac{\alpha}{p_2} + \frac{M_3}{p_2} - \frac{M_4}{p_1} C \geq M_3$ and $(\Gamma x)(t) \leq \frac{\alpha}{p_1} + \frac{M_4}{p_1} + \frac{M_4}{p_1} C \leq M_4$, thus $\Gamma\Omega \subset \Omega$.

For any $x, y \in \Omega$ and $t \geq T$, we have

$$\begin{aligned}
|(\Gamma x)(t) - (\Gamma y)(t)| &\leq \frac{1}{p(t+\tau)}|x(t+\tau) - y(t+\tau)| \\
&\quad + \frac{1}{p(t+\tau)} \sum_{i=1}^{2} \int_{t+\tau}^{\infty} (s-t-\tau)|f_i(s, x(\sigma_i(s)) - f_i(s, y(\sigma_i(s))|ds \\
&\leq \frac{1}{p_1}[(1 + \sum_{i=1}^{2} \int_{t+\tau}^{\infty} (s-t-\tau) q_i(s) ds] \|x - y\| \\
&\leq \frac{1+2C}{p_1} \|x - y\|.
\end{aligned}$$

Hence $\|\Gamma x - \Gamma y\| \leq \frac{1+2C}{p_1}\|x-y\|$, where $\|\cdot\|$ denotes the sup norm. Since $0 < \frac{1+2C}{p_1} < 1$, Γ is a contraction mapping. By Banach's Contraction Mapping Principle, there exists a unique $x \in \Omega$ such that $\Gamma x = x$. Then $x(t)$ is a bounded positive solution of (1.1).

Case 3. $\quad -1 < -p \leq p(t) \leq 0.$

Set

$$\Omega = \{x \in BC : N_1 \leq x(t) \leq N_2,\ t \geq t_0\},$$

where N_1 and N_2 are positive constants with $N_1 < (1-p)N_2$, $N_2 \leq b$.

Obviously, Ω is a bounded, closed and convex subset of BC. Choose $\alpha, C > 0$ such that

$$N_1 + pN_2 < \alpha < N_2,$$

$$C = \min\{\frac{1-p}{3},\ \frac{\alpha - N_1 - pN_2}{N_2},\ \frac{N_2 - \alpha}{N_2}\}.$$

In view of (1.5), there exists a $T \geq t_0$ large enough such that

$$\int_t^{\infty} (s-t) q_i(s) ds \leq C, \quad for\ t \geq T.$$

Define a mapping $\Gamma : \Omega \to BC$ as

$$(\Gamma x)(t) = \begin{cases} \alpha + p(t)x(t-\tau) + \int_t^{\infty} (s-t)[f_2(s, x(\sigma_2(s))) - f_1(s, x(\sigma_1(s)))]ds, & t \geq T, \\ (\Gamma x)(T), & t_0 \leq t < T. \end{cases}$$

The next steps are the same as in case 1.

Case 4. $-\infty < -p_1 \le p(t) \le -p_2 < -1.$

Set

$$\Omega = \{x \in BC : N_3 \le x(t) \le N_4,\, t \ge t_0\},$$

where N_3 and N_4 are positive constants with $(p_2 - \frac{p_1}{p_2})N_4 > (p_1 - \frac{p_2}{p_1})N_3$, $N_4 \le b$.

It is easy to see that Ω is a bounded, closed and convex subset of BC. Choose $\alpha, C > 0$ such that

$$\frac{p_1}{p_2}N_4 + p_1 N_3 < \alpha < \frac{p_2}{p_1}N_3 + p_2 N_4,$$

$$C = \min\{\frac{p_2 - 1}{3},\ \frac{1}{N_4}\cdot(\frac{p_2}{p_1}\alpha - N_4 - p_2 N_3),\ \frac{1}{N_4}(\frac{p_2}{p_1}N_3 - \alpha + p_2 N_4)\}.$$

In view of (1.5), there exists a $T \ge t_0$ large enough such that

$$\int_t^\infty (s-t)q_i(s)ds \le C, \ \ for\ t \ge T.$$

Define a mapping $\Gamma : \Omega \to BC$ as follows:

$$(\Gamma x)(t) = \begin{cases} -\frac{\alpha}{p(t+\tau)} + \frac{x(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)}\int_{t+\tau}^\infty (s-t-\tau)[f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))]ds & t \ge T \\ (\Gamma x)(T) & t_0 \le t < T \end{cases}$$

The next steps are the same as in case 2. The proof of Theorem 2.1 is completed. □

***Theorem* 2.2** — *Assume that (i)-(iii) and one of the four conditions (a)-(d) hold, then eq. (1.2) has a bounded nonoscillatory solution.*

PROOF : Let $g_+(t) = \max\{g(t), 0\}$, and $g_-(t) = \max\{-g(t), 0\}$, then $g(t) = g_+(t) - g_-(t)$, and eq. (1.2) is equivalent to

$$(x(t) - p(t)x(t-\tau))'' + [f_1(t, x(\sigma_1(t))) + g_-(t)] - [f_2(t, x(\sigma_2(t))) + g_+(t)] = 0. \qquad (1.2')$$

It is easy to show that (1.2′) satisfies all the conditions of Theorem 2.1, so we omit the proof. □

***Theorem* 2.3** — *Assume that (iv), (v) and one of the four conditions (a)-(d) in Theorem 2.1 hold, then eq. (1.3) has a bounded nonoscillatory solution.*

PROOF : Without loss of generality we suppose that the condition (a) holds.

Set

$$\Omega = \{x \in BC : M_1 \le x(t) \le M_2,\, t \ge t_0\},$$

where M_1 and M_2 are positive constants with $M_1 < (1-p)M_2$, $M_2 \le b$.

It is obvious that Ω is a bounded, closed and convex subset of BC. Choose $\alpha, C > 0$ such that

$$M_1 < \alpha < (1-p)M_2,$$

$$C = min\{\frac{\alpha - M_1}{nM_2}, \frac{(1-p)M_2 - \alpha}{nM_2}\}. \tag{2.5}$$

In view of (1.7), there exists a $T \geq t_0$ large enough such that

$$\int_T^\infty (s-t)|A_i(s)|ds \leq C, \ \ for\ t \geq T. \tag{2.6}$$

Define operators Γ_1 and Γ_2 on Ω as

$$(\Gamma_1 x)(t) = \begin{cases} \alpha + p(t)x(t-\tau), & t \geq T, \\ (\Gamma_1 x)(T), & t_0 \leq t < T; \end{cases}$$

$$(\Gamma_2 y)(t) = \begin{cases} -\int_t^\infty (s-t)\sum_{i=1}^n A_i(s)y(\sigma_i(s))ds, & t \geq T, \\ (\Gamma_2 x)(T), & t_0 \leq t < T. \end{cases}$$

In view of (2.5) and (2.6), for any $x, y \in \Omega$ and $t \geq T$, we have

$$(\Gamma_1 x)(t) + (\Gamma_2 y)(t) \geq \alpha - nM_2 C \geq \alpha - nM_2 \tfrac{\alpha - M_1}{nM_2} = M_1,$$

$$(\Gamma_1 x)(t) + (\Gamma_2 y)(t) \leq \alpha + pM_2 + nM_2 C = M_2.$$

Thus $\Gamma_1 x + \Gamma_2 y \in \Omega$ for any $x, y \in \Omega$. From $0 \leq p(t) \leq p < 1$, we know that Γ_1 is a contraction mapping. We show now that Γ_2 is continuous. In fact, choose $\{y_k\}_{k=1}^\infty \subset \Omega$ such that for every fixed t, $\|y_k - y\| \to 0$ as $k \to \infty$. In view of (2.5), we get

$$\|(\Gamma_2 y_k)t - (\Gamma_2 y)t\| \leq \sup_{t \geq t_0} \|(\Gamma_2 y_k)t - (\Gamma_2 y)t\| \leq nC\|y_k - y\| \to 0, \ \ as\ k \to \infty,$$

which implies that Γ_2 is continuous. We can also show that

$$|\frac{d(\Gamma_2 y)t}{dt}| = |\int_t^\infty \sum_{i=1}^n A_i(s)y(\sigma_i(s))ds| \leq |\int_t^\infty s\sum_{i=1}^n A_i(s)y(\sigma_i(s))ds| \leq nCM_2, \quad for\ t \geq T$$

and

$$|\frac{d(\Gamma_2 y)t}{dt}| = 0, \ \ for\ \ t_0 \leq t < T.$$

Hence $\Gamma_2\Omega$ is equi-continuous.

On the other hand, it is easy to see that the family of $\Gamma_2\Omega$ is uniformly bounded. Therefore, Γ_2 is completely continuous. By Krasnoselskii's fixed point theorem, there exists $x \in \Omega$, such that $\Gamma_1 x + \Gamma_2 x = x$. Clearly, $x(t)$ is a bounded positive solution of eq. (1.3). □

3. ILLUSTRATIVE EXAMPLES

In this section, four examples are offered to illustrate our results.

Example 3.1 — Consider

$$(x(t)-\frac{1}{t}x(t-1))''+\frac{8(t-2)^3}{t^3(2t-5)^3}x^3(t-2)-\frac{2(t-3)^{\frac{3}{2}}}{(t-1)^3(2t-7)^{\frac{3}{2}}}x^{\frac{3}{2}}(t-3)=0,\ \ (t\geq 4) \quad (3.1)$$

where $p=\frac{1}{t}\leq\frac{1}{4}<1$, $\tau=1$, $\sigma_1(t)=t-2$, $\sigma_2(t)=t-3$, $f_1(t,u)=\dfrac{8(t-2)^3}{t^3(2t-5)^3}u^3$, and $f_2(t,u)=\dfrac{2(t-3)^{\frac{3}{2}}u^{\frac{3}{2}}}{(t-1)^3(2t-7)^{\frac{3}{2}}}$, for $u,v\in[1,2]$. Furthermore,

$$|f_1(t,u)-f_1(t,v)|\leq\frac{96(t-2)^3}{t^3(2t-5)^3}|u-v|,$$

$$|f_2(t,u)-f_2(t,v)|\leq\frac{12(t-3)^{\frac{3}{2}}}{(t-1)^3(2t-7)^{\frac{3}{2}}}|u-v|.$$

Then we have

$$\int_4^\infty\frac{96(t-2)^3(t-4)}{t^3(2t-5)^3}dt<\infty,\quad and\ \int_4^\infty\frac{12(t-3)^{\frac{3}{2}}(t-4)}{(t-1)^3(2t-7)^{\frac{3}{2}}}dt<\infty.$$

It is easy to see that the conditions of Theorem 2.1 are all satisfied. Therefore, eq. (3.1) has a bounded nonoscillatory solution. In fact, $x(t)=2-\dfrac{1}{t}$ is such a solution.

Example 3.2 — Consider

$$(x(t)-(2+\frac{1}{t})x(t-1))''+\frac{6(t-2)}{t^3(t-3)}x(t-2)-\frac{6\sqrt{t-4}}{(t-1)^3\sqrt{t-5}}\sqrt{x(t-4)}=0,\ \ (t\geq 6)\quad (3.2)$$

where $p(t)=2+\frac{1}{t}>1$, $f_1(t,u)=\dfrac{6(t-2)}{t^3(t-3)}u$, $f_2(t,u)=\dfrac{6\sqrt{t-4}}{(t-1)^3\sqrt{t-5}}\sqrt{u}$. We can show that the conditions of Theorem 2.1 are all satisfied. Therefore, eq. (3.2) has a bounded nonoscillatory solution. In fact, $x(t)=1-\dfrac{1}{t}$ is such a solution.

Example 3.3 — Consider

$$(x(t)+\frac{1}{t}x(t-1))''+\frac{2(t-2)^5}{(t-1)^3(t-3)^5}x^5(t-2)-\frac{2(t-1)^3}{t^3(t-2)^3}x^3(t-1)=0,\ \ (t\geq 4)\quad (3.3)$$

where $-1<p(t)=-\frac{1}{t}\leq 0$, $f_1(t,u)=\dfrac{2(t-2)^5}{(t-1)^3(t-3)^5}u^5$, $f_2(t,u)=\dfrac{2(t-1)^3}{t^3(t-2)^3}u^3$. We can show that the conditions of Theorem 2.1 are all satisfied. Therefore, eq. (3.3) has a bounded nonoscillatory solution. In fact, $x(t)=1-\dfrac{1}{t}$ is such a solution.

Example 3.4 — Consider

$$(x(t) + 2x(t-2))'' + \frac{t-1}{t^3(2t-1)}x(t-1) - \frac{4x(t-2)}{(t-2)^2(2t-3)} = \frac{3}{t^3}, \quad (t \geq 4) \tag{3.4}$$

where $p(t) = 2 > 1$, $f_1(t,u) = \dfrac{t-1}{t^3(2t-1)}u$, $f_2(t,u) = \dfrac{4u}{(t-2)^2(2t-3)}$. We can check that the conditions of Theorem 2.2 are all satisfied. Hence eq. (3.4) has a bounded nonoscillatory solution. In fact, $x(t) = 2 + \dfrac{1}{t}$ is such a solution.

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