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A NOTE ON BOUNDARY CONDITIONS FOR NONLINEAR OPERATORS

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We investigate boundary conditions for strict-ψ-contractive and ψ-condensing operators. We derive results on the existence of eigenvectors with positive and negative eigenvalues and we obtain fixed point theorems for classes of noncompact operators.

Key words: Measure of noncompactness; k-ψ-contraction; ψ-condensing operator; fixed point index

1. INTRODUCTION

The fixed point index for ψ-condensing operators (see [12] and [1] for an axiomatic approach) has proved to be relevant to the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear integral equations and nonlinear differential equations. In particular, if A is a compact operator the fixed point index agrees with the Leray-Schauder degree, deg$(I - A, \Omega, 0)$, of $I - A$ on Ω relative to the point zero.

A well known result of Guo in [10, Lemma 1] states that if a compact operator $A : \overline{\Omega} \to X$, defined on the closure of a bounded open set Ω of an infinite-dimensional Banach space X, satisfies (i) $\inf_{x \in \partial\Omega} ||Ax|| > 0$ and (ii) $Ax \neq \lambda x$ for $x \in \partial\Omega$ and $0 < \lambda \leq 1$, then the Leray-Schauder degree deg$(I - A, \Omega, 0) = 0$.

This theorem has been generalized to strict-ψ-contractive operators, by strengthening the boundary condition (i) (see, for example, [13, 14]). In [7] the result has been generalized replacing (i) with the following condition

$$\inf_{x\in\partial\Omega}\|Ax\| > kW_\psi(X)\sup_{x\in\partial\Omega}\|x\| \tag{1}$$

which depends on the Wośko constant $W_\psi(X)$ of the space X and is optimal when $W_\psi(X) = 1$. Though it has been shown that $W_\psi(X) = 1$ in certain classes of Banach spaces, the problem whether this is true in any Banach space X is open. Aim of this note is to give a formulation of Guo's theorem for strict-ψ-contractive (and ψ-condensing) operators under an hypothesis that looks weaker than (1) in Banach spaces X in which the known estimate of $W_\psi(X)$ is greater than 1. Then we derive results on the existence of eigenvectors with positive and negative eigenvalues and we obtain fixed point theorems for classes of noncompact operators (see also [11] for related results).

2. Preliminaries

Throughout this note X is an infinite-dimensional real Banach space. We denote by $B_r(X) = \{x \in X : \|x\| \le r\}$ and by $S_r(X) = \{x \in X : \|x\| = r\}$, respectively, the closed ball and the sphere of radius $r > 0$ in X. We use the shortcut $B(X)$ and $S(X)$ instead of $B_1(X)$ and $S_1(X)$. All the maps considered in what follows are assumed to be continuous.

We recall that for a bounded set M in X: the Kuratowski measure of noncompactness $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M admits a finite covering by sets of diameter at most ε; the lattice measure of noncompactness $\beta(M)$ is the supremum of all $\varepsilon > 0$ such that M contains a sequence $\{x_n\}$ with $||x_m - x_n|| \ge \varepsilon$, for $m \ne n$; the Hausdorff measure of noncompactness $\gamma(M)$ is the infimum of all $\varepsilon > 0$ such that M admits a finite ε-net in X.

We refer to [3] for all details. In the following $\psi \in \{\alpha, \beta, \gamma\}$.

Given an operator $F : dom(F) \subseteq X \to X$ we denote by

$$\psi(F) = \inf\{\, k \ge 0 : \psi(FM) \le k\psi(M) \text{ for every bounded } M \subseteq dom(F)\}$$

the ψ-norm of F. The operator F is called a k-ψ-contraction if $\psi(F) \le k$, in particular F is called a strict-ψ-contraction if it is a k-ψ-contraction for some $k < 1$. The operator F is called ψ-condensing if $\psi(FM) < \psi(M)$ for each bounded $M \subseteq dom(F)$ which is not relatively compact. Assuming $dom(F)$ bounded, then every strict-ψ-contractive operator is ψ-condensing.

Throughout Ω is a bounded open subset of X containing the origin 0. Denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary of Ω, respectively. When $A : \overline{\Omega} \to X$ is ψ-condensing and has no fixed points on $\partial\Omega$ the fixed point index $\mathrm{ind}(A, \Omega)$ of A on Ω (see [1]) has the basic properties as follows:

(P1) (Homotopy invariance) If $H : [0,1]\times\overline{\Omega} \to X$ is continuous and such that $\psi(H([0,1]\times M)) < \psi(M)$ for each $M \subset \overline{\Omega}$ with $\psi(M) > 0$, i.e., H is ψ-condensing, and if $H(s,x) \neq x$ for all $(s,x) \in [0,1] \times \partial\Omega$, then ind$(H(0,\cdot),\Omega) =$ ind$(H(1,\cdot),\Omega)$.

(P2) (Additivity) If U_i, $i = 1, 2, \cdots$, are pairwise disjoint open subsets of Ω such that A has no fixed points on $\overline{\Omega}\setminus\cup_{i=1}^{\infty}U_i$, then the indices $\mathrm{ind}(A, U_i)$ are defined for all i, only finitely many of them are different from zero, and ind$(A,\Omega) = \sum_{i=1}^{\infty}$ ind(A, U_i).

(P3) If $Ax \equiv x_0$, with $x_0 \in \Omega$, then $\mathrm{ind}(A,\Omega) = 1$.

(P4) If $Ax \equiv x_0$, with $x_0 \notin \Omega$, then $\mathrm{ind}(A,\Omega) = 0$.

(P5) (Solution property) If $\mathrm{ind}(A,\Omega) \neq 0$, then A has at least one fixed point in Ω.

Moreover we recall the following result on the invariance of the index.

***Theorem* 2.1** [Theorem 3.4.3] — Let $A,\ F : \overline{\Omega} \to X$ be ψ-condensing. If A and F coincide on $\partial\Omega$ and have no fixed points on $\partial\Omega$, then $\mathrm{ind}(A,\Omega) = \mathrm{ind}(F,\Omega)$.

It is well-known that in the infinite-dimensional setting there is always a retraction R from $B(X)$ onto $S(X)$. Moreover we observe that given such a retraction R, then for any $r > 0$ the map R_r defined by $R_r(x) = rR(\frac{x}{r})$ is a retraction from $B_r(X)$ onto $S_r(X)$ such that $\psi(R) = \psi(R_r)$. In this connection, the quantitative characteristic

$$W_\psi(X) = \inf\{k \geq 1 : \exists \text{ a retraction } R : B(X) \to S(X) \text{ with } \psi(R) \leq k\}$$

has been introduced by Wośko in [16]. The estimate of $W_\psi(X)$ is of interest in problems of nonlinear analysis (see, for example, [2, 7, 9]). Concerning general results, in [15] it was proved that $W_\psi(X) \leq 6$ for any infinite-dimensional Banach space X, reaching the value 4 or 3 depending on the geometry of the space. Moreover, it has been proved that $W_\gamma(X) = 1$ in some Banach spaces of continuous functions [5, 16] and in some classical Banach spaces of measurable functions [6]. In [2] it is proved that $W_\psi(X) = 1$ in Banach spaces whose norm is monotone with respect to some basis.

We observe that there is no unified method to evaluate $W_\psi(X)$, most of the evaluations have required individual constructions in each space X. However, a standard way to construct a retraction from $B(X)$ onto $S(X)$ is that of normalizing a map which coincides with the identity on $S(X)$ and maps $B(X)$ out of a ball $B_r(X)$ of radius $r < 1$ (or possibly maps $B(X)$ into $X \setminus \{0\}$). In this connection it is of some interest for any $0 < \rho \leq \beta$ to define another geometrical characteristic $c_\psi(\rho,\beta,X)$ of the space X. Denote by $S_{\rho,\beta}$ the set of all continuous maps $G_{\rho,\beta} : B_\beta(X) \to X$ satisfying the following conditions:

(1) $G_{\rho,\beta}\, x = x$ for all $x \in S_\beta(X)$,

(2) $\|G_{\rho,\beta}\, x\| \geq \rho$ for all $x \in B_\beta(X)$.

We observe that since there is always a retraction from $B_\beta(X)$ onto $S_\beta(X)$, the set $S_{\rho,\beta}$ is nonempty. Moreover $S_{\rho_2,\beta} \subseteq S_{\rho_1,\beta}$ whenever $0 < \rho_1 < \rho_2 \leq \beta$.

Now we put

$$c_\psi(\rho, \beta, X) = \inf_{G_{\rho,\beta} \in S_{\rho,\beta}} \psi(G_{\rho,\beta}).$$

As no confusion can arise we write briefly $c_{\rho,\beta}$ instead of $c_\psi(\rho, \beta, X)$. The map $\rho \to c_{\rho,\beta}$ is clearly nondecreasing, moreover it is right-continuous. Indeed, given $0 < \rho_0 < \beta$ and $G_{\rho_0,\beta} \in S_{\rho_0,\beta}$, we choose for any $\rho_0 < \rho \leq \beta$ the map $G_{\rho,\beta} \in S_{\rho,\beta}$ defined by

$$G_{\rho,\beta}\, x = \begin{cases} G_{\rho_0,\beta}\, x & \text{if} \quad \|G_{\rho_0,\beta}\, x\| \geq \rho \\ \rho \frac{G_{\rho_0,\beta}\, x}{\|G_{\rho_0,\beta}\, x\|} & \text{if} \quad \rho_0 \leq \|G_{\rho_0,\beta}\, x\| < \rho. \end{cases}$$

For any $M \subseteq B_\beta(X)$ we set

$$M_1 = \{x \in M : \ \|G_{\rho_0,\beta}\, x\| \geq \rho\} \ \text{ and } M_2 = \{x \in M : \ \rho_0 \leq \|G_{\rho_0,\beta}\, x\| < \rho\},$$

then

$$G_{\rho,\beta} M_1 = G_{\rho_0,\beta} M_1 \ \text{ and } \ G_{\rho,\beta} M_2 \subseteq \left[0, \frac{\rho}{\rho_0}\right] \cdot G_{\rho_0,\beta} M_2.$$

Being $\psi(G_{\rho,\beta} M) = \max\{\psi(G_{\rho,\beta} M_1),\ \psi(G_{\rho,\beta} M_2)\}$ we easily obtain $\psi(G_{\rho,\beta} M) \leq \dfrac{\rho}{\rho_0} \psi(G_{\rho_0,\beta} M)$. Hence $\psi(G_{\rho,\beta}) \leq \dfrac{\rho}{\rho_0} \psi(G_{\rho_0,\beta})$ and $\lim_{\rho \to \rho_0^+} \psi(G_{\rho,\beta}) \leq \psi(G_{\rho_0,\beta})$ from which the right-continuity follows.

We observe that for $0 < \rho \leq \beta$ we have $1 \leq c_{\rho,\beta} \leq W_\psi(X)$ and $c_{\beta,\beta} = W_\psi(X)$ in any infinite-dimensional Banach space X. Moreover, in some Banach spaces X (cfr. [5, 6, 8, 16]) there are examples, for any $0 < \rho < 1$, of 1-γ-contractive maps $G_{\rho,1}$. In contrast, in the same spaces, for any $\varepsilon > 0$ there is a retraction $R : B(X) \to S(X)$ such that $\gamma(R) \leq 1 + \varepsilon$, but it is unknown whether there is a retraction with $\gamma(R) = 1$.

Lemma 2.2 — Let $0 < \alpha < \beta$. Set $\alpha_t = \dfrac{\alpha}{t}$ and $\beta_t = \dfrac{\beta}{t}$ for some $t > 0$.

Then $c_{\alpha_t,\beta_t} = c_{\alpha,\beta}$.

PROOF : To each $\rho \in [\alpha, \beta]$ associate $\rho_t = \dfrac{\rho}{t} \in [\alpha_t, \beta_t]$ and for any $G_{\rho,\beta} \in S_{\rho,\beta}$ define the map $G_{\rho_t,\beta_t} : B_{\beta_t}(X) \to X$ by $G_{\rho_t,\beta_t}(x) = \dfrac{1}{t} G_{\rho,\beta}(tx)$. Then we have

(1) $G_{\rho_t,\beta_t}\, x = x$ for all $x \in S_{\beta_t}(X)$,

(2) $\|G_{\rho_t,\beta_t}\, x\| \geq \rho_t$ for all $x \in B_{\beta_t}(X)$,

so that $G_{\rho_t,\beta_t} \in S_{\rho_t,\beta_t}$ and moreover $\psi(G_{\rho_t,\beta_t}) = \psi(G_{\rho,\beta})$. Therefore we find $c_{\alpha_t,\beta_t} = c_{\alpha,\beta}$. □

3. Main Results

We generalize Guo's result to strict-ψ-contractions and ψ-condensing operators under a condition which depends on the parameter $c_{\alpha,\beta}$ (for suitable numbers α and β). The condition seems to arise in a natural way from the geometry of the space X.

Lemma 3.1 — Let $A : \overline{\Omega} \to X$ with $\psi(A) < 1$ and

$$\inf_{x\in\partial\Omega} \|Ax\| > \sup_{x\in\partial\Omega} \|x\|.$$

Set $\alpha = \sup_{x\in\partial\Omega} \|x\|$ and $\beta = \inf_{x\in\partial\Omega} \|Ax\|$ and assume that $\psi(A)c_{\alpha,\beta} < 1$.

Then $\operatorname{ind}(A, \Omega) = 0$.

Proof : Since $\psi(A)c_{\alpha,\beta} < 1$ and the map $\rho \to c_{\rho,\beta}$ is right-continuous we can find $\rho \in (\alpha, \beta]$ and $G_{\rho,\beta} \in S_{\rho,\beta}$ such that $\psi(A)\psi(G_{\rho,\beta}) < 1$. Define $L : X \to X$ by

$$Lx = \begin{cases} G_{\rho,\beta}\, x & \text{if} \quad x \in B_\beta(X) \\ x & \text{if} \quad x \in X \setminus B_\beta(X) \end{cases}$$

and $F : \overline{\Omega} \to X$ by setting $Fx = L(Ax)$. Then $\psi(F) = \psi(G_{\rho,\beta})\psi(A) < 1$. On the other hand we have $\|Fx\| \geq \rho$ for all $x \in \overline{\Omega}$. Since $\rho > \alpha$, we get

$$\inf_{x\in\overline{\Omega}} \|Fx\| > \sup_{x\in\partial\Omega} \|x\|,$$

hence F has no fixed points on $\overline{\Omega}$. By the solution property $\operatorname{ind}(F, \Omega) = 0$. Since A and F coincide on $\partial\Omega$, by the invariance of the index we find $\operatorname{ind}(A, \Omega) = \operatorname{ind}(F, \Omega)$ which completes the proof.

The following is the main result of the section.

***Theorem* 3.2** — *Let* $A : \overline{\Omega} \to X$ with $\psi(A) = k < 1$. *Let* $\beta = \inf_{x\in\partial\Omega} ||Ax||$ *and assume that there is* α' *such that*

$$\inf_{x\in\partial\Omega} \|Ax\| \geq \alpha' > kc_{\alpha',\beta} \sup_{x\in\partial\Omega} \|x\|.$$

Assume also that $Ax \neq \lambda x$ for $x \in \partial\Omega$ and $kc_{\alpha',\beta} < \lambda \leq 1$, if $kc_{\alpha',\beta} < 1$.

Then $\operatorname{ind}(A, \Omega) = 0$.

PROOF : Choose $\varepsilon > 0$ such that

$$\inf_{x\in\partial\Omega} \|Ax\| \geq \alpha' > (kc_{\alpha',\beta} + \varepsilon) \sup_{x\in\partial\Omega} \|x\|, \tag{2}$$

and let $A_0 : \overline{\Omega} \to X$ be defined by

$$A_0 x = \frac{1}{kc_{\alpha',\beta} + \varepsilon} Ax.$$

We show that the operator A_0 satisfies the hypotheses of Lemma 3.1.

First of all $\psi(A_0) = \dfrac{k}{kc_{\alpha',\beta} + \varepsilon} < \dfrac{1}{c_{\alpha',\beta}} \leq 1$, and using (2) we get

$$\inf_{x\in\partial\Omega} \|A_0 x\| > \sup_{x\in\partial\Omega} \|x\|.$$

Now set $\beta_\varepsilon = \inf_{x\in\partial\Omega} \|A_0 x\|$, then $\beta_\varepsilon = \dfrac{\beta}{kc_{\alpha',\beta} + \varepsilon}$. Set also $\alpha = \sup_{x\in\partial\Omega} \|x\|$. To see that $\psi(A_0)c_{\alpha,\beta_\varepsilon} < 1$ we put $\alpha'_\varepsilon = \dfrac{\alpha'}{kc_{\alpha',\beta} + \varepsilon}$, then by (2) we have $\alpha < \alpha'_\varepsilon \leq \beta_\varepsilon$ and thus $c_{\alpha,\beta_\varepsilon} \leq c_{\alpha'_\varepsilon,\beta_\varepsilon}$. Since by Lemma 2.2, we have $c_{\alpha'_\varepsilon,\beta_\varepsilon} = c_{\alpha',\beta}$, we obtain $c_{\alpha,\beta_\varepsilon} \leq c_{\alpha',\beta}$. Thus we find $\psi(A_0)c_{\alpha,\beta_\varepsilon} = \dfrac{kc_{\alpha,\beta_\varepsilon}}{kc_{\alpha',\beta} + \varepsilon} < 1$, and by Lemma 3.1, we infer $\operatorname{ind}(A_0, \Omega) = 0$.

Then we set $H(s,x) = sAx + (1-s)A_0 x$ for $(s,x) \in [0,1] \times \overline{\Omega}$. Then H is ψ-condensing and $H(s,x) \neq x$ for $(s,x) \in [0,1] \times \partial\Omega$. Consequently, by the homotopy invariance of the index we obtain $\operatorname{ind}(A,\Omega) = \operatorname{ind}(A_0,\Omega)$, hence the result. □

As a corollary we obtain [7, Theorem 3.2].

Corollary 3.3 — Let $A : \overline{\Omega} \to X$ be a k-ψ-contraction ($k < 1$), satisfying

$$\inf_{x\in\partial\Omega} ||Ax|| > kW_\psi(X) \sup_{x\in\partial\Omega} ||x||.$$

Assume that one of the following conditions holds:

(i) $kW_\psi(X) < 1$ and $Ax \neq \lambda x$ for $x \in \partial\Omega$ and $kW_\psi(X) < \lambda \leq 1$;

(ii) $kW_\psi(X) \geq 1$.

Then $\operatorname{ind}(A,\Omega) = 0$.

As for the estimates of the parameter $c_{\alpha,\beta}$ and of the Wośko constant $W_\psi(X)$ we observe what follows.

Remark 3.4 : Given $0 < \alpha < \beta$ and a map $G_{\alpha,\beta} \in S_{\alpha,\beta}$ we can define a retraction $R : B_\beta(X) \to S_\beta(X)$ by $Rx = \beta \frac{G_{\alpha,\beta}\, x}{\|G_{\alpha,\beta}\, x\|}$. Then for any $M \subseteq B_\beta(X)$ we have

$$RM \subseteq \left[0, \frac{\beta}{\rho}\right] \cdot G_{\rho,\beta} M,$$

which implies $\psi(R) \leq \frac{\beta}{\rho}\psi(G_{\rho,\beta})$. Therefore we have $W_\psi(X) \leq \frac{\beta}{\alpha} c_{\alpha,\beta}$, which shows that starting from the map $G_{\alpha,\beta}$ the estimate of the Wośko constant $W_\psi(X)$ increases, with respect to the estimate of the constant $c_{\alpha,\beta}$, of a factor $\frac{\beta}{\alpha} \geq 1$.

Evidently Theorem 3.2 and Corollary 3.3 are equivalent in any Banach space X in which $W_\psi(X) = 1$. Next we see that Theorem 3.2 extends to ψ-condensing operators.

***Theorem* 3.5** — *Let $A : \overline{\Omega} \to X$ be ψ-condensing and set $\beta = \inf_{x\in\partial\Omega} \|Ax\|$. Suppose that there exists α' such that*

$$\inf_{x\in\partial\Omega} \|Ax\| \geq \alpha' > c_{\alpha',\beta} \sup_{x\in\partial\Omega} \|x\|. \tag{3}$$

Then $\operatorname{ind}(A, \Omega) = 0$.

PROOF : The proof is analogous to that of Theorem 3.2.

4. EIGENVECTORS AND FIXED POINT THEOREMS

The following corollary generalizes the Birkhoff-Kellogg [4] theorem to k-ψ-contractions.

Corollary 4.1 — Let $A : \overline{\Omega} \to X$ with $\psi(A) = k$ (for any $k > 0$). Let $\beta = \inf_{x\in\partial\Omega} \|Ax\|$ and assume that there exists α' such that

$$\inf_{x\in\partial\Omega} \|Ax\| \geq \alpha' > k c_{\alpha',\beta} \sup_{x\in\partial\Omega} \|x\|.$$

Then there exist $\lambda > k c_{\alpha',\beta}$ and $x_\lambda \in \partial\Omega$ such that $\lambda x_\lambda = A x_\lambda$, and also there exist $\mu < -k c_{\alpha',\beta}$ and $x_\mu \in \partial\Omega$ such that $\mu x_\mu = A x_\mu$.

PROOF : Assume by contradiction that

$$\lambda x \neq Ax \tag{4}$$

for $x \in \partial\Omega$ and $\lambda > k c_{\alpha',\beta}$. Choose $\varepsilon > 0$ such that

$$\inf_{x\in\partial\Omega} ||Ax|| > (k c_{\alpha',\beta} + \varepsilon) \sup_{x\in\partial\Omega} ||x||$$

and let $A_0 : \overline{\Omega} \to X$ be defined by

$$A_0 x = \frac{1}{k c_{\alpha',\beta} + \varepsilon} Ax.$$

Then A_0 is a strict-ψ-contraction with $\psi(A_0) = \dfrac{k}{kc_{\alpha',\beta}+\varepsilon}$. Set $\beta_\varepsilon = \inf_{x\in\partial\Omega}\|A_0x\|$, then $\beta_\epsilon = \dfrac{\beta}{kc_{\alpha',\beta}+\epsilon}$. Setting $\alpha'_\epsilon = \dfrac{\alpha'}{kc_{\alpha',\beta}+\epsilon}$ we find

$$\inf_{x\in\partial\Omega}||A_0x|| \geq \alpha'_\epsilon > \psi(A_0)c_{\alpha'_\epsilon,\beta_\epsilon}\sup_{x\in\partial\Omega}||x||.$$

Moreover, $A_0x \neq \lambda x$ for $x \in \partial\Omega$ and $0 < \lambda \leq 1$. Thus the operator A_0 satisfies the hypotheses of Theorem 3.2, hence $\mathrm{ind}(A_0, \Omega) = 0$.

Set now $H(s,x) = (1-s)A_0x$ for $(s,x) \in [0,1]\times\overline{\Omega}$. Then H is ψ-condensing. Since $((1-s)/(kc_{\alpha',\beta}+\varepsilon))^{-1} > kc_{\alpha',\beta}$, for $s \in [0,1)$, using (4) we have $Ax \neq ((1-s)/(kc_{\alpha',\beta}+\varepsilon))^{-1}x$ for $(s,x)\in[0,1)\times\partial\Omega$. Moreover $H(1,x)\equiv 0$ for $x\in\overline{\Omega}$, thus on the one hand $H(s,x)\neq x$ for $(s,x)\in[0,1]\times\partial\Omega$ and on the other hand from property (P_3) it follows $\mathrm{ind}(H(1,\cdot),\Omega) = 1$. By the homotopy invariance of the index we get $\mathrm{ind}(A_0,\Omega) = 1$, which is a contradiction.

The last assertion follows by considering in the above proof $-A$ in place of A. □

We end this note by establishing some fixed point results.

Corollary 4.2 — Let Ω_1 and Ω_2 be bounded open sets in X, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $A : \overline{\Omega}_2 \to X$ with $\psi(A) = k < 1$. Let $\beta_i = \inf_{x\in\partial\Omega_i}\|Ax\|$, for $i = 1, 2$. Suppose that one of the following conditions holds:

(a) there exists α'_1 such that $kc_{\alpha'_1,\beta_1} < 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_1}\|Ax\| \geq \alpha'_1 > & kc_{\alpha'_1,\beta_1}\sup_{x\in\partial\Omega_1}\|x\| \\ Ax \neq \lambda x & x\in\partial\Omega_1,\;\; kc_{\alpha'_1,\beta_1} < \lambda < 1 \\ Ax \neq \nu x & x \in \partial\Omega_2,\;\; \nu > 1 \end{cases}$$

(b) there exists α'_2 such that $kc_{\alpha'_2,\beta_2} < 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_2}\|Ax\| \geq \alpha'_2 > & kc_{\alpha'_2,\beta_2}\sup_{x\in\partial\Omega_2}\|x\| \\ Ax \neq \lambda x & x\in\partial\Omega_2,\;\; kc_{\alpha'_2,\beta_2} < \lambda < 1 \\ Ax \neq \nu x & x \in \partial\Omega_1,\;\; \nu > 1 \end{cases}$$

(c) there exists α'_1 such that $kc_{\alpha'_1,\beta_1} \geq 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_1}\|Ax\| \geq \alpha'_1 > & kc_{\alpha'_1,\beta_1}\sup_{x\in\partial\Omega_1}\|x\| \\ Ax \neq \nu x & x \in \partial\Omega_2,\;\; \nu > 1. \end{cases}$$

(d) there exists α'_2 such that $kc_{\alpha'_2,\beta_2} \geq 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| \geq \alpha'_2 > & kc_{\alpha'_2,\beta_2} \sup_{x\in\partial\Omega_2} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_1, \quad \nu > 1. \end{cases}$$

Then A has at least one fixed point on $\overline{\Omega}_2 \setminus \Omega_1$.

PROOF : We prove the result in the first case of (a). If A has no fixed points on $\partial\Omega_1 \cup \partial\Omega_2$, then by Theorem 3.2, we have ind$(A, \Omega_1) = 0$. On the other hand, by the Leray-Schauder Theorem (see [1, Theorem 3.2.3]) it follows that ind$(A, \Omega_2) = 1$. Consequently we get

$$\mathrm{ind}(A, \Omega_2 \setminus \overline{\Omega}_1) = \mathrm{ind}(A, \Omega_2) - \mathrm{ind}(A, \Omega_1) = 1.$$

By the solution property of the index, A has at least one fixed point on $\Omega_2 \setminus \overline{\Omega}_1$, hence the thesis.

The other cases can be proved similarly. □

The cases (a) and (b) of Corollary 4.2 can be reformulated in terms of the norms.

Corollary 4.3 — Let Ω_1 and Ω_2 be bounded open sets in X, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : \overline{\Omega}_2 \to X$ with $\psi(A) = k < 1$. Let $\beta_i = \inf_{x\in\partial\Omega_i} \|Ax\|$, for $i = 1, 2$.

Suppose either there exists α'_1 such that $kc_{\alpha'_1,\beta_1} < 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_1} \|Ax\| \geq \alpha'_1 > & kc_{\alpha'_1,\beta_1} \sup_{x\in\partial\Omega_1} \|x\| \\ \|Ax\| \geq \|x\| & x \in \partial\Omega_1 \\ \|Ax\| \leq \|x\| & x \in \partial\Omega_2, \end{cases}$$

or there exists α'_2 such that $kc_{\alpha'_2,\beta_2} < 1$ and

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| \geq \alpha'_2 > & kc_{\alpha'_2,\beta_2} \sup_{x\in\partial\Omega_2} \|x\| \\ \|Ax\| \geq \|x\| & x \in \partial\Omega_2 \\ \|Ax\| \leq \|x\| & x \in \partial\Omega_1. \end{cases}$$

Then A has at least one fixed point on $\overline{\Omega}_2 \setminus \Omega_1$.

Finally reasoning as in Corollary 4.2, we get the following:

Corollary 4.4 — Let Ω_1 and Ω_2 be bounded open sets in X, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $A : \overline{\Omega}_2 \to X$ be a ψ-condensing operator. Let $\beta_i = \inf_{x\in\partial\Omega_i} \|Ax\|$. Suppose that one of the following conditions holds:

There exists α'_1 such that

$$\begin{cases} \inf_{x\in\partial\Omega_1} \|Ax\| \geq \alpha'_1 > & c_{\alpha'_1,\beta_1} \sup_{x\in\partial\Omega_1} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_2, \ \ \nu > 1 \end{cases}$$

or there exists α'_2 such that

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| \geq \alpha'_2 > & c_{\alpha'_2,\beta_2} \sup_{x\in\partial\Omega_2} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_1, \ \ \nu > 1. \end{cases}$$

Then A has at least one fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

The results of this section yield the corresponding results of [7].

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