

THE RELATIVE SECOND FOX AND THIRD DIMENSION SUBGROUP OF ARBITRARY
GROUPS

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Let $I_R(G)$ denote the augmentation ideal of the group algebra $R(G)$ of a group G with coefficients in a commutative ring R . We give a complete description of the third relative dimension subgroup $G \cap (1 + I_R(K)I_R(G) + I_R^3(G))$ and the second relative Fox subgroup $G \cap (1 + I_R(K)I_R(H) + I_R^2(G)I_R(H))$ for any subgroups K and H of G .

Key words : Dimension subgroup; Fox subgroup; augmentation quotient; homological methods in group theory

1. INTRODUCTION

It is a classical problem to study the link between filtrations of groups and of their group algebras. There are mainly two cases much studied in the literature, namely dimension subgroups and Fox subgroups. As to the first, fix a commutative ring of coefficients, R , with unit 1_R . Now an appropriate filtration of a group G (by an N-series $\mathcal{N} = \{N_i\}$ or a restricted N-series) gives rise to a descending filtration of the group algebra $R(G)$ by ideals $I_{R,\mathcal{N}}^n(G)$, see [16, III.1.5] or section 2 below. For example, the lower central series $\gamma: G = G_1 \supset G_2 \supset \dots$ of G induces the

filtration by the powers $I_R^n(G)$ of the augmentation ideal $I_R(G)$ of $R(G)$. Pulling back the induced filtration $\{I_{R,\mathcal{N}}^n(G)\}$ to G defines the series of dimension subgroups with respect to R, \mathcal{N} , $D_{n,R}^{\mathcal{N}}(G) = G \cap (1_R + I_{R,\mathcal{N}}^n(G))$. Now the so-called dimension subgroup problem asks whether the canonical inclusion $N_n \subset D_{n,R}^{\mathcal{N}}(G)$ is an equality. For classical dimension subgroups $D_n^\gamma(G)$ this is true for $n \leq 3$ and is now known to be false for all $n \geq 4$, by means of counterexamples due to Rips [18] ($n = 4$) and later by Gupta [5] ($n \geq 4$). While their method is combinatorial, a homological approach to the problem was inaugurated by Passi, from where the notion of dimension subgroups relative to a subgroup K of G emerged in a natural way: these are defined by $D_{n,R}^{\mathcal{N}}(G, K) = G \cap (1_R + I_R(K)I_R(G) + I_{R,\mathcal{N}}^n(G))$. Since then, they proved more and more to be an appropriate tool for the study of the classical case $K = \{1\}$. Many qualitative properties of relative dimension subgroups were established, notably by Kuz'min [12]. Nevertheless, the main problem, concerning their explicit computation, remains difficult, even in low dimensions n . With regard to the ring of coefficients, only $D_{n,R}^\gamma(G)$, $n \leq 3$, seems to be known for all commutative rings R , by work of Sandling [20]. Focussing on $R = \mathbb{Z}$, the “relative” analogue of the classical dimension subgroup problem asks when the inclusion $K_2 N_n \subset D_{n,\mathbb{Z}}^{\mathcal{N}}(G, K)$ is an equality. This is true for $n \leq 2$, and various conditions on G and K were exhibited in the literature ensuring that it also holds for $n = 3$. In the first part of this paper we compute $D_{3,R}^{\mathcal{N}}(G, K)$ in general and show that equality does not always hold, the minimal counterexample for $\mathcal{N} = \gamma$ and $R = \mathbb{Z}$ being of order 2^6 . We note that the computation of $D_{3,\mathbb{Z}}^\gamma(G, K)$, achieved and distributed by the author in 1994, was reproved by Tahara, Vermani and Razdan by different methods in [17].

In the second part of the paper we unify the study of relative dimension subgroups with the one of Fox subgroups $G \cap (1_R + I_R^n(G)I_R(H))$ for a subgroup H of G . After a long history (for a review see [4] or [3]), they are now completely known for free groups G ($\mathcal{N} = \gamma$, $R = \mathbb{Z}$), thanks to work of Gupta, Hurley and Yunus. The case of arbitrary groups is much harder; for $n = 2$ (and $R = \mathbb{Z}$) the problem was solved by Curzio and Gupta in [3] but seems to be completely open for $n > 2$. On the other hand, a first case of a relative version of Fox subgroups was considered in [17] where the group $G \cap (1 + I_{\mathbb{Z}}(K)I_{\mathbb{Z}}(H) + I_{\mathbb{Z}}^2(G)I_{\mathbb{Z}}(H))$ is determined for H normal and K being a specific subgroup of G containing $[H, G]$. Generalizing the two last-mentioned results we here determine the group $G \cap (1_R + I_R(K)I_R(H) + I_R^2(G)I_R(H))$ for any subgroups H, K of G and coefficient rings R .

Although the latter result formally includes the case $H = G$ treated in section 2 by a different method, the result obtained there is much simpler and does not seem to be easily deducible from the more general formula in Section 3. Nor it seems possible to generalise the method of section 2 to the latter case, because of the essential difference of the behaviour of additive and of quadratic functors (like the tensor square \otimes^2 and the symmetric square SP^2 of abelian groups) with respect to subgroups.

2. THE THIRD RELATIVE DIMENSION SUBGROUP

Recall the notation from the introduction. In particular, recall that an N-series $\mathcal{N} = \{N_i\}$ is a descending chain of subgroups

$$G = N_1 \supset N_2 \supset \dots \supset 1$$

such that $[N_i, N_j] \subset N_{i+j}$ (with $[a, b] = aba^{-1}b^{-1}$ for $a, b \in G$). An N-series \mathcal{N} induces a descending chain of two-sided ideals

$$R(G) \supset I_{R,\mathcal{N}}^1 \supset I_{R,\mathcal{N}}^2 \supset \dots \supset 0$$

by defining $I_{R,\mathcal{N}}^n$ to be the R -submodule of $R(G)$ generated by the elements

$$(a_1 - 1) \cdots (a_r - 1), \quad r \geq 1, a_i \in N_{k_i}, \text{ such that } k_1 + \dots + k_r \geq n.$$

For a subgroup K of G and $n \geq 1$ define

$$D_{n,R}^{\mathcal{N}}(G, K) = G \cap (1_R + I_R(K)I_R(G) + I_{R,\mathcal{N}}^n(G)).$$

Note that for $n = 3$ the case of an arbitrary N-series \mathcal{N} can be reduced to the case $\mathcal{N} = \gamma$ by the identity $D_{3,R}^{\mathcal{N}}(G, K)/N_3 = D_3^{\gamma}(G/N_3, KN_2/N_3)$, but we do not make use of this reduction as our method genuinely works for any N-series.

The main goal of this section is to prove the following result.

Theorem 2.1 — *Let G be a group, K a subgroup, \mathcal{N} an N-series of G and R a commutative ring with unit 1_R . Then*

$$D_{3,R}^{\mathcal{N}}(G, K) = U_0N_3Z_2 \prod_{\substack{p \in \sigma(R) \\ p \text{ odd}}} t_p(G \bmod U_0N_3) \cap (U_{p^e}N_3G^{p^e})$$

where

- $U_m = \text{sgp}\{[a, b^k] \mid a, b \in G, k \in \mathbb{Z}, a^k, b^k \in KN_2G^m\}$;
- $\sigma(R) = \{p \mid p \text{ is a prime and } p^n R = p^{n+1} R \text{ for some } n \geq 0\}$, and for $p \in \sigma(R)$, $p^e = p^{e(p)}$ is the smallest power of p for which $p^e R = p^{e+1} R$;
- $t_p(G \bmod U_0N_3) = \{g \in G \mid g^{p^k} \in U_0N_3 \text{ for some } k \geq 0\}$;
- $Z_2 = \{1\}$ if $2 \notin \sigma(R)$, else $Z_2 = t_2(G \bmod U_0N_3) \cap (U_{2^{e(2)}}N_3G^{2^{e(2)+1}}V^{2^{e(2)}})$, where $V = \{g \in G \mid g^{e(2)-1} \in KN_2G^{2^{e(2)}}\}$ if $e(2) > 0$ and $V = G$ else.

For the proof one first reduces to the case $R = \mathbb{Z}/m\mathbb{Z}$ via the "universal coefficient decomposition" obtained in Theorem 1 of [8]. Then what remains to prove is the following.

Theorem 2.2 — *Let G be a group, K a subgroup, \mathcal{N} an N -series of G and $m \geq 0$. If m is even let $V = \{a \in G \mid a^{m/2} \in KN_2G^m\}$. Then*

$$D_{3,\mathbb{Z}}^{\mathcal{N}}(G, K) = U_0N_3$$

$$D_{3,\mathbb{Z}/m\mathbb{Z}}^{\mathcal{N}}(G, K) = U_mN_3G^m \quad \text{if } m \text{ is odd}$$

$$D_{3,\mathbb{Z}/m\mathbb{Z}}^{\mathcal{N}}(G, K) = U_mN_3G^{2m}V^m \quad \text{if } m \text{ is even.} \quad \square$$

For $K = \{1\}$ one rediscovers a result of Sandling [20]. Some other special cases are resembled in the following

Corollary 2.3 — The inclusion $K_2N_3 \subset D_{3,\mathbb{Z}}^{\gamma}(G, K)$ is an equality if one of the following conditions holds:

1. $[K, G] \subset G_3$ (cf. [16, Theorem 5.9]);
2. there exists a normal subgroup $N \subset G$ such that $G = NK$ and $N \cap K$ is central in G (cf. [11]);
3. K is normal and G/K is cyclic (cf. [16, Theorem V.5.4]);
4. one of the following three groups is torsion-free: G/K_2G_3 (cf. [11]), $[G, K]G_3/K_2G_3$ or G/KG_2 ;
5. the abelian group KG_2/G_2 is divisible. □

These facts are easily derived from Theorem 2.2 or the – essentially equivalent – Theorem 2.6.

Nevertheless, Theorem 2.2 surprisingly shows that the inclusion $K_2G_3 \subset D_{3,\mathbb{Z}}^{\gamma}(G, K)$ is not always an equality, as was suggested by the known partial results reviewed in 2.3. Indeed, we find counterexamples which are p -groups for any prime p , see 2.4 below; this is in contrast to the case of classical dimension subgroups (i.e. $K = 1$) which coincide with the terms of the lower central series of G unless $p = 2$ (due to a recent result of Gupta).

Example 2.4 — Let p be a prime and $0 < r \leq s$. Define

$$G = \langle x, y \mid 1 = x^{p^{s+1}} = y^{p^{s+1}} = [x, [x, y]] = [y, [x, y]] \rangle .$$

Let $K = \text{sgp}\{x^{p^r}, y^{p^s}, [x, y]\}$. Then $z = [x, y]^{p^s} = [x, y^{p^s}] \in D_{3,\mathbb{Z}}^{\gamma}(G, K)$ by 2.2, but z has order p modulo $K_2G_3 = \{1\}$. □

In order to prove Theorem 2.2, we need to study a related quotient of the group algebra. The relative polynomial group is defined by

$$P_{n,R}^{\mathcal{N}}(G, K) = I_R(G)/(I_R(K)I_R(G) + I_{R,\mathcal{N}}^{n+1}(G)).$$

It generalizes the well-known polynomial group of Passi, $P_{n,R}^{\mathcal{N}}(G) = P_{n,R}^{\mathcal{N}}(G, \{1\})$, see [14] or also [16]. Also the relative version is implicit in the work of Passi and various other places in the literature. For a discussion and more properties, in particular of the torsion subgroup and the torsion-free quotient of $P_{n,\mathbb{Z}}^{\gamma}(G, K)$, see [7]. Here we only resemble some elementary properties in the following

Lemma 2.5 — Let $K \triangleleft G$ be a normal subgroup. Then $P_{n,R}^{\mathcal{N}}(G, K)$ admits a left $R(G/K)$ -module structure induced by multiplication in $I_R(G)$, which makes the canonical map

$$p_{n,R}^{\mathcal{N}} : G \rightarrow P_{n,R}^{\mathcal{N}}(G, K), \quad a \mapsto (a - 1) + I_R(K)I_R(G) + I_{R,\mathcal{N}}^{n+1}(G)$$

into a (left) derivation, $p_{n,R}^{\mathcal{N}}(ab) = ap_{n,R}^{\mathcal{N}}(b) + p_{n,R}^{\mathcal{N}}(a)$ for $a, b \in G$. Moreover, there is an exact sequence of $R(G/K)$ -linear homomorphisms

$$R \otimes (KN_{n+1}/K_2N_{n+1}) \xrightarrow{p_{n,R}^{\mathcal{N}}i} P_{n,R}^{\mathcal{N}}(G, K) \xrightarrow{P_{n,R}^{\mathcal{N}}(\pi)} P_{n,R}^{\pi\mathcal{N}}(G/K) \rightarrow 0,$$

where i is induced by the inclusion $K \hookrightarrow G$, $p_{n,R}^{\mathcal{N}}i$ also denotes its R -linear extension, $\pi\mathcal{N}$ is the \mathcal{N} -series $\pi\mathcal{N}_i = \pi(N_i)$ with $\pi : G \rightarrow G/K$, $P_{n,R}^{\mathcal{N}}(\pi)\{a - 1\} = \{aK - 1\}$, and where the $R(G/K)$ -action on the left-hand term is induced by R -linear extension of the (G/K) -action induced by conjugation in G . □

Note that by definition,

$$\text{Ker}(p_{n,\mathbb{Z}}^{\mathcal{N}}i) = \frac{KN_{n+1} \cap D_{n+1,\mathbb{Z}}^{\mathcal{N}}(G, K)}{K_2N_{n+1}}$$

which is nontrivial in general; indeed, in the case $K = N_n$ one rediscovers the classical dimension subgroup problem, so that in taking G to be the group of Rips [18], $K = G_3$ and $\mathcal{N} = \gamma$ we have $\text{Ker}(p_{3,\mathbb{Z}}^{\gamma}i) \neq \{1\}$. In the next theorem we calculate $\text{Ker}(p_{2,\mathbb{Z}}^{\mathcal{N}}i)$ by extending the sequence in Lemma 2.5 on the left, as follows.

Consider the following part of a six-term exact sequence for the tensor and torsion product of abelian groups.

$$\begin{array}{ccccccc} \text{Tor}_1^{\mathbb{Z}}(G/KN_2, G/KN_2) & \xrightarrow{\tau} & (G/KN_2) \otimes (KN_2/N_2) & \xrightarrow{id \otimes j} & (G/KN_2) \otimes (G/N_2) & & \\ & & & & \downarrow id \otimes q & & \\ & & 0 & \leftarrow & (G/KN_2) \otimes (G/KN_2) & & \end{array}$$

Here j, q are the canonical inclusion and quotient map, respectively. Moreover, commutation in G induces a homomorphism

$$[,]: (G/KN_2) \otimes (KN_2/N_2) \rightarrow KN_3/K_2N_3.$$

Theorem 2.6 — *Let K be a normal subgroup of a group G . Then the following sequence of natural homomorphisms is exact:*

$$\mathrm{Tor}_1^{\mathbb{Z}}(G/KN_2, G/KN_2) \xrightarrow{[,]\tau} KN_3/K_2N_3 \xrightarrow{p_{2,\mathbb{Z}}^{\mathcal{N}^i}} P_{2,\mathbb{Z}}^{\mathcal{N}}(G, K) \xrightarrow{P_{2,\mathbb{Z}}^{\mathcal{N}}(\pi)} P_{2,\mathbb{Z}}^{\pi\mathcal{N}}(G/K) \rightarrow 0.$$

We remark that this result leads to an application in group cohomology with respect to the variety of nilpotent groups of class 2 [10].

The proof of Theorem 2.6 requires a homological lemma which is useful also elsewhere.

Lemma 2.7 — *Let A be an abelian group and $B \xrightarrow{j} A$ a subgroup. Consider the following homomorphisms*

$$(A/B) \otimes B \xleftarrow{q \otimes id} A \otimes B \xrightarrow{\nu} A \wedge A \xrightarrow{\ell} A \otimes A \xrightarrow{q \otimes id} (A/B) \otimes A$$

where $A \wedge A = A \otimes A / \mathrm{sgp}\{a \otimes a \mid a \in A\}$, $a \wedge a' = \{a \otimes a'\}$, $\nu(a \otimes b) = a \wedge b$, $l(a \otimes a') = a \otimes a' - a' \otimes a$, and where q is the quotient map. Then

$$\mathrm{Ker}((q \otimes id)\ell) = \nu(q \otimes id)^{-1} \mathrm{Im}(\tau: \mathrm{Tor}_1^{\mathbb{Z}}(A/B, A/B) \rightarrow (A/B) \otimes B),$$

where τ appears in the following part of a six-term exact sequence,

$$\mathrm{Tor}_1^{\mathbb{Z}}(A/B, A/B) \xrightarrow{\tau} (A/B) \otimes B \xrightarrow{id \otimes j} (A/B) \otimes A \xrightarrow{id \otimes q} (A/B) \otimes (A/B) \rightarrow 0.$$

PROOF : Consider the following commutative square

$$\begin{array}{ccccc} A \wedge A & \xrightarrow{\ell} & A \otimes A & \xrightarrow{q \otimes id} & A/B \otimes A \\ \downarrow q \otimes q & & & & \downarrow id \otimes q \\ (A/B) \wedge (A/B) & \xrightarrow{\ell} & & & (A/B) \otimes (A/B) \end{array}$$

As the map ℓ is injective for all abelian groups (see [1]), $\mathrm{Ker}((q \otimes id)\ell)$ is contained in $\mathrm{Ker}(q \otimes q) = \mathrm{Im}(\nu)$, whence

$$\begin{aligned} \mathrm{Ker}((q \otimes id)\ell) &= \nu \mathrm{Ker}((q \otimes id)\ell\nu) \\ &= \nu \mathrm{Ker}(q \otimes j) \\ &= \nu \mathrm{Ker}((id \otimes j)(q \otimes id)) \\ &= \nu(q \otimes id)^{-1} \mathrm{Ker}(id \otimes j) \\ &= \nu(q \otimes id)^{-1} \mathrm{Im}(\tau). \end{aligned}$$

□

PROOF OF THEOREM 2.6 : Consider the following commutative diagram of homomorphisms

$$\begin{array}{ccccc}
 (G/N_2) \wedge (G/N_2) & \xrightarrow{\ell} & (G/N_2) \otimes (G/N_2) & \xrightarrow{q \otimes id} & (G/KN_2) \otimes (G/N_2) \\
 \downarrow c & & \downarrow \mu & & \downarrow \bar{\mu} \\
 N_2/N_3 & \xrightarrow{p_{2,\mathbb{Z}}^{\mathcal{N}}} & I_{\mathbb{Z},\mathcal{N}}^2(G)/I_{\mathbb{Z},\mathcal{N}}^3(G) & \xrightarrow{\bar{q}} & \frac{I_{\mathbb{Z},\mathcal{N}}^2(G)}{I_{\mathbb{Z}}(K)I_{\mathbb{Z}}(G) + I_{\mathbb{Z},\mathcal{N}}^3(G)}
 \end{array}$$

where

$$c(aN_2 \wedge bN_2) = [a, b]N_3, \quad \mu(aN_2 \otimes bN_2) = (a - 1)(b - 1) + I_{\mathcal{N}}^3(G),$$

and where \bar{q} is the canonical quotient map and $\bar{\mu}$ is induced by μ . Indeed, $\text{Ker}(\bar{q}) = \mu \text{Ker}(q \otimes id)$, so the right-hand square is a pushout of abelian groups (cf. [19]), as is the left-hand square by the identity $Q_{2,\mathbb{Z}}^{\mathcal{N}}(G) = U_2L^{\mathcal{N}}(G) = (G/N_2) \otimes (G/N_2)/l \text{Ker}(c)$ obtained in [6], see also [2] (and which can also be derived from Passi's theorem that $D_3^{\mathcal{N}}(G, \zeta_1(G)) = G_3$, see [16, V.5.9], by using his technique in [16, VIII.8.7]). So by general nonsense (gluing of pushouts, which is easily verified by using the universal property), also the exterior rectangle is a pushout. Therefore,

$$(G \cap (1 + I_{\mathbb{Z}}(K)I_{\mathbb{Z}}(G) + I_{\mathbb{Z},\mathcal{N}}^3(G)))/N_3 = \text{Ker}(\bar{q}p_{2,\mathbb{Z}}^{\mathcal{N}}) = c \text{Ker}((q \otimes id)\ell)$$

where the first identity follows from the elementary relations

$$G \cap (1 + I_{\mathbb{Z}}(N)I_{\mathbb{Z}}(G) + I_{\mathbb{Z},\mathcal{N}}^3(G)) \subset G \cap (1 + I_{\mathbb{Z},\mathcal{N}}^2(G)) = N_2. \tag{1}$$

Now apply Lemma 2.7 for $A = G/N_2$ and $B = KN_2/N_2$. Then one has

$$c\nu \text{Ker}(q \otimes id) = c\nu((KN_2/N_2) \otimes (KN_2/N_2)) = (K_2N_3)/N_3,$$

so the result follows from commutativity of the following diagram.

$$\begin{array}{ccccc}
 (G/KN_2) \otimes (KN_2/N_2) & \xleftarrow{q \otimes id} & (G/N_2) \otimes (KN_2/N_2) & \xrightarrow{\nu} & (G/N_2) \wedge (G/N_2) \\
 \downarrow [,\cdot] & & \searrow & & \downarrow c \quad \square \\
 ((KN_3) \cap N_2)/K_2N_3 & \hookrightarrow & N_2/K_2N_3 & \longleftarrow & N_2/N_3
 \end{array}$$

PROOF OF THEOREM 2.2 : We first observe that K may be replaced by the normal subgroup KN_2G^m , indeed,

$$I_R(KN_2G^m) \equiv I_R(K) + I_R(N_2) + m I_R(G) \equiv I_R(K) \pmod{I_{R,\mathcal{N}}^2(G)},$$

whence

$$D_{3,R}^{\mathcal{N}}(G, K) = D_{3,R}^{\mathcal{N}}(G, KN_2G^m) = N_2G^m \cap (1_R + I_R(KN_2G^m)I_R(G) + I_{R,\mathcal{N}}^3(G)) \tag{2}$$

where the second equation is obtained from the estimate

$$G \cap (1_R + I_R(KN_2G^m)I_R(G) + I_{R,\mathcal{N}}^3(G)) \subset G \cap (1_R + I_{R,\mathcal{N}}^2(G)) = G \cap (1_R + I_R^2(G) + I_R(N_2))$$

together with the isomorphisms

$$I_R(G)/(I_R^2(G) + I_R(N_2)) \cong R \otimes (G/N_2) \cong G/N_2G^m,$$

cf. (11) and (5) below. Together with (1) and Theorem 2.6 we obtain identities

$$\begin{aligned} \frac{D_{3,\mathbb{Z}}^{\mathcal{N}}(G, KN_2G^m)}{K_2N_3(G^m)_2} &= \frac{KN_2G^m \cap (1 + I_{\mathbb{Z}}(KN_2G^m)I_{\mathbb{Z}}(G) + I_{\mathbb{Z},\mathcal{N}}^3(G))}{K_2N_3(G^m)_2} \\ &= \text{Ker}(p_2^{\mathcal{N}}i) \\ &= \text{Im}\left([\cdot, \cdot]\tau: \text{Tor}_1^{\mathbb{Z}}(G/(KN_2G^m), G/(KN_2G^m)) \longrightarrow \right. \\ &\quad \left. (KN_2G^m)/(K_2N_3(G^m)_2)\right) \end{aligned} \tag{3}$$

To make the last term explicit we use the description of the torsion product of abelian groups and the connecting homomorphism given in [13, V.6]. In fact, let $\langle \bar{x}_1, k, \bar{x}_2 \rangle$ be a generator of $\text{Tor}_1^{\mathbb{Z}}(G/KN_2G^m, G/KN_2G^m)$, i.e., $k \in \mathbb{Z}$, $\bar{x}_i = x_iKN_2G^m$ for $x_i \in G$ such that $x_i^k \in KN_2G^m$, $i = 1, 2$. Then $[\cdot, \cdot]\tau\langle \bar{x}_1, k, \bar{x}_2 \rangle = [x_1, x_2^k]$, so

$$\text{Im}([\cdot, \cdot]\tau) = U_mN_3/K_2N_3(G^m)_2, \tag{4}$$

noting that U_m contains $K_2(G^m)_2$. By (2) and (3) this proves the assertion for $m = 0$.

The case $m > 0$. In the sequel we shall frequently use the canonical identifications

$$R \otimes X \cong X/mX \cong X \otimes R, \quad 1_R \otimes x \mapsto x + mX \mapsto x \otimes 1_R \tag{5}$$

for abelian groups X , and the fact that $\binom{m}{2}(x \otimes 1_R) = -\binom{m}{2}(x \otimes 1_R)$.

We start with the observation that $P_{2,R}^{\mathcal{N}}(G, KN_2G^m) \cong R \otimes P_{2,\mathbb{Z}}^{\mathcal{N}}(G, KN_2G^m)$. Now consider the following commutative diagram.

$$\begin{array}{ccccc} \text{Tor}_1^{\mathbb{Z}}(R, \text{SP}^2(G/KN_2G^m)) & \xrightarrow{\tau_1} & R \otimes \frac{(G/N_2) \wedge (G/N_2)}{\text{Ker}((q \otimes id)\ell)} & \xrightarrow{R \otimes (q \otimes id)\ell} & R \otimes (G/KN_2G^m) \otimes (G/N_2) \\ \downarrow \mu_* & & \downarrow R \otimes c & & \downarrow R \otimes \bar{\mu} \\ \text{Tor}_1^{\mathbb{Z}}(R, P_{2,\mathbb{Z}}^{\gamma}(G/KN_2G^m)) & \xrightarrow{\tau_2} & R \otimes (KN_2G^m/U_mN_3) & \xrightarrow{R \otimes p_{2,\mathbb{Z}}^{\mathcal{N}}i} & R \otimes P_{2,\mathbb{Z}}^{\mathcal{N}}(G, KN_2G^m) \\ \parallel & & & & \\ \text{Tor}_1^{\mathbb{Z}}(R, P_{2,\mathbb{Z}}^{\gamma}(G/KN_2G^m)) & \xrightarrow{\rho_*} & \text{Tor}_1^{\mathbb{Z}}(R, G/KN_2G^m) & \xrightarrow{\tau_3} & R \otimes \text{SP}^2(G/KN_2G^m). \end{array}$$

The lines are parts of the six-term exact sequences associated with the short exact sequences

$$\begin{aligned}
 ((G/N_2) \wedge (G/N_2))/\text{Ker}((q \otimes id)\ell) &\xrightarrow{(q \otimes id)\ell} (G/KN_2G^m) \otimes (G/N_2) \xrightarrow{q} \text{SP}^2(G/KN_2G^m) \\
 (KN_2G^m)/(U_mN_3) &\xrightarrow{p_{2,\mathbb{Z}}^{\mathcal{N}i}} P_{2,\mathbb{Z}}^{\mathcal{N}}(G, KN_2G^m) \xrightarrow{P_{2,\mathbb{Z}}^{\mathcal{N}}(\pi)} P_{2,\mathbb{Z}}^{\gamma}(G/KN_2G^m) \\
 \text{SP}^2(G/KN_2G^m) &\xrightarrow{\mu} P_{2,\mathbb{Z}}^{\gamma}(G/KN_2G^m) \xrightarrow{\rho} G/KN_2G^m.
 \end{aligned} \tag{6}$$

In the first sequence, q denotes the canonical quotient map. The second sequence is obtained from 2.5, (3) and (4). The third sequence is due to Passi [15], where SP^2 denotes the symmetric tensor product, and $\rho p_{2,\mathbb{Z}}^{\gamma}(gKN_2G^m) = gKN_2G^m$.

Our goal is to compute $\text{Ker}(R \otimes p_{2,\mathbb{Z}}^{\mathcal{N}i}) = \text{Im}(\tau_2)$. First note that tensoring with $R = \mathbb{Z}/m\mathbb{Z}$ leaves the first sequence unchanged by (5); this implies that $\tau_1 = 0$. So τ_2 factors through a map

$$\bar{\tau}_2 : \text{Ker}(\tau_3) = \text{Im}(\rho_*) \cong \text{coker}(\mu_*) \rightarrow R \otimes (KN_2G^m/U_mN_3),$$

so that

$$\text{Im}(\tau_2) = \text{Im}(\bar{\tau}_2). \tag{7}$$

In order to calculate $\text{Ker}(\tau_3)$ we use the canonical identification of $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A)$ with the subgroup $A_{(m)}$ of m -torsion elements of an abelian group A .

Lemma 2.8 — For an abelian group A and $m \geq 0$, one has

$$\text{Ker}(\tau_3 : A_{(m)} \rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes \text{SP}^2(A)) = \begin{cases} A_{(m)} & \text{if } m \text{ is odd;} \\ (A_{(m)} \cap A^2)A_{(m/2)} & \text{if } m \text{ is even.} \end{cases}$$

PROOF: The assertion is true for $m = 0$ by exactness of sequence (6) and since for $m = 0$, $A_{(m)} = A = A_{(\frac{m}{2})}$. So suppose $m > 0$. For $a \in A_{(m)}$ one has

$$\begin{aligned}
 \tau_3(a) &= 1_R \otimes \mu^{-1}(mp_{2,\mathbb{Z}}^{\gamma}(a)) \\
 &= 1_R \otimes \mu^{-1}(p_{2,\mathbb{Z}}^{\gamma}(a^m) - \binom{m}{2} p_{2,\mathbb{Z}}^{\gamma}(a) p_{2,\mathbb{Z}}^{\gamma}(a)) \\
 &= 1_R \otimes \binom{m}{2} a \hat{\otimes} a.
 \end{aligned}$$

If m is odd, $\binom{m}{2}$ is divisible by m , so $\tau_3(a) = 0$. Now suppose that m is even. If $a = xy$, $y \in A_{(m/2)}$, $x \in A_{(m)}$ such that $x = \tilde{x}^2$ for some $\tilde{x} \in A$, then

$$\begin{aligned}
 \tau_3(a) &= 1_R \otimes 2 \binom{m}{2} x \hat{\otimes} \tilde{x} + 1_R \otimes 2 \binom{m}{2} x \hat{\otimes} y + 1_R \otimes (m-1) \frac{m}{2} y \hat{\otimes} y \\
 &= 0.
 \end{aligned}$$

To prove the converse inclusion, $\text{Ker}(\tau_3) \subset (A_{(m)} \cap A^2)A_{(m/2)}$, we may by a standard argument suppose that A is finitely generated. Choose a decomposition $t(A) = \bigoplus_{p,r} \mathbb{Z}/p^r \mathbb{Z} \cdot a_{p,r}$ of the torsion subgroup $t(A)$ of A . Then any $a \in A_{(m)}$ can be written in the form $a = x^2 \prod_r a_{2,r}^{c_r}$ with $x \in A$ such that $x^2 \in A_{(m)}$, and $c_r \in \mathbb{Z}$ such that c_r is odd if it is non-zero. As above, we get

$$\begin{aligned} \tau_3(a) &= \sum_r 1_R \otimes \binom{m}{2} c_r^2 a_{2,r} \widehat{\otimes} a_{2,r} + \sum_{r < s} 1_R \otimes 2 \binom{m}{2} c_r c_s a_{2,r} \widehat{\otimes} a_{2,s} \\ &= \sum_r 1_R \otimes \binom{m}{2} c_r^2 a_{2,r} \widehat{\otimes} a_{2,r}. \end{aligned}$$

Now suppose $a \in \text{Ker}(\tau_3)$. Then it follows from the decomposition $\text{SP}_2(A) = \bigoplus_{p,r \leq s} \mathbb{Z}/p^r \mathbb{Z} \cdot a_{p,r} \widehat{\otimes} a_{p,s}$ that $\binom{m}{2} c_r^2 \equiv 0 \pmod{(2^r, m)}$ for all r . This implies that $\frac{m}{2} c_r \equiv 0 \pmod{2^r}$ for all r , whence $(\prod_r a_{2,r}^{c_r})^{\frac{m}{2}} = 1$. Thus $a \in (A_{(m)} \cap A^2)A_{(m/2)}$, as asserted. \square

For $A = G/KN_2G^m$ we have $A_{(m)} = A$. Write $\bar{g} = gKN_2G^m$ for $g \in G$. Let $a \in \text{Ker}(\tau_3)$. Then by the lemma, $a = \bar{x}^2 \bar{y}$ where $x, y \in G$ such that $y = 1$ if m is odd, and $y^{\frac{m}{2}} \in KN_2G^m$ if m is even. One has

$$\rho_*(2p_{2,\mathbb{Z}}^\gamma(\bar{x}) + p_{2,\mathbb{Z}}^\gamma(\bar{y})) = a$$

and in $P_{2,\mathbb{Z}}^{\mathcal{N}}(G, KN_2G^m)$,

$$\begin{aligned} m(2p_{2,\mathbb{Z}}^{\mathcal{N}}(x) + p_{2,\mathbb{Z}}^{\mathcal{N}}(y)) &= p_{2,\mathbb{Z}}^{\mathcal{N}}(x^{2m}) - \binom{2m}{2} \bar{\mu}(\bar{x} \otimes xN_2) \\ &\quad + p_{2,\mathbb{Z}}^{\mathcal{N}}(y^m) - \binom{m}{2} \bar{\mu}(\bar{y} \otimes yN_2) \\ &= p_{2,\mathbb{Z}}^{\mathcal{N}}(x^{2m}) + p_{2,\mathbb{Z}}^{\mathcal{N}}(y^m) \\ &= p_{2,\mathbb{Z}}^{\mathcal{N}}i(x^{2m}y^m). \end{aligned}$$

Whence

$$2p_{2,\mathbb{Z}}^\gamma(\bar{x}) + p_{2,\mathbb{Z}}^\gamma(\bar{y}) = P_{2,\mathbb{Z}}^{\mathcal{N}}(\pi)(2p_{2,\mathbb{Z}}^{\mathcal{N}}(x) + p_{2,\mathbb{Z}}^{\mathcal{N}}(y)) \in (P_{2,\mathbb{Z}}^\gamma(G/KN_2G^m))_{(m)} \cap \rho_*^{-1}\{a\},$$

and

$$\begin{aligned} \bar{\tau}_2(a) = \tau_2(2p_{2,\mathbb{Z}}^\gamma(\bar{x}) + p_{2,\mathbb{Z}}^\gamma(\bar{y})) &= 1_R \otimes (x^{2m}y^m U_m N_3) \\ &= 2 \cdot 1_R \otimes (x^m U_m N_3) + 1_R \otimes (y^m U_m N_3). \end{aligned} \tag{8}$$

The latter equation shows that for odd m , $\text{Im}(\bar{\tau}_2) = 1_R \otimes (G^m U_m N_3 / U_m N_3)$. Now abbreviate $V^{(m)} = G^m$ if m is odd, and $V^{(m)} = G^{2m} V^m$ if m is even. Then by (7) and (8) we obtain

$$\text{Ker}(R \otimes p_{2,\mathbb{Z}}^{\mathcal{N}} i) = \text{Im}(\tau_2) = \text{Im}(\bar{\tau}_2) = 1_R \otimes (V^{(m)} U_m N_3 / U_m N_3).$$

Then the factorization

$$p_{2,\mathbb{Z}}^{\mathcal{N}} i: KN_2 G^m \twoheadrightarrow R \otimes (KN_2 G^m / U_m N_3) \xrightarrow{R \otimes p_{2,\mathbb{Z}}^{\mathcal{N}} i} R \otimes P_{2,\mathbb{Z}}^{\mathcal{N}}(G, KN_2 G^m) \cong P_{2,R}^{\mathcal{N}}(G, KN_2 G^m)$$

shows that

$$KN_2 G^m \cap (1 + I_R(KN_2 G^m)I_R(G) + I_{R,\mathcal{N}}^3(G)) = \text{Ker}(p_{2,R}^{\mathcal{N}} i) = U_m N_3 V^{(m)},$$

also noting that $V^{(m)}$ contains $(KN_2 G^m)^m$. Together with (2) this proves the theorem. □

3. THE THIRD RELATIVE FOX SUBGROUP

We consider the following common generalization of Fox subgroups and of relative dimension subgroups:

Definition 3.1 — Let G be a group and H, K be subgroups of G . For a commutative ring R with unit 1_R let $I_R(G)$ denote the augmentation ideal of the group algebra $R(G)$. Then define the “ n th relative Fox subgroup with respect to H, K, R ” to be the term

$$G \cap (1_R + R(G)I_R(K)I_R(H) + I_R^n(G)I_R(H)), \tag{9}$$

with $n \geq 0$ and $I_R^0(G) = R(G)$.

Note that for $H = G$ and K normal in G this group is the $n + 1$ st relative dimension subgroup with respect to K introduced by Passi (cf. [16]). On the other hand, for $K = \{1\}$ we rediscover the classical n th Fox subgroup of G with respect to H . The mixed case ($H \neq G$ and $K \neq \{1\}$) seems to have first been studied by Tahara, Vermani and Razdan [17] where the group (9) is determined for $n = 2$, $R = \mathbb{Z}$, H normal and a specific subgroup K of $[H, G]$.

We determine the group (9) for $n \leq 2$ in full generality, as follows. The case $n = 0$ is elementary; here

$$G \cap (1_R + R(G)I_R(H)) = H, \tag{10}$$

cf. [8, Lemma 7]. For $n = 1$ we have the following.

Proposition 3.2 — Let n_R denote the characteristic of R . Then the following is true, where we use the notation of Theorem 2.1.

1. If $n_R = 0$ then

$$G \cap (1_R + I_R(G)I_R(H)) = H_2 \prod_{p \in \sigma(R)} p^e t_p(H \bmod H_2).$$

2. If $n_R > 0$ then

$$G \cap (1_R + I_R(G)I_R(H)) = H_2 H^{n_R}.$$

PROOF : Consider the following sequence of homomorphisms

$$H/H_2 \rightarrow \frac{R(G)I_R(H)}{I_R(G)I_R(H)} \cong \frac{I_R(H)}{I_R^2(H)} \cong \frac{R \otimes I_{\mathbb{Z}}(H)}{\text{Im}(R \otimes I_{\mathbb{Z}}^2(H))} \cong R \otimes \left(\frac{I_{\mathbb{Z}}(H)}{I_{\mathbb{Z}}^2(H)} \right) \cong R \otimes (H/H_2) \tag{11}$$

where the first one is given by $hH_2 \mapsto h - 1 + I_R(G)I_R(H)$. The composition is the canonical morphism $j_R: H/H_2 \rightarrow R \otimes (H/H_2)$. By (10) we have $(G \cap (1_R + I_R(G)I_R(H)))/H_2 = \text{Ker}(j_R)$, which was computed in [8, Lemma 6]. The formula provided there gives the result. \square

For $n = 2$ one uses the universal coefficient decomposition obtained in [8, Corollary 3] to reduce from an arbitrary coefficient ring R to the case that $R = \mathbb{Z}/m\mathbb{Z}$, $m \geq 0$. Then the computation is completed by the following result.

Theorem 3.3 — *Let G be a group and H, K be subgroups of G . Let $m \geq 0$ and $R = \mathbb{Z}/m\mathbb{Z}$. Then*

$$(i) \quad G \cap (1_R + I_R(G)I_R(H)) + I_R^2(G)I_R(H) = S_m,$$

$$S_m \stackrel{\text{def}}{=} \text{sgp} \left\{ \prod_{h,k \in H} [h, k]^{a_{hk}} g^m \mid g = \prod_{l \in H} l^{b_l}, \text{ all } a_{hk}, b_l \in \mathbb{Z}, \text{ and } \forall k \in H: \right.$$

$$\left. \exists d_k \geq 0: k^{d_k} \in H_2 H^m \text{ and } \prod_{h \in H} h^{a_{hk} - a_{kh} + \binom{m}{2} b_h b_k} \in K G_2 G^{d_k} \right\}.$$

(ii) If H/H_2 is finitely generated we have the following improvement of (i). Choose a decomposition $H/H_2 H^m \cong \bigoplus_{k=1}^r \mathbb{Z}/d_k \mathbb{Z} \cdot (h_k H_2 H^m)$, $h_k \in H$. Then

$$G \cap (1_R + I_R(K)I_R(H) + I_R^2(G)I_R(H)) = S_m^{fg} H_3 H^{m^2},$$

$$S_m^{fg} \stackrel{\text{def}}{=} \text{sgp} \left\{ \prod_{1 \leq i < j \leq r} [h_i, h_j]^{a_{ij}} \left(\prod_{l=1}^r h_l^{b_l} \right)^m \mid a_{ij}, b_l \in \mathbb{Z}, \forall 1 \leq k \leq r: \right.$$

$$\left. h_k^{\binom{m}{2} b_k^2} \prod_{i < k} h_i^{a_{ik} + \binom{m}{2} b_i b_k} \prod_{j > k} h_j^{-a_{kj} + \binom{m}{2} b_j b_k} \in K G_2 G^{d_k} \right\}.$$

Remark 3.4 : (1) It is easy to check directly that $G \cap (1_R + I_R(K)I_R(H) + I_R^2(G)I_R(H))$ contains the canonical subgroup $H_3V_H^{(m)}T_1T_2$ where $V_H^{(m)} = H^m$ if m is odd and $V_H^{(m)} = H^{2m}W^m$ with $W = \{h \in H \mid h^{m/2} \in KG_2G^m\}$ if m is even,

$$T_1 = \text{sgp}\{ [h, k^q] \mid h, k \in H, q \in \mathbb{Z}, h^q, k^q \in KG_2G^m \},$$

$$T_2 = \text{sgp}\{ [h, k] \mid h \in H \cap KG_2G^m, k \in H \cap KG_2G^mG^q, q \in \mathbb{Z}, h^q \in H_2 \}.$$

Note that T_1 contains $[H \cap KG_2G^m, H \cap KG_2G^m]$.

(2) Compared to the description of $G \cap (1_{\mathbb{Z}} + I_{\mathbb{Z}}^2(G)I_{\mathbb{Z}}(H))$ in [3] for finitely generated G the one given in Theorem 3.3(ii) - apart from being more general (only H/H_2H^m finitely generated instead of G/G_2 , arbitrary R and K) - has the advantage not to require the choice of elementary-divisor-compatible generators of G/G_2 and HG_2/G_2 but just the choice of any basis of H/H_2H^m .

PROOF OF THEOREM 3.3 : In the sequel we shall frequently use the remarks around (5). Consider the following commutative diagram, where we abbreviate $(H/H_2)^{\wedge 2} = (H/H_2) \wedge (H/H_2)$.

$$\begin{array}{ccccc} (H/H_2)^{\wedge 2} & \xrightarrow{\ell} & (H/H_2) \otimes (H/H_2) \otimes R & \xrightarrow{i \otimes \text{id} \otimes R} & (G/KG_2) \otimes (H/H_2) \otimes R \\ \downarrow c & & \downarrow \mu_{H,R} & & \downarrow \mu_{G,R} \\ H_2/H_3 & \xrightarrow{p_{2,R}} & I_R^2(H)/I_R^3(H) & \xrightarrow{j} & \frac{I_R(G)I_R(H)}{I_R(K)I_R(H) + I_R^2(G)I_R(H)} \end{array} \quad (12)$$

where for $a, b, h \in H, g \in G$, and $r \in R$

$$l((aH_2) \wedge (bH_2)) = (aG_2) \otimes (bH_2) \otimes 1_R - (bG_2) \otimes (aH_2) \otimes 1_R,$$

$$c((aH_2) \wedge (bH_2)) = aba^{-1}b^{-1}H_3,$$

$$\mu_{H,R}((aG_2) \otimes (bH_2) \otimes r) = r(a-1)(b-1) + I_R^3(H),$$

$$\mu_{G,R}((gG_2) \otimes (hH_2) \otimes r) = r(g-1)(h-1) + I_R(K)I_R(H) + I_R^2(G)I_R(H),$$

and where i and j are induced by the inclusion $H \hookrightarrow G$.

By proposition 3.2, $G \cap (1_R + I_R(G)I_R(H)) = H_2H^m$. Modulo $I_R(K)I_R(H) + I_R^2(G)I_R(H)$, we have for $h' \in H_2, h \in H$

$$\begin{aligned} h'h^m - 1_R &\equiv (h' - 1_R) + m(h - 1_R) + \binom{m}{2}(h - 1_R)^2 \\ &= j\left(p_{2,R}(h') + \binom{m}{2}\mu_{H,R}(hH_2 \otimes hH_2 \otimes 1_R)\right). \end{aligned} \quad (13)$$

Now we use the crucial fact ([9], Proof of Theorem 3.1) that the right-hand square of diagram (12) is a cocartesian (or pushout) square which implies that $\text{Ker}(j) = \mu_{H,R} \text{Ker}(i \otimes id \otimes R)$. Using this and Proposition 3.2 we conclude that $x \in G \cap (1_R + I_R(K)I_R(H) + I_R^2(G)I_R(H))$ if and only if $x = h'h^m$ as above such that in $I_R^2(H)/I_R^3(H)$,

$$p_{2,R}(h'h^m) = p_{2,R}(h') + \binom{m}{2} \mu_{H,R}(hH_2 \otimes hH_2 \otimes 1_R) \in \mu_{H,R} \text{Ker}(i \otimes id \otimes R). \quad (14)$$

This situation is further analyzed in the following two lemmas.

Lemma 3.5 — The preimage of the subset $p_{2,R}(H_2H^m) \subset I_R^2(H)/I_R^3(H)$ under the map $\mu_{H,R}$ is

$$Y \stackrel{def}{=} \text{Im}(\ell) + \left\{ \binom{m}{2} (hH_2 \otimes hH_2 \otimes 1_R) \mid h \in H \right\} \subset (H/H_2) \otimes (H/H_2) \otimes R.$$

PROOF : First of all, $\mu_{H,R}(Y) = p_{2,R}(H_2H^m)$; this follows from (13) and the diagram above. Secondly, we show that Y contains $\text{Ker}(\mu_{H,R})$. For this purpose, recall the isomorphism $I_{\mathbb{Z}}^2(H)/I_{\mathbb{Z}}^3(H) \cong \text{U}_2\text{L}(G) = ((H/H_2) \otimes (H/H_2))/\ell \text{Ker}(c)$ from [2]. Using the six-term exact sequence for the tensor product one deduces exact sequences

$$\text{Ker}(c) \xrightarrow{\ell} (H/H_2) \otimes (H/H_2) \otimes R \xrightarrow{\mu_{H,\mathbb{Z}} \otimes R} (I_{\mathbb{Z}}^2(H)/I_{\mathbb{Z}}^3(H)) \otimes R \rightarrow 0$$

$$\text{Tor}_1^{\mathbb{Z}}(H/H_2, R) \xrightarrow{\tau} (I_{\mathbb{Z}}^2(H)/I_{\mathbb{Z}}^3(H)) \otimes R \xrightarrow{\nu \otimes R} (I_{\mathbb{Z}}(H)/I_{\mathbb{Z}}^3(H)) \otimes R \xrightarrow{\pi \otimes R} (H/H_2) \otimes R \rightarrow 0,$$

where ν is the inclusion and π is the quotient map modulo $\text{Im}(\nu)$, combined with the canonical isomorphism $I_{\mathbb{Z}}(H)/I_{\mathbb{Z}}^2(H) \cong H/H_2$. Identifying $\text{Tor}_1^{\mathbb{Z}}(H/H_2, R)$ with the set of m -torsion elements of H/H_2 , let $h \in H$ such that $h^m \in H_2$. Then

$$\tau(hH_2) = \nu^{-1}(mp_{2,\mathbb{Z}}(h)) \otimes 1_R = \left(p_{2,\mathbb{Z}}(h^m) - \binom{m}{2} \mu_{H,\mathbb{Z}}(hH_2 \otimes hH_2) \right) \otimes 1_R.$$

Using the canonical isomorphism $(I_{\mathbb{Z}}(H)/I_{\mathbb{Z}}^3(H)) \otimes R \cong (I_R(H)/I_R^3(H))$, we deduce that $\text{Ker}(\mu_{H,R}) = \text{Ker}((\nu \otimes R)(\mu_{H,\mathbb{Z}} \otimes R)) \subset Y$.

It remains to show that Y is a subgroup. Let $x, y \in (H/H_2) \wedge (H/H_2)$ and $h, k \in H$. Then

$$\begin{aligned} & \left(\ell(x) + \binom{m}{2} (hH_2 \otimes hH_2 \otimes 1_R) \right) - \left(\ell(y) + \binom{m}{2} (kH_2 \otimes kH_2 \otimes 1_R) \right) \\ &= \ell(x - y) + \binom{m}{2} (hkH_2 \otimes hkH_2 \otimes 1_R) + \binom{m}{2} (hH_2 \otimes kH_2 \otimes 1_R) \\ & \quad - \binom{m}{2} (kH_2 \otimes hH_2 \otimes 1_R) - 2 \binom{m}{2} (kH_2 \otimes kH_2 \otimes 1_R) \end{aligned}$$

$$= \ell\left(x - y + \binom{m}{2}(hH_2 \wedge kH_2)\right) + \binom{m}{2}(hkH_2 \otimes hkH_2 \otimes 1_R)$$

$$\in Y.$$

Thus the lemma is proved. □

Lemma 3.6 — One has $Y \cap \text{Ker}(i \otimes id \otimes R) = T_m$, with

$$T_m = \left\{ \ell\left(\sum_{h,k \in H} a_{hk}(hH_2 \wedge kH_2)\right) + \binom{m}{2}(gH_2 \otimes gH_2 \otimes 1_R) \parallel g = \prod_{l \in H} l^{b_l}, \text{ all } a_{hk}, b_l \in \mathbb{Z}, \right.$$

$$\left. \text{such that } \forall k \in H: \exists d_k \geq 0: k^{d_k} \in H_2H^m \text{ and } \prod_{h \in H} h^{a_{hk} - a_{kh} + \binom{m}{2}b_h b_k} \in KG_2G^{d_k} \right\}.$$

If H is finitely generated, we have (with the notation of 3.3 (ii))

$$Y \cap \text{Ker}(i \otimes id \otimes R) = T_m^{fg},$$

$$T_m^{fg} = \left\{ \ell\left(\sum_{1 \leq i < j \leq r} a_{ij}(h_iH_2 \wedge h_jH_2)\right) + \binom{m}{2}(gH_2 \otimes gH_2 \otimes 1_R) \parallel g = \sum_{l=1}^r h_l^{b_l} \text{ such that} \right.$$

$$\left. \forall 1 \leq k \leq r: h_k^{\binom{m}{2}b_k^2} \prod_{i < k} h_i^{a_{ik} + \binom{m}{2}b_i b_k} \prod_{j > k} h_j^{-a_{kj} + \binom{m}{2}b_j b_k} \in KG_2G^{d_k} \right\}.$$

PROOF : Let $y \in Y$, i.e. $y = \ell(\sum_{h,k \in H} a_{hk}(hH_2 \wedge kH_2)) + \binom{m}{2}(gH_2 \otimes gH_2 \otimes 1_R)$ for some $a_{hk} \in \mathbb{Z}$, $g \in H$. Consider an arbitrary decomposition $g = \prod_{l \in H} l^{b_l}$, $b_l \in \mathbb{Z}$. Using the isomorphism $\alpha: (G/KG_2) \otimes (H/H_2) \otimes R \cong (G/KG_2) \otimes (H/H_2H^m)$ we obtain

$$\begin{aligned} \alpha(i \otimes id \otimes R)(y) &= \sum_{h,k \in H} a_{hk} \left((hKG_2) \otimes (kH_2H^m) - (kKG_2) \otimes (hH_2H^m) \right) \\ &\quad + \binom{m}{2} \sum_{l_1, l_2 \in H} b_{l_1} b_{l_2} (l_1KG_2) \otimes (l_2H_2H^m) \\ &= \sum_{k \in H} \left(\prod_{h \in H} h^{a_{hk} - a_{kh} + \binom{m}{2}b_h b_k} KG_2 \right) \otimes (kH_2H^m). \end{aligned} \tag{15}$$

So if y and the chosen decomposition of g satisfy the condition in the definition of T_m then it follows from bilinearity of the tensor product that $(i \otimes id \otimes R)(y) = 0$. Whence $T_m \subset Y \cap \text{Ker}(i \otimes id \otimes R)$. To prove the converse inclusion we may by a standard argument suppose that H is finitely generated. Then $y \in Y$ can be written in the form

$$y = \ell\left(\sum_{1 \leq i < j \leq r} a_{ij}(h_iH_2 \wedge h_jH_2)\right) + \binom{m}{2}(gH_2 \otimes gH_2 \otimes 1_R)$$

with $g = \prod_{l=1}^r h_l^{b_l}$, all $a_{ij}, b_l \in \mathbb{Z}$. To see this just observe that $(H/H_2) \otimes (H/H_2) \otimes R \cong (H/H_2H^m) \otimes (H/H_2H^m)$ and that ℓ factors through $(H/H_2H^m) \wedge (H/H_2H^m)$. Now we use the composite isomorphism

$$\begin{aligned} \Psi : (G/KG_2) \otimes (H/H_2) \otimes R &\stackrel{\alpha}{\cong} (G/KG_2) \otimes (H/H_2H^m) \cong \\ &\bigoplus_{k=1}^r (G/KG_2) \otimes (\mathbb{Z}/d_k\mathbb{Z} \cdot (h_k H_2 H^m)) \cong \bigoplus_{k=1}^r G/(KG_2G^{d_k}). \end{aligned}$$

By (15) we have

$$\Psi(i \otimes id \otimes R)(y) = \bigoplus_{k=1}^r \left(h_k^{\binom{m}{2} b_k^2} \prod_{i < k} h_i^{a_{ik} + \binom{m}{2} b_i b_k} \prod_{j > k} h_j^{-a_{kj} + \binom{m}{2} b_j b_k} KG_2G^{d_k} \right).$$

So $Y \cap \text{Ker}(i \otimes id \otimes R) = T_m^{fg}$. Moreover, $T_m^{fg} \subset T_m$; to see this, let $a_{hk} = b_l = 0$ unless $h, k, l \in \{h_1, \dots, h_r\}$, and $a_{h_i h_j} = 0$ if $i > j$. Then an element of T_m^{fg} satisfies the conditions defining T_m in taking $d_k = m$ if $k \notin \{h_1, \dots, h_r\}$ and $d_{h_k} = d_k$ for $1 \leq k \leq r$. Thus also the inclusion $T_m \subset Y \cap \text{Ker}(i \otimes id \otimes R)$ already proved is an equality, as asserted. \square

Summarizing the above we obtain equations

$$p_{2,R}(G \cap (1_R + I_R(K)I_R(H) + I_R^2(G)I_R(H))) = \mu_{H,R}(T_m) = p_{2,R}(S_m),$$

analogously for T_m^{fg} and S_m^{fg} , where the first equation follows from (14), Lemmas 3.5 and 3.6 and the second one from (12) and (13). So it only remains to observe that $\text{Ker}(p_{2,R}: H_2H^m \rightarrow P_{2,R}(H)) = D_{3,R}(H)$ is contained in $V_H^{(m)}$ by Sandling's formula for $D_{3,R}(H)$, see [20] (or Theorem 2.2 above for $K = \{1\}$) and that $V_H^{(m)}$ is contained in S_m and S_m^{fg} by Remark 3.4. Thus, Theorem 3.3 is proved. \square

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