

SINGULAR SELF-ADJOINT BOUNDARY CONDITIONS FOR PRODUCTS OF STURM-LIOUVILLE DIFFERENTIAL OPERATORS

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(Received 2 July 2000; after revision 21 February 2002; accepted 21 June 2002)

In this paper, the second-order symmetric Sturm-liouville differential expressions $\tau_1, \tau_2, \dots, \tau_n$ with real coefficients are considered on the interval $I = (a, b)$, $-\infty \leq a < b \leq \infty$. It is shown that the characterization of singular self-adjoint boundary conditions involves the sesquilinear form associated with the product of Sturm-Liouville differential expressions and elements of the maximal domain of the product operators, and it is an exact parallel of the regular case. This characterization is an extension of those obtained in [6], [8], [11], [12], [14] and [15].

Key Words : Sturm-Liouville Differential Expressions; Product Differential Expressions; Deficiency Indices; Integrable Square Solutions; Regular and Singular End-Points; Boundary Conditions

1. INTRODUCTION

Krall and Zettl¹² considered the Sturm-Liouville differential expression

$$\tau[u] = [-(p u')' + qu] \text{ on } I = (a, b) - \infty \leq a < b \leq \infty, \quad \dots (1.1)$$

with real-valued Lebesgue measurable functions p and q assumed to satisfy the following basic conditions :

$$p^{-1}, q \in L_{loc}(I), w(x) > 0 \text{ a.e.}, \quad \dots (1.2)$$

and proved that the characterization of the singular self-adjoint boundary conditions is identical to that in the regular case provided that u and $p u'$ are replaced by certain Wronskians involving u and two linearly independent solutions of $\tau[u] = 0$.

The relationship between the deficiency index of a symmetric differential expression (1.1) and its powers $\tau^2, \tau^3, \dots$ has recently been studied by Chaudhuri and Everitt¹ and the relationship between the number of linearly independent $L^2(0, \infty)$ solutions of the equations $\tau_j[u] = 0$ and of the product equation $(\tau_1 \tau_2 \dots \tau_n)u = 0$ has been investigated by Everitt⁶. These results are extension of those recently obtained in [5, 16, 17, 18] for the special case $\tau_j = \tau$ for $j = 1, \dots, n$ and τ is a real second-order symmetric differential expression.

Our objective in this paper is to show that, the characterization of the singular self-adjoint boundary conditions is identical to that in the regular case provided that u and its quasi-derivatives

are replaced by sesquilinear forms associated with the product of Sturm-Liouville differential expressions, involving u and elements of the maximal domain of the product operators. This characterization is an extension of those by Everitt and Zettl⁶ and in [7, 8 & 11-15].

In the regular case, these conditions can be interpreted as linear combinations of the values of the unknown function u and its quasi-derivatives at the end-points a and b .

In the singular case, these conditions are given in terms of sesquilinear forms involving u and linearly independent solutions of the product equation $(\tau_1 \tau_2 \dots \tau_n) u = 0$ which given by Everitt and Zettl in [6].

2. PRELIMINARIES

We begin with a brief summary of adjoint pairs of operators and products operators, a full treatment may be found in [2, Chapter III].^{5-7 & 9-11}

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T , written $nul(T)$, is the dimension of $N(T)$ and the deficiency of T , written $def(T) = nul(T^*)$. The Fredholm domain of T is (in the notation of [2]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values $x \in \mathbb{C}$ which are such that $(I - \lambda T)$ is a Fredholm Operator. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency, I being the identity operator on H .

A closed operator A in a Hilbert space H has Property (C), if it has closed range and $\lambda = 0$ is not an eigenvalue; i.e., there is some positive number r such that $\|Ax\| \geq r\|x\|$ for all $x \in D(A)$.

Note that, Property (C) is equivalent to $\lambda = 0$ being a regular type point of A . This in turn is equivalent to the existence of A^{-1} as a bounded operator on the range of A (which need not be all of H).

Given two operators A and B , both acting in a Hilbert space H , we wish to consider the product operator AB . This is defined as follows

$$D(AB) = \{x \in D(A) \mid Bx \in D(A)\} \text{ and } (AB)x = A(Bx) \text{ for all } x \in D(AB).$$

It may happen in general that $D(AB)$ contains only the null element of H . However, in the case of many differential operators the domains of the product will be dense in H .

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors.

Lemma 2.1 (cf. [6, 17, Theorem A]) — Let A and B be closed operators with dense domains in a Hilbert space H . Suppose that $\lambda = 0$ is a regular type point for both operators and $def A$ and $def B$ are finite. Then AB is a closed operator with dense domain, has $\lambda = 0$ as a regular type point and

$$def AB = def A + def B.$$

Evidently Lemma 2.1 extends to the product of any finite number of operators $A_1, A_2, \dots, A_n$.

Let the interval I have end-points a, b ($-\infty \leq a < b \leq \infty$), and let $w : I \rightarrow \mathbb{R}$ be a non-negative weight function with $w \in L_{loc}^1(I)$ and $w(x) > 0$ (for almost all $x \in I$). Then $H = L_w^2(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_I w |f|^2 < \infty$; the inner-product is defined by :

$$(f, g) := \int_I w(x) f(x) \overline{g(x)} dx \quad (f, g \in L_w^2(I)).$$

We shall consider the Sturm-Liouville differential equation of the form :

$$\tau[u] = -p u' + qu = \lambda w u \text{ on } I, \quad \dots (2.1)$$

where the real-valued Lebesgue measurable functions p, q and w from I into $\mathbb{R}$ satisfying the conditions (1.2) which are taken to hold throughout this paper. Under this assumptions, τ is interpreted as a quasi-differential expression, u being a solution of (2.1) if u and $p u'$ are in $AC_{loc}(a, b)$, the space of functions which are absolutely continuous on compact subsets of (a, b) , and (2.1) is satisfied almost everywhere on (a, b) . Also, $p u' = u^{[1]}$ is called the quasi-derivative of u .

Eq. (2.1) is said to be regular at the left end-point $a \in \mathbb{R}$, if for all $X \in (a, b)$,

$$a \in \mathbb{R}; p^{-1}, q, w \in L^1(a, X),$$

Otherwise (2.1) is said to be singular at a . If (2.1) is regular at both end-points a and b , then it is said to be regular, in this case we have,

$$a, b \in \mathbb{R}; p^{-1}, q, w \in L^1(a, b)$$

We will be concerned with the second-order symmetric differential expression on I and when both end-points a and b may be either regular or singular end-points of (2.1). Note that, in view of (1.2), an end-point of I is regular for (2.1), if and only if it is regular for the equation,

$$\tau^+[v] = \lambda w v \quad (\lambda \in \mathbb{C}) \text{ on } I, \quad \dots (2.2)$$

where τ^+ is the formal, or Lagrangian adjoint of τ given by :

$$\tau^+[v] = -(p v') + q v \text{ on } I. \quad \dots (2.3)$$

The maximal domain $D(\tau)$ is defined by,

$$D(\tau) : \{f : f, p f' \in AC_{loc}(I) \text{ and } w^{-1} \tau[f] \in L_w^2(a, b)\},$$

is a subspace of $L_w^2(a, b)$. The maximal operator $T(\tau)$ is defined by,

$$T(\tau) u := w^{-1} \tau[u] \quad u \in D(\tau).$$

It is well known that $D(\tau)$ is dense in $L_w^2(a, b)$; see [9], [10], [11] and [12].

In the regular problem the minimal operator $T_0(\tau)$ is the restriction of $w^{-1}\tau[u]$ to the subspace:

$$D_0(\tau) := \{u : u \in D(\tau), u^{[r-1]}(a) = u^{[r-1]}(b) = 0, r = 1, 2\}, \quad \dots (2.4)$$

The subspace $D_0(\tau)$ is dense and closed in $L_w^2(a, b)$; see [2], [15] and [19].

In the singular problem we first introduce the operator $T'_0(\tau)$; $T'_0(\tau)$ being the restriction of $w_p^{-1}\tau[\cdot]$ to the subspace

$$D'_0(\tau) := \{u : u \in D(\tau), \text{supp } u \subset (a, b)\}. \quad \dots (2.5)$$

This operator is densely-defined and closable in $L_w^2(a, b)$; and we defined the minimal operator $T_0(\tau)$, to be its closure (see [2], [15] and [19, Section 5]). We denote the domain of $T_0(\tau)$ by $D_0(\tau)$. It can be shown that :

$$u \in D_0(\tau) \Rightarrow u^{[r-1]}(a) = 0, (r = 1, 2),$$

whenever we assume a to be regular end-point and b to be singular end-point.

For $f, g \in D(\tau)$ and $\alpha, \beta \in I$, Green's formula is given by :

$$\int_{\alpha}^{\beta} \{ \tau[f] \bar{g} - f \overline{\tau[g]} \} dx = [f, g](\beta) - [f, g](\alpha), \quad \dots (2.6)$$

where $[f, g] := f \bar{g}^{[1]} - f^{[1]} \bar{g}, f, g \in D(\tau). \quad \dots (2.7)$

For $f, g \in D(\tau)$, the limits $\lim_{\alpha \rightarrow a^+} [f, g](\alpha)$ and $\lim_{\beta \rightarrow b^-} [f, g](\beta)$ exist and are finite. These are denoted by $[f, g](a)$ and $[f, g](b)$, respectively.

For $f, g \in AC_{loc}(a, b)$, let

$$W(f, g) = fp'g' - gpf'. \quad \dots (2.8)$$

Choose two solutions θ and ϕ of $\tau[u] = 0$ satisfying,

$$W(\theta, \phi)(x) = 1 \text{ for all } x \in I. \quad \dots (2.9)$$

Clearly such θ and ϕ exist, i.e., they can be determined by the initial conditions:

$$\theta(c) = 1, (p\theta')(c) = 1, \phi(c) = 0, (p\phi')(c) = 1 \text{ for all } c \text{ in } I.$$

Note that, the sesquilinear form $[f, g]$ in (2.7) can be written as :

$$\begin{aligned} [f, g] &= f p \bar{g}' - \bar{g} p f' \\ &= (\bar{g}, p \bar{g}') \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ p f' \end{pmatrix}. \end{aligned} \quad \dots (2.10)$$

From (2.8) and (2.9), we get

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta & \phi \\ p \theta' & p \phi' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta & p \theta' \\ \phi & p \phi' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and hence the sesquilinear form in (2.10) can also be written as:

$$\begin{aligned} [f, g] &= (W(\bar{g}, \theta), W(\bar{g}, \phi)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W(f, \theta) \\ W(f, \phi) \end{pmatrix} \\ &= W(\bar{g}, \phi) W(f, \theta) - W(\bar{g}, \theta) W(f, \phi) \\ &= \det \begin{pmatrix} W(f, \theta) & W(f, \phi) \\ W(\bar{g}, \theta) & W(\bar{g}, \phi) \end{pmatrix}; \end{aligned} \quad \dots (2.11)$$

see [11] and [12].

Lemma 2.2 — If for some $\lambda_0 \in \mathbb{C}$, there are two linearly independent solutions of $\tau[u] = \lambda_0 w u$ in $L_w^2(a, b)$. Then all solutions of $\tau[u] = \lambda w u$ are in $L_w^w(a, b)$ for all $\lambda \in \mathbb{C}$; see [2, Chapter 3] for more details.

Theorem 2.3 — (cf. Theorem 3.10.1) — Let $f \in L_{loc}^1(a, b)$ and suppose that the conditions (1.2) are satisfied. Then given any complex numbers c_0 and c_1 and any $x_0 \in (a, b)$ there exist a unique solution of $\tau[\phi] = f$ in (a, b) which satisfies $\phi(x_0) = c_0, \phi^{[1]}(x_0) = c_1$.

A simple consequence of Theorem 2.3 is that the solution of (2.1) form a 2-dimensional vector space over $\mathbb{C}$. If (α_0, α_1) and (β_0, β_1) are linearly independent vectors in $\mathbb{C}^2$ then the solutions $\phi_1(\cdot, \lambda), \phi_2(\cdot, \lambda)$ of the eq. (2.1) which satisfy $\phi_1(x_0, \lambda) = \alpha_0, \phi_1^{[1]}(x_0, \lambda) = \alpha_1, \phi_2(x_0, \lambda) = \beta_0, \phi_2^{[1]}(x_0, \lambda) = \beta_1$ for some $x_0 \in (a, b)$, form a basis for the space of solutions of the eq. (2.1).

Note that, an important distinction between a regular end-point and a singular end-points is the fact that at a regular end-point x_0 , all initial value problems $\phi(x_0, \lambda) = c_0, \phi^{[1]}(x_0, \lambda) = c_1, c_0, c_1 \in \mathbb{C}$ have a unique solutions. This is not true when x_0 is singular end-point (see [2] and [10]).

Assume that a and b are singular end-points. For any α and β in the open interval (a, b) and any $\lambda \in \mathbb{C}$, the conditions (1.2) imply that any solution ϕ of the eq. (2.1) is in $L_w^2(a, b)$. If ϕ is in $L_w^2(a, \beta)$ for some β in (a, b) , then we say that $\tau[\cdot]$ is in the limit-circle case at a , or simply that a is *LC*. Otherwise, $\tau[\cdot]$ is in the limit-point case at a or a is *LP*. Similarly, b is *LC* means that all solutions of (2.1) are in $L_w^2(\alpha, b)$, $2 < \alpha < b$. This classification is independent of λ in (2.1); (see [11], [12], [15] and [18]). Otherwise b is *LP*. The limit-point, limit circle terminology are used for historical reasons.

The classification of the self-adjoint extensions of $T_0(\tau)$ depends, in an essential way, on the deficiency index of $T_0(\tau)$. We briefly recall the definition of this notion for abstract symmetric operators in a separable Hilbert space.

A linear operator A from a Hilbert space H into H is said to be symmetric if its domain $D(A)$ is dense in H and $(Af, g) = (f, Ag)$ for all $f, g \in D(A)$. Any such operator has associated with it a pair (d^+, d^-) , where each of d^+, d^- is a non-negative or $+\infty$. The extended integers are called the deficiency indices of A and we have the following :

For $\lambda \in \mathbb{C}$, the set of complex numbers, let R_λ denote the range of $T_0(\tau) - \bar{\lambda}I$, $N_\lambda = R_\lambda^\perp$ and let

$$N^+ = N_i, N^- = N_{-i}, i = \sqrt{-1}; \quad \dots (2.12)$$

$d^+ = \text{dimension of } N^+$ and $d^- = \text{dimension of } N^-$. The spaces N^+, N^- are called the deficiency spaces of $T_0(\tau)$ and d^+, d^- are called the deficiency indices of $T_0(\tau)$. These are related to the eq. (2.1) as follows :

$$N_\lambda = \{f \in D[T_0^*(\tau)] \mid [T_0^*(\tau)]f = [T(\tau)]f = w^{-1} \tau[f] = \lambda f\}.$$

Thus, N^+, N^- consist of the solutions of the eq. (2.1) which lie in the space $H = L_w^2(I)$ for $\lambda = -i$, respectively. It is clear for a symmetric differential operator $T_0(\tau)$ that :

$$0 \leq d^+ = d^- \leq 2. \quad \dots (2.13)$$

We denote the common value by d and call d the deficiency index of τ on I . From the above discussion we see that there are only three possibilities for d as : $d = 0, 1, 2$.

Note that, in the literature the maximal and minimal deficiency cases are often referred to as the limit-circle and limit-point cases. Strictly these latter terms are only suitable for the now classical second order differential expressions; in this case the terminology was originally introduced by Hermann Weyl. The term limit-point does give an acceptable description of the minimal deficiency case for real, and hence even-order, symmetric expressions.

Now, we recall the following results :

For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and for a symmetric differential operator $T_0(\tau)$, we have from the general theory that,

$$D(\tau) = D_0(\tau) \dot{+} N^+ \dot{+} N^-, \quad \dots (2.14)$$

where $D_0(\tau)$, N^+ and N^- are linearly independent subspaces and the sum is direct (which we indicate with the symbol $\dot{+}$); see [2], [7], [11] and [15].

Any self-adjoint extension S of the symmetric differential operator $T_0(\tau)$, satisfies

$$T_0(\tau) \subset S = S^* \subset T_0^*(\tau)$$

and hence is completely determined by specifying its domain $D(S)$

$$D_0(\tau) \subset D(S) \subset D(\tau).$$

This can be proved using formula (2.13); (see [1], [2], [7], [11] and [15]).

Theorem 2.4 — *The operator $T_0(\tau)$ is a closed symmetric operator from H into H and*

$$T_0^*(\tau) = T(\tau) \text{ and } T^*(\tau) = T_0(\tau), \quad D(\tau) = \text{domain of } T(\tau), \quad \dots (2.15)$$

PROOF : See [11], [12] and [15, Section 17.4]

Some of the basic facts are summarized in :

Theorem 2.5 — (cf [12, Proposition 1])

(a) $D_0(\tau) = \{f \in D(\tau) : [f, g](b) - [f, g](a) = 0 \text{ for all } f, g \in D(\tau)\}.$

(b) If $\tau[\cdot]$ is in the limit point case at an end-point c , then $[f, g](c) = 0$ for all $f, g \in D(\tau)$, $c = a$ or $c = b$.

(c) If an end-point c is regular, then for any solution u , u and $u^{[1]}$ are continuous at c .

(d) If a and b are both regular end-points, then for any $\alpha, \beta, \gamma, \delta$ in $\mathbb{C}$, there exists a function f in $D(\tau)$ such that

$$f(a) = \alpha, f^{[1]}(a) = \beta,$$

$$f(b) = \gamma, f^{[1]}(b) = \delta.$$

(e) If a is regular and b singular, then a function f from $D(\tau)$ is an $D_0(\tau)$ if and only if the following conditions are satisfied :

$$(i) f(a) = 0, f^{[1]}(a) = 0,$$

$$(ii) [f, g](b) = 0 \text{ for all } f, g \in D(\tau).$$

The analogous results holds when a is singular and b is regular; also see [8], [10] and [12].

Lemma 2.6 (cf. [11] and [12, Lemma 2]) — Given α, β, γ and δ in $\mathbb{C}$, then there exists a $\Psi \in D(\tau) D_0(\tau)$ such that

$$W(\Psi, \theta)(a) = \alpha, \quad W(\Psi, \phi)(a) = \beta,$$

$$W(\Psi, \theta)(b) = \gamma, \quad W(\Psi, \phi)(b) = \delta,$$

where θ and ϕ are solutions of $\tau[u] = 0$ satisfying (2.9). Furthermore, Ψ can be taken to be a linear combination of θ and ϕ near each end-points.

3. SOME TECHNICAL LEMMAS

The proof of general theorem will be based on the results in this section. We start by listing some properties and results of Sturm-liouville differential expressions $\tau_1, \tau_2, \dots, \tau_n$ each of order two. For proofs the reader is referred to [6], [9], [10], [11], [16], [17] and [18].

$$(\tau_1 + \tau_2)^+ = \tau_1^+ + \tau_2^+ \quad \dots (3.1)$$

$$(\tau_1 \tau_2)^+ = \tau_2^+ \tau_1^+, (\lambda \tau)^+ = \bar{\lambda} \tau^+ \text{ for } \lambda \text{ a complex number.} \quad \dots (3.2)$$

A consequence of Properties (3.1) and (3.2) is that if $\tau^+ = \tau$ then $P(\tau)^+ = P(\tau^+)$ for P any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence the product of regular differential expressions is regular. The next Lemma shows under conditions that the deficiency indices of a product is the sum of the deficiencies of the factors.

Lemma 3.1 (cf. [6, Theorem 1]) — Suppose τ_j is a regular differential expression on the interval $[a, b]$ such that the minimal operator $T_0(\tau_j)$ has Property (C) for $j = 1, 2, \dots, n$. Then

(i) The product operator $\prod_{j=1}^n [T_0(\tau_j)]$ is closed, have dense domain, Property (C), and

$$\text{def} \left[\prod_{j=1}^n T_0(\tau_j) \right] = \sum_{j=1}^n \text{def} [T_0(\tau_j)]; \quad \dots (3.3)$$

also (ii) $[T_0(\tau_1 \tau_2 \dots \tau_n)] \subseteq \prod_{j=1}^n [T_0(\tau_j)]$.

In part (ii) the containment may be proper, i.e., the operators $T_0(\tau_1 \tau_2 \dots \tau_n)$ and $\prod_{j=1}^n [T_0(\tau_j)]$ are not equal in general.

Note that, for symmetric differential operator $T_0(\tau_j)$ which satisfies Property (C) and by (2.13), then (3.3) is constant on $[0, 2n]$. In the problem with one singular end-point this constant is in $[n, 2n]$, while in the regular problem it is equal $2n$; see [2].

Lemma 3.2 (cf. [6, Theorem 2]) — Let $\tau_1, \tau_2, \dots, \tau_n$ be regular differential expressions on $[a, b]$. Suppose that $T_0(\tau_j)$ satisfies Property (C) for $j = 1, 2, \dots, n$. Then

$$T_0(\tau_1 \tau_2 \dots \tau_n) = \prod_{j=1}^n T_0(\tau_j) \quad \dots (3.4)$$

if and only if the following partial separation condition is satisfied.

$\{f \in L_w^2(a, b), f^{[s-1]} \in AC_{loc}[a, b], \text{ where } s \text{ is the order of product expression } (\tau_1 \tau_2 \dots \tau_n) \text{ and } (\tau_1 \tau_2 \dots \tau_n)^+ f \in L_w^2(a, b) \text{ together imply that}$

$$\left(\prod_{j=1}^+ (\tau_j^+) \right) f \in L_w^2(a, b), k = 1, \dots, n-1. \quad \dots (3.5)$$

Therefore (3.4) and (3.5) are equivalent.

We will say that the product $(\tau_1 \tau_2 \dots \tau_n)$ is a partially separated expressions in $L_1^2(a, b)$ whenever Property (3.5) holds.

Lemma 3.3 — Let τ_j be a regular differential expression on $[a, b]$ for $j = 1, \dots, n$. If all solutions of the differential equation $(\tau_j)u = 0$ and $(\tau_j^+)v = 0$ on $[a, b]$ are in $L_w^2(a, b)$ for $j = 1, \dots, n$; then all solutions of $(\tau_1 \tau_2 \dots \tau_n)u = 0$ and $(\tau_1 \tau_2 \dots \tau_n)^+ v = 0$ are in $L_w^2(a, b)$.

PROOF : Since order of $\tau_j = 2$, for $j = 1, \dots, n$. Then $\text{def } [T_0(\tau_j)] = n_2$. Hence $T_0(\tau_j)$ has property (C). By Theorem 3.1, we have

$$\text{def } [T_0(\tau_1 \tau_2 \dots \tau_n)] \geq \text{def } \left[\prod_{j=1}^n T_0(\tau_j) \right] = \sum_{j=1}^n n_j = \text{order of } (\tau_1 \tau_2 \dots \tau_n).$$

$\text{def } [T_0(\tau_1 \tau_2 \dots \tau_n)] = \text{order of } (\tau_1 \tau_2 \dots \tau_n)$ and consequently all solutions of $\left(\prod_{j=1}^n \tau_j \right) u = 0$ are in $L_w^2(a, b)$, we refer to [6] for more details.

The special case of Lemma 3.3 when $\tau_j = \tau$ for $j = 1, 2, \dots, n$ and τ is symmetric was established in [17]. In this case it is easy to see that the converse also holds. If all solutions of $\tau^n u = 0$ are in $L_w^2(a, b)$ then all solutions of $\tau u = 0$ must be in $L_w^2(a, b)$. In general, if all solutions of $(\tau_1 \tau_2 \dots \tau_n)u = 0$ are in $L_w^2(a, b)$ then all solutions of $\tau_n u = 0$ are in $L_w^2(a, b)$ since these also

solutions of $(\tau_1 \tau_2 \dots \tau_n) u = 0$. If all solutions of the adjoints equation $(\tau_1 \tau_2 \dots \tau_n)^+ v = 0$ are also in $L_w^2(a, b)$ then it follows similarly that all solutions of $\tau_1^+ v = 0$ are in $L_w^2(a, b)$, So in particular for $n = 2$ we have established the following Corollary.

Corollary 3.4 — Suppose τ_1, τ_2 and τ_1, τ_2 are all regular symmetric expressions on $[a, b]$. Then the product is in the maximal deficiency case at b if and only if both τ_1, τ_2 are in the maximal deficiency case at b (i.e. if τ_1 and τ_2 are in the classical limit-circle case at b , then the fourth-order expression τ_1, τ_2 is in the limit-circle case at b (i.e., $d^+ = d^- = 4$), we refer to [6, Corollary 2] for more details.

In connection with the application of Theorem 3.1 to get information about the deficiency indices of symmetric differential expressions, we note that the product of symmetric expressions is not symmetric in general. However, any power of symmetric expression is symmetric and so called symmetric such as $\tau_1 \tau_2 \tau_1, \tau_1 \tau_2 \tau_3 \tau_2 \tau_1$ etc., of symmetric expressions as symmetric.

Remark 3.5 : In the case of product operators, the sesquilinear (bilinear) form $[f, g]$ can be written similar to that in (2.7) and (2.11) as follows: For $f, g \in D(\tau_1 \tau_2 \dots \tau_n)$,

$$\begin{aligned} [f, g](x) &= \sum_{k=1}^n (-1)^{(k-1)} (f^{[k-1]} \bar{g}^{[2n-k]} - f^{[2n-k]} \bar{g}^{[k-1]})(x) \\ &= (\bar{g}, \bar{g}^{[1]}, \dots, \bar{g}^{[2n-1]}) J_{2n \times 2n} (f, f^{[1]}, \dots, f^{[2n-1]})^T(x), \\ &= ([\bar{g}, \phi_1], [\bar{g}, \phi_2], \dots, [\bar{g}, \phi_{2n}]) J_{2n \times 2n} ([f, \phi_2], \dots, [f, \phi_{2n}])^T(x), \dots \quad (3.6) \end{aligned}$$

T for transposed matrix, where $f^{[2n-k]}$, $k = 1, \dots, 2n$, are the quasi derivatives of f , $J_{2n \times 2n} = ((-1)^r \delta_{r, 2n+1-s})$ ($1 \leq r, s \leq 2n$) and $\phi_1, \phi_2, \dots, \phi_{2n}$ are linearly independent solutions of the equation $\left[\prod_{j=1}^n (\tau_j) \right] u = 0$. We refer to [11], [12] and [13] for more details.

The next result is straightforward extension of Theorem 4 in [15, Section 18.1]; see also [2], [8] and [11].

Theorem 3.6 — If the operator S with $D(S)$ is a self-adjoint extension of the minimal

operator $T_0(\tau_1 \tau_2 \dots \tau_n) = \prod_{j=1}^n [T_0(\tau_j)]$ with $\text{def} \prod_{j=1}^n [T_0(\tau_j)] \equiv d \in [0, 2n]$, then there exists

$\Psi_1, \dots, \Psi_d$ in $D(S) \subset D[T(\tau_1 \tau_2 \dots \tau_n)]$ satisfying the following conditions:

(i) $\Psi_1, \dots, \Psi_d$ are linearly independent modulo $D[T_0(\tau_1 \tau_2 \dots \tau_n)]$.

$$(ii) [\Psi_j, \Psi_k]_a^b = 0, j, k = 1, \dots, d. \quad \dots (3.7)$$

(iii) $D(S)$ consists precisely of those y in $D[T(\tau_1 \tau_2 \dots \tau_n)]$ which satisfy,

$$[y, \Psi_j]_a^b = 0, j = 1, \dots, d. \quad \dots (3.8)$$

Conversely, given $\Psi_1, \dots, \Psi_d$ in $D[T(\tau_1 \tau_2 \dots \tau_n)]$ which satisfy (i) and (ii), the set $D(S)$ defined by (iii) is a self-adjoint domain.

PROOF : The proof is entirely similar to that in [15, Theorem 18.1.4] and therefore omitted.

Remark 3.7 : It is well known from Naimark [15] that no boundary condition is needed at a limit-point end-point in order to get a self-adjoint realization of $\prod_{j=1}^n (\tau_j) u = 0$. If both end-points are *LP*, then no boundary conditions are necessary and hence the minimal (maximal) operator associated with $\prod_{j=1}^n (\tau_j)$ in $L_w^2(a, b)$ is itself self-adjoint and has no proper self-adjoint extensions (restrictions). On the other hand, a boundary condition is needed for each limit-circle end-point.

The self-adjoint extensions are determined by boundary conditions imposed at the end-points of the interval I . The type of these boundary conditions depends on the nature of the problem in the interval I .

Theorem 3.8 -- Let $\prod_{j=1}^n \tau_j$ be regular symmetric differential expression on the interval $[a, b]$. Then the boundary conditions determine the domain of self-adjoint extension S of $T_0(\tau_1 \tau_2 \dots \tau_n)$ is the set of functions $y \in D[T(\tau_1 \tau_2 \dots \tau_n)]$ which are such that

$$MY(a) + NY(b) = 0, \quad \dots (3.9)$$

where $M = (\alpha_{jk})_{1 \leq j, k \leq 2n}, N = (\beta_{jk})_{1 \leq j, k \leq 2n} \quad \dots (3.10)$

are $2n \times 2n$ matrices over $\mathbb{C}$, $Y(\cdot) = (y, y^{[1]}, \dots, y^{[2n-1]})^T(\cdot)$, T for transposed matrix, and α_{jk}, β_{jk} are complex numbers satisfying,

$$MJM^* = NJN^*, J_{2n \times 2n} = (-1)^r \delta_r, 2n \dots + 1 - s \quad (1 \leq r, s \leq 2n) \quad \dots (3.11)$$

Conversely, if S is self-adjoint extension of $T_0(\tau_1 \tau_2 \dots \tau_n)$, then there exist $2n \times 2n$ matrices M and N over $\mathbb{C}$ such that the conditions (3.9) and (3.11) are satisfied and $D(S)$ is the set of functions $y \in D[T(\tau_1 \tau_2 \dots \tau_n)]$ satisfying (3.9).

PROOF : Let the boundary conditions (3.9) and (3.11) be given. By Theorem 2.5, there are functions $\Psi_1, \dots, \Psi_{2n}$ in $D[T(\tau_1 \tau_2 \dots \tau_n)]$ which satisfy the conditions

$$\overline{\Psi_j}^{[2n-k]}(a) = (-1)^k \alpha_{jk}, \quad \overline{\Psi_j}^{[2n-k]}(b) = (-1)^{(k-1)} \beta_{jk}, \quad j, k = 1, \dots, 2n. \quad \dots (3.12)$$

Given (3.12), it is not difficult to show that (3.11) and (3.9) can be restated in the forms (3.7) and (3.8) respectively. It then follows from Theorem 3.6 that the domain determined by (3.9) and (3.11) is the domain of self-adjoint extension of $T_0(\tau_1 \tau_2 \dots \tau_n)$.

Conversely, if S is self-adjoint extension of $T_0(\tau_1 \tau_2 \dots \tau_n)$, then by Theorem 3.6, $D(S)$ is determined by the functions $\Psi_1, \dots, \Psi_{2n}$ in $D[T(\tau_1 \tau_2 \dots \tau_n)]$ satisfying (3.7) and (3.8). If α_{jk} and β_{jk} , $1 \leq j, k \leq 2n$ are then defined by (3.12), it is clear that $D(S)$ is determined by (3.9) and (3.11); see [9], [11] and [15] for more details.

In the following cases, the self-adjoint extension S of $T_0(\tau_1 \tau_2 \dots \tau_n)$ is determined by boundary conditions in terms of certain Wronskians (sesquilinear forms) involving y and $2n$ linearly independent solutions of the equation $\left(\prod_{j=1}^n \tau_j \right) u = 0$ at the singular end-points.

Case (i) — Assume both end-points a and b are singular LC. By (3.6), (3.7) and Lemma 2.6, if we put,

$$[\overline{\Psi_j}, \phi_k](a) = (-1)^k \alpha_{jk}, \quad [\overline{\Psi_j}, \phi_k](b) = (-1)^{(k-1)} \beta_{jk}, \quad j, k = 1, \dots, 2n, \quad \dots (3.13)$$

then the boundary conditions of the functions $y \in D[T(\tau_1 \tau_2 \dots \tau_n)]$ have the same form (3.9), where M, N satisfy (3.10) and (3.11), and $Y(\cdot) = ([y, \phi_1], \dots, [y, \phi_{2n}])^T(\cdot)$.

Case (ii) — (a) Assume the left end-point a is regular and the right end-point b is singular LC. Then the boundary conditions of the functions $y \in D[T(\tau_1 \tau_2 \dots \tau_n)]$ in this case are given by (3.9), where

$$y(a) = (y, y^{[1]}, \dots, y^{[2n-1]})^T(a) \quad \text{and} \quad Y(b) = ([y, \phi_1], \dots, [y, \phi_{2n}])^T(b) \quad \dots (3.14)$$

and the matrices M and N satisfy (3.11)

(b) If the left end-point a is singular LC and the right end-point b is regular, then let,

$$Y(a) = ([y, \phi_1], \dots, [y, \phi_{2n}])^T(a) \quad \text{and} \quad y(b) = (y, y^{[1]}, \dots, y^{[2n-1]})^T(b)$$

and the rest is the same as in (a).

Case (iii) — Assume one end-point is *LP* end-point and the other is either regular or singular *LC* end-point, then have,

(a) Suppose a is *LP*. Then the boundary conditions in this case on the functions $y \in D[T(\tau_1 \tau_2 \dots \tau_n)]$ are (3.9) with $M = 0$; i.e.,

$$NY(b) = 0, \quad \dots (3.15)$$

where, $Y(b) = (y, y^{[1]}, \dots, y^{[2n-1]})^T(b)$, if b is regular,

$$Y(b) = ([y, \phi_1], \dots, [y, \phi_{2n}])^T(b) \text{ if } b \text{ is singular and } LC.$$

(b) if b is *LP*, then it suffices to reverse the roles of a and b in (a).

Case (iv) — If both end-points a and b are *LP*, then no boundary conditions are necessary, see Remark 3.7 above.

DISCUSSION

In this section, we show how Cases (i), (ii), (iii) and (iv) follow from the sesquilinear form (3.6), Lemma 2.6 and Theorem 3.6. The Cases $d = 0, n, 2n$ are considered separately.

Case 1 — $d = 0$. In this case, both end-points are *LP* end-points and the minimal operator $T_0(\tau_1 \tau_2 \dots \tau_n)$ is itself self-adjoint and has no proper self-adjoint extensions.

Case 2 — $d = n$. In this case, one end-point must be *LP* and the other either regular or *LC* end-point.

(2a) Assume a is *LP* and b is regular. In this case Condition (iii) becomes,

$$\begin{aligned} [y, \Psi_j]_a^b &= [y, \Psi_j](b) \\ &= \sum_{k=1}^n (-1)^{(k-1)} [y^{[k-1]} \overline{\Psi_j^{[2n-k]}} - y^{[2n-k]} \overline{\Psi_j^{[k-1]}}](b) = 0, j = 1, \dots, n. \end{aligned} \quad \dots (4.1)$$

If b is regular, then $\Psi_j(b), \Psi_j^{[1]}(b), \dots, \Psi_j^{[2n-1]}(b)$ can take an arbitrary values and so (3.9) can be rewritten as :

$$NY(b) = 0, \quad \dots (4.2)$$

where $N = (\beta_{jk})_{1 \leq j \leq n, 1 \leq k \leq 2n}$ and $Y(b) = (y, y^{[1]}, \dots, y^{[2n-1]})^T(b)$

From Theorem 3.6 (i), we have that not all of $\beta_{j,1}, \dots, \beta_{j,2n}$ can be zero since this would imply by Theorem 3.6 that $\Psi_j \in D_0(\tau_1 \tau_2 \dots \tau_n), j = 1, \dots, n$. Condition (ii) becomes,

$$NJ_{2n \times 2n} N^* = 0, \quad J_{2n \times 2n} = (-1)^r \delta_{r, 2n+1-s} \quad (1 \leq r, s \leq 2n). \quad \dots (4.3)$$

Hence, the self-adjoint "boundary conditions" are of the form (4.2) with real $\beta_{j,1}, \dots, \beta_{j,2n}$, not all zero $j = 1, \dots, n$.

We have similar result if a is regular and b is LP .

(2b) Assume a is LP and b is LC . In this case, Condition (iii) becomes (4.1), which is equivalent to

$$([\overline{\Psi}_j, \phi_1], \dots, [\overline{\Psi}_j, \phi_{2n}]) J_{2n \times 2n} ([y, \phi_1], \dots, [y, \phi_{2n}])^T = 0, j = 1, \dots, n \quad \dots (4)$$

Set

$$[\overline{\Psi}_j, \phi_k] (b) = (-1)^{(k-1)} \beta_{jk}, j = 1, \dots, n; k = 1, \dots, 2n. \quad \dots (4.5)$$

Then, the "boundary conditions" (iii) can be expressed as:

$$NY(b) = 0, \quad \dots (4.6)$$

where $N = (\beta_{jk})_{1 \leq j \leq n, 1 \leq k \leq 2n}$ and $Y(b) = ([y, \phi_1], \dots, [y, \phi_{2n}])^T (b)$. Again by Theorem 3.6 (i), $\beta_{j,1}, \dots, \beta_{j,2n}, j = 1, \dots, n$, are real and not all zero.

Similarly for the case when a is LC and b is LP .

Remark 4.1 : Assume that a is LP . Comparing (4.6) with (4.2), note that when $y^{[k-1]}(b)$ is replaced by $[y, \phi_k] (b), k = 1, \dots, 2n$ then the singular case when b is LC is an exact parallel to the case when b is regular.

Case (3) $d = 2n$. In this case, each end-point is either regular or LC . By (3.9), (3.12) and proceeding as in Case (2) above, we find that the condition (iii) is equivalent to the equations :

$$\sum_{k=1}^{2n} \alpha_{jk} [y, \phi_k] (a) + \sum_{k=1}^{2n} \beta_{jk} [y, \phi_k] (b) = 0, j = 1, \dots, 2n. \quad \dots (4.7)$$

Theorem 3.6 (i) guarantee the linear independence of $2n$ equations in (4.7), and Condition (ii) reduces to the following conditions:

$$\begin{aligned} & \sum_{s=1}^n \alpha_{js} \overline{\alpha}_{k, 2n-s+1} - \sum_{s=1}^n \alpha_{j, 2n-s+1} \overline{\alpha}_{ks} \\ & = \sum_{s=1}^n \beta_{js} \overline{\beta}_{k, 2n-s+1} - \sum_{s=1}^n \beta_{j, 2n-s+1} \overline{\beta}_{ks}, j, k = 1, \dots, 2n. \quad \dots (4.9) \end{aligned}$$

We refer to [7], [8], [11] and [12] for more details.

Remark 4.2: It remains an open question as to characterize the singular non self-adjoint boundary conditions provided that u and its quasi-derivatives are replaced by certain Woronskains (sesquilinear form) associated with non-symmetric differential expressions involving u and elements of the maximal domain.

ACKNOWLEDGEMENT

I am grateful to the referee for reading the manuscript carefully and making helpful comments.

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