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ON THE OSCILLATION OF THIRD ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Some new criteria for the oscillation of third order functional differential equations of the form

$$\left(a(t)\left(x'(t)\right)^{\alpha}\right)'' + q(t)f(x[g(t)]) = 0$$

and

$$\left(a(t)\left(x'(t)\right)^{\alpha}\right)'' = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]),$$

where

$$\int^{\infty} a^{-1/\alpha}(s)ds < \infty$$

are established.

Key words: Functional differential equation, oscillation, nonoscillation, comparison

1. Introduction

This paper deals with the oscillatory behaviour of solutions of third order nonlinear functional differential equations

$$\left(a(t)\left(x'(t)\right)^{\alpha}\right)'' + q(t)f(x[g(t)]) = 0 \tag{1.1}$$

and

$$\left(a(t)\left(x'(t)\right)^{\alpha}\right)'' = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]), \tag{1.2}$$

subject to the hypotheses:

(i) α is the ratio of two positive odd integers;

(ii) $a,\ p,\ q : [t_0,\infty) \to (0,\infty)$ are continuous;

(iii) $g,\ \sigma \in C^1([t_0,\infty),\mathbb{R}),\ g(t) < t,\ \sigma(t) > t,\ g'(t) \geq 0$ and $\sigma'(t) \geq 0$ for $t \geq t_0$ and $\lim_{t\to\infty} g(t) = \infty$;

(iv) $f,\ h \in C^1(\mathbb{R},\mathbb{R}),\ xf(x) > 0,\ xh(x) > 0,\ f'(x) \geq 0$ and $h'(x) \geq 0$ for $x \neq 0$ and

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0 \tag{1.3}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for} \quad xy > 0. \tag{1.4}$$

By a solution of equation (1.1) (respectively, equation (1.2)) is meant a function $x : [T_x,\infty) \to \mathbb{R},\ T_x \geq t_0$ such that $x(t),\ x'(t)$ and $\left(a(t)\left(x'(t)\right)^{\alpha}\right)'$ are continuously differentiable and satisfy equation (1.1) (respectively, equation (1.2)) on $[T_x,\infty)$. Our attention will be restricted to solutions $x(t)$ of equations (1.1) and (1.2) which satisfy $\sup\{|x(t)| :\ t \geq T\} > 0$ for any $T \geq T_x$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity, and nonoscillatory otherwise.

The oscillatory behaviour of equations (1.1) and (1.2) when $\alpha = 1,\ a(t) = 1$ and/or higher order functional differential equations has received a great deal of attention in the last three decades. For recent contributions to this study we refer to reader [1-5, 8, 10] and the references cited therein, however, the study of oscillatory behaviour of equations (1.1) and (1.2) when $\alpha \neq 1$ and

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)ds = \infty \tag{1.5}$$

has received considerably less attention, see [4-6, 9]. Therefore, the main purpose of this paper is to establish some criteria for the oscillation of equations (1.1) and (1.2) when

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)ds \;<\; \infty \tag{1.6}$$

holds.

In Section 2, we present oscillation criteria for equation (1.1) and Section 3 is devoted to the study of the oscillation of equation (1.2). The results of this paper are presented in a form which is essentially new and of high degree of generality. Some special cases of the obtained results unify and improve many existing oscillation criteria appeared in the literature.

2. Oscillation of Equation (1.1)

Here we are interested to discuss the situation in which all solutions of equation (1.1) are oscillatory. In order to prove our main result we will compare the following first order delay differential inequalities

$$y'(t) + q(t)f(y[g(t)]) \;\leq\; 0\,, \tag{2.1}$$

$$y'(t) - q(t)f(y[g(t)]) \;\geq\; 0\,, \tag{2.2}$$

with the corresponding differential equations

$$y'(t) + q(t)f(y[g(t)]) \;=\; 0\,, \tag{2.3}$$

$$y'(t) - q(t)f(y[g(t)]) \;=\; 0\,. \tag{2.4}$$

We have the following lemma, which is given in [3, 7].

Lemma 2.1 — If the inequality (2.1) (the inequality (2.2)) has eventually positive solution, then the equation (2.3) (the equation (2.4)) also has eventually positive solution.

***Theorem* 2.1** — Suppose that conditions (i) – (iv), (1.3), (1.4) and (1.6) hold and assume that there exists a nondecreasing function $\eta : [t_0, \infty) \to \mathbb{R}$ such that $g(t) < \eta(t) < t$ for $t \geq t_0$. If both first order delay equations

$$z'(t) + c\,q(t)f\left(\int_T^{g(t)} \left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) f\left(z^{1/\alpha}[g(t)]\right) \;=\; 0 \tag{2.5}$$

for any constant $c,\ 0 < c < 1$ and for all $T \geq t_0$, and

$$w'(t) + q(t)f\Big(A[g(t)]\Big) f\left([\eta\,(t) - g\,(t)]^{1/\alpha}\right) f\left(w^{1/\alpha}[\eta(t)]\right) \;=\; 0 \tag{2.6}$$

are oscillatory, and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(s)} \int_T^s \int_T^u q(v) f\Big(A[g(v)]\Big) dv\, du \right)^{1/\alpha} ds \;=\; \infty, \tag{2.7}$$

for any $T \geq t_0$, where $A(t) = \displaystyle\int_t^{\infty} a^{-1/\alpha}(s)ds,$ then equation (1.1) is oscillatory.

PROOF : Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0 \geq 0$. Now $(a(t)\,(x'(t))^{\alpha})'' \leq 0$ for $t \geq t_0$. There exists a $t_1 \geq t_0$ such that $x'(t)$ and $(a(t)\,(x'(t))^{\alpha})'$ are of constant sign for $t \geq t_1$. There are four possibilities to consider:

(I) $(a(t)\,(x'(t))^{\alpha})' > 0$ and $x'(t) > 0$ for $t \geq t_1$;

(II) $(a(t)\,(x'(t))^{\alpha})' > 0$ and $x'(t) < 0$ for $t \geq t_1$;

(III) $(a(t)\,(x'(t))^{\alpha})' < 0$ and $x'(t) < 0$ for $t \geq t_1$; and

(IV) $(a(t)\,(x'(t))^{\alpha})' < 0$ and $x'(t) > 0$ for $t \geq t_1$.

The case (IV) cannot hold. In fact, if we let $y(t) = a(t)\,(x'(t))^{\alpha}$ for $t \geq t_1,$ then we see that $y''(t) < 0$ and $y'(t) < 0$ for $t \geq t_1$ and hence $\lim_{t\to\infty} y(t) = -\infty,$ which contradicts the positivity of $x'(t)$. Next, we consider:

Case (I): Let $y(t) = a(t)\,(x'(t))^{\alpha}$. Then, since $y(t) > 0$ and $y'(t)$ is decreasing on $[t_1, \infty)$, we have

$$y(t) - y(t_1) = \int_{t_1}^{t} y'(s)\, ds \geq y'(t)(t - t_1)\,.$$

Then, there exist a $t_2 \geq t_1$ and a constant $b,\ 0 < b < 1$ such that

$$y(t) \;\geq\; b\, t\, y'(t) \quad \text{for} \quad t \geq t_2\,.$$

Now

$$x'(t) \geq b^{1/\alpha} \left(\frac{t}{a(t)} \right)^{1/\alpha} (y'(t))^{1/\alpha} \quad \text{for} \quad t \geq t_2.$$

Integrating both sides of the above inequality from t_2 to $t,$ we get

$$x(t) \;\geq\; b^{1/\alpha} \left(\int_{t_2}^{t} \left(\frac{s}{a(s)} \right)^{1/\alpha} ds \right) (y'(t))^{1/\alpha} \quad \text{for} \quad t \geq t_2.$$

There exists a $t_3 \geq t_2$ such that

$$x[g(t)] \ \geq \ \bar{b}\left(\int_{t_2}^{g(t)}\left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) y'\left([g\,(t)]\right)^{1/\alpha} \quad \text{for} \quad t \geq t_3, \tag{2.8}$$

where $\bar{b} = b^{1/\alpha}$.

Using (2.8) and (1.3) in equation (1.1), we have

$$\begin{aligned} -y''(t) \quad &= \quad q(t)f(x[g(t)]) \ \geq \ f(\bar{b})q(t)f\left(\int_{t_2}^{g(t)}\left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) \\ &\qquad f\left(y'\left([g\,(t)]\right)^{1/\alpha}\right) \quad \text{for} \quad t \geq t_3. \end{aligned}$$

Set $z(t) = y'(t) > 0$ for $t \geq t_3$ in the above inequality, to obtain

$$z'(t) + f(\bar{b})q(t)f\left(\int_{t_2}^{g(t)}\left(\frac{s}{a(s)}\right)^{1/\alpha} ds\right) f\left(z^{1/\alpha}[g(t)]\right) \ \leq \ 0 \quad \text{for} \quad t \geq t_3.$$

By Lemma 2.1, we see that the equation (2.5) has an eventually positive solution, which contradicts the hypothesis.

Case (II): Since $a(t)\left(x'(t)\right)^{\alpha}$ is increasing, for $t \geq s \geq t_1,$ we obtain

$$a(t)\left(x'(t)\right)^{\alpha} \ \geq \ a(s)\left(x'(s)\right)^{\alpha} \ \text{ or } \ a(s)\left(-x'(s)\right)^{\alpha} \ \geq \ a(t)\left(-x'(t)\right)^{\alpha},$$

which by integration from t to ∞ implies

$$x\,(t) \geq \left[a\,(t)\left(-x'\,(t)\right)^{\alpha}\right]^{1/\alpha}\int_t^{\infty} a^{-1/\alpha}(s)ds.$$

Putting that $z(t) = -a(t)\left(x'\,(t)\right)^{\alpha}$ and replacing t by $g(t)$, we find

$$\begin{aligned} x[g(t)] \quad &\geq \quad \left[a\left(g\,(t)\right)\left(-x'\left(g\,(t)\right)\right)^{\alpha}\right]^{1/\alpha} A\left(g\,(t)\right) \quad \text{for} \quad t \geq t_2 \geq t_1 \\ &= \quad z^{1/\alpha}\left[g\,(t)\right] A\left(g\,(t)\right) \quad \text{for} \quad t \geq t_2\,. \end{aligned} \tag{2.9}$$

Using (2.9) and (1.3) in equation (1.1), we have

$$z''(t) \ \geq \ q(t)f\Big(A(g(t))\Big) f\left(z^{1/\alpha}\left[g\,(t)\right]\right) \quad \text{for} \quad t \geq t_2. \tag{2.10}$$

Clearly, $z(t) > 0,$ $z'(t) < 0$, $z''(t) > 0$ for $t \geq t_2$. Then, for $t \geq s \geq t_2,$ we get

$$-z(s) \leq z(t) - z(s) = \int_s^t z'(u)\,du \leq z'(t)(t-s)\,,$$

or

$$z(s) \ \geq \ (t-s)(-z'(t)).$$

Replacing s and t by $g(t)$ and $\eta(t)$ respectively, we find

$$z[g(t)] \ \geq \ (\eta(t)-g(t))(-z'[\eta(t)]) \quad \text{for} \quad t \geq t_3 \geq t_2. \tag{2.11}$$

Using (2.11) and (1.3) in (2.10), and setting $w(t)=-z'(t),\ t \geq t_3,$ we get

$$-w'(t) \ \geq \ q(t)f\Big(A(g(t))\Big)f\left([\eta(t)-g(t)]^{1/\alpha}\right)f\left(w^{1/\alpha}[\eta(t)]\right)$$

or

$$w'(t)+q(t)f\Big(A(g(t))\Big)f\left([\eta(t)-g(t)]^{1/\alpha}\right)f\left(w^{1/\alpha}[\eta(t)]\right) \ \leq \ 0 \quad \text{for} \quad t \geq t_3.$$

By applying Lemma 2.1, we see that the equation (2.6) has an eventually positive solution, which is contradiction.

Case (III): For $s \geq t \geq t_1,$ one can easily find

$$a(s)\left(-x'(s)\right)^{\alpha} \ \geq \ a(t)\left(-x'(t)\right)^{\alpha},$$

or

$$-x'(s) \ \geq \ -\left(a^{1/\alpha}(t)x'(t)\right)\left(a^{-1/\alpha}(s)\right).$$

Integrating the above inequality from t to $u \geq t$ and letting $u \to \infty,$ we get

$$x(t) \geq -a^{1/\alpha}(t)\,x'(t)\int_t^{\infty} a^{-1/\alpha}(s)ds = -a^{1/\alpha}(t)\,x'(t)\,A(t) \quad \text{for} \quad t \geq t_1 \tag{2.12}$$

and

$$x(t) \geq -a^{1/\alpha}(t_1)\,x'(t_1)\,A(t) \quad \text{for} \quad t \geq t_1\,.$$

Now, there exist a $t_2 \geq t_1$ and a constant $c>0$ such that

$$x[g(t)] \ \geq \ c\,A[g(t)] \quad \text{for} \quad t \geq t_2. \tag{2.13}$$

Using (2.13) and (1.3) in equation (1.1), we get

$$-\left(a(t)\left(x'(t)\right)^{\alpha}\right)'' \ = \ q(t)f(x[g(t)]) \ \geq \ \bar{c}\,q(t)f\Big(A[g(t)]\Big) \quad \text{for} \quad t \geq t_2, \tag{2.14}$$

where $\bar{c}=f(c)$. Integrating (2.14) from t_2 to $t,$ we have

$$-\left(a(t)\left(x'(t)\right)^{\alpha}\right)' + (a(t_2)(x'(t_2))^{\alpha})' \ \geq \ \bar{c}\int_{t_2}^{t} q(s)f\Big(A[g(s)]\Big)ds,$$

or

$$-\left(a(t)\left(x'(t)\right)^{\alpha}\right)' \;\geq\; \overline{c}\int_{t_2}^{t} q(s)f\Big(A[g(s)]\Big)ds \quad \text{for} \quad t \geq t_2. \tag{2.15}$$

Integrating (2.15) from t_2 to t, we have

$$-a(t)\left(x'(t)\right)^{\alpha} \;\geq\; \overline{c}\int_{t_2}^{t}\int_{t_2}^{s} q(u)f\Big(A[g(u)]\Big)\,du\,ds,$$

or

$$-x'(t) \;\geq\; \tilde{c}\left(\frac{1}{a(t)}\int_{t_2}^{t}\int_{t_2}^{s} q(u)f\Big(A[g(t)]\Big)\,du\,ds\right)^{1/\alpha}, \tag{2.16}$$

where $\tilde{c} = (\overline{c})^{1/\alpha}$. Integrating (2.16) from t_2 to t, and using the assumption (2.7) we have

$$\infty \;>\; x(t_2) \;\geq\; x(t_2) - x(t) \;\geq\; \tilde{c}\int_{t_2}^{t}\left(\frac{1}{a(v)}\int_{t_2}^{v}\int_{t_2}^{s} q(u)f\Big(A[g(u)]\Big)\,du\,ds\right)^{1/\alpha} dv \;\to\; \infty \quad \text{as} \quad t \;\to\; \infty$$

which is a contradiction. This completes the proof. ■

We state some known results for the first order functional differential equations

$$y'(t) + Q(t)y[g(t)] \;=\; 0\,, \tag{2.17}$$

$$y'(t) - Q(t)y[g(t)] \;=\; 0\,, \tag{2.18}$$

and

$$y'(t) + Q(t)F(y[g(t)]) \;=\; 0\,, \tag{2.19}$$

$$y'(t) - Q(t)F(y[g(t)]) \;=\; 0\,. \tag{2.20}$$

***Theorem* A** — *Let the functions* $g, Q \in C([t_0,\infty),\mathbb{R})$, $Q(t) \geq 0$ *eventually,* $g(t) \leq t$, $g'(t) \geq 0$ *for* $t \geq t_0$ and $\lim_{t\to\infty} g(t) = \infty$.

(i) If $\liminf\limits_{t\to\infty}\displaystyle\int_{g(t)}^{t} Q(s)\,ds > \frac{1}{e}$, *then the equation (2.17) is oscillatory.*

(ii) If $\liminf\limits_{t\to\infty}\displaystyle\int_{t}^{g(t)} Q(s)\,ds > \frac{1}{e}$, *then the equation (2.18) is oscillatory.*

(iii) If $\limsup\limits_{t\to\infty}\displaystyle\int_{g(t)}^{t} Q(s)\,ds > 1$, *then the equation (2.17) is oscillatory.*

(iv) If $\limsup_{t\to\infty}\int_t^{g(t)} Q(s)\,ds > 1$*, then the equation (2.18) is oscillatory.*

***Theorem* B** — *Let the functions* $g, Q \in C([t_0,\infty),\mathbb{R})$, $Q(t)\ge 0$ *eventually,* $g(t)\le t$, $g'(t)\ge 0$ *for* $t\ge t_0$ *and* $\lim_{t\to\infty} g(t)=\infty$, $F\in C^1(\mathbb{R},\mathbb{R})$, $x\,F(x)>0,\ F'(x)\ge 0$.

(i) If $\dfrac{u}{F(u)}\to 0$ *as* $u\to 0$, *and* $\limsup_{t\to\infty}\int_{g(t)}^{t} Q(s)\,ds > 0$*, then the equation (2.19) is oscillatory.*

(ii) If $\dfrac{u}{F(u)}\to 0$ *as* $u\to 0$, *and* $\liminf_{t\to\infty}\int_t^{g(t)} Q(s)\,ds > 0$*, then the equation (2.20) is oscillatory.*

(iii) If $\displaystyle\int_{\pm 0}\frac{du}{F(u)}<\infty$ *and* $\displaystyle\int^{\infty} Q(s)\,ds=\infty$*, then the equation (2.19) is oscillatory.*

(iv) If $\displaystyle\int^{\pm\infty}\frac{du}{F(u)}<\infty$ *and* $\displaystyle\int^{\infty} Q(s)\,ds=\infty$*, then the equation (2.20) is oscillatory.*

By combining Theorem A., Theorem B. and Theorem 2.1. we have the following result:

***Theorem* 2.2** — *Let conditions (i) – (iv), (1.6) and (2.7) hold and suppose that there exists a nondecreasing function* $\eta:[t_0,\infty)\to\mathbb{R}$ *such that* $g(t)<\eta(t)<t$ for $t\ge t_0$. *Equation (1.1) is oscillatory if one of the following conditions holds*:

(I_1) $\dfrac{f(u^{1/\alpha})}{u}\ge k$ *for* $u\ne 0$ *and some* $k>0$

$$\limsup_{t\to\infty}\int_{g(t)}^{t} q(s)\left(\int_T^{g(s)}\left(\frac{u}{a(u)}\right)^{1/\alpha}du\right)^{\alpha} ds \;>\; \frac{1}{c\,k^2} \tag{2.21}$$

for any constant c, $0<c<1$ *and every* $T\ge t_0$, *and*

$$\limsup_{t\to\infty}\int_{\eta(t)}^{t} q(s)\,(\eta(s)-g(s))\Big(A[g(s)]\Big)^{\alpha} ds \;>\; \frac{1}{k^3}.$$

(I_2) $\dfrac{f(u^{1/\alpha})}{u}\ge k$ *for* $u\ne 0$ *and some* $k>0$

$$\liminf_{t\to\infty}\int_{g(t)}^{t} q(s)\left(\int_T^{g(s)}\left(\frac{u}{a(u)}\right)^{1/\alpha}du\right)^{\alpha} ds \;>\; \frac{1}{c\,e\,k^2} \tag{2.22}$$

for any constant c, $0 < c < 1$ *and any* $T \geq t_0$, *and*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} q(s)(\eta(s) - g(s))\Big(A[g(s)]\Big)^{\alpha} ds \;>\; \frac{1}{e\,k^3}.$$

(I$_3$) $\dfrac{u}{f(u^{1/\alpha})} \to 0$ *as* $u \to 0$,

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s) f\left(\int_{T}^{g(s)} \left(\frac{u}{a(u)}\right)^{1/\alpha} du\right) ds \;>\; 0 \qquad \text{for} \quad T \geq t_0,$$

and

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} q(s) f\Big(A[g(s)]\Big) f\left([\eta(s) - g(s)]^{1/\alpha}\right) ds \;>\; 0.$$

(I$_4$) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$,

$$\int^{\infty} q(s) f\left(\int_{T}^{g(s)} \left(\frac{u}{a(u)}\right)^{1/\alpha} du\right) ds \;=\; \infty\,, \qquad \text{for} \quad T \geq t_0,$$

and

$$\int^{\infty} q(s) f\Big(A[g(s)]\Big) f\left([\eta(s) - g(s)]^{1/\alpha}\right) ds \;=\; \infty.$$

When the condition (1.5) holds, we can also eliminate the case (III). Indeed, from (2.12) we have

$$x(t) \;\leq\; x(t_1) - K \int_{t_1}^{t} a^{-1/\alpha}(s)\, ds \;\to -\infty, \qquad t \to \infty\,,$$

where $K = -a^{1/\alpha}(t_1)\, x'(t_1) > 0$, which contradicts the positivity of $x(t)$. Therefore, we have the following result:

***Theorem* 2.3** — *Suppose that conditions (i) – (iv), (1.3), (1.4) and (1.5) hold and assume that there exists a nondecreasing function* $\eta : [t_0, \infty) \to \mathbb{R}$ *such that* $g(t) < \eta(t) < t$ *for* $t \geq t_0$. *If both first order delay equations (2.5) and (2.6) are oscillatory, then equation (1.1) is oscillatory*.

The following example is illustrative.

Example 2.1 — Consider the third order equation

$$\left(t^4\, (x'\,(t))^3\right)'' + \frac{1}{\ln t}\, x^{\beta}[\sqrt{t}] \;=\; 0,\; t > 5, \tag{2.23}$$

where β is the ratio of two positive odd integers, $0 < \beta \leq 3$. Here $a(t) = t^4$ and $\alpha = 3$ and so

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)ds \;=\; \int_{5}^{\infty} s^{-4/3}ds \;<\; \infty.$$

Also, $g(t) = \sqrt{t}$ and so, we let $\eta(t) = 2\sqrt{t}$ for all $t > 5$. It is easy to check that all conditions of Theorem 2.2 are satisfied and hence equation (2.23) is oscillatory.

Similarly, we see that the equation

$$\left(t^{4/3}x'(t)\right)'' + \frac{1}{\ln t}\, x^{\beta}[\sqrt{t}] \;=\; 0$$

is oscillatory by Theorem 2.2 with $\alpha = 1$.

3. Oscillation of Equation (1.2)

The main objective of this section is to establish some criteria for the oscillation of all solutions of equation (1.2) of mixed nonlinearities and arguments. Now, we we will compare the following inequality

$$y''(t) + q(t)f(y[g(t)]) \;\leq\; 0\,, \tag{3.1}$$

with the corresponding second order differential equations

$$y''(t) + q(t)f(y[g(t)]) \;=\; 0\,, \tag{3.2}$$

We have the following lemma, which is given in [3, 7].

Lemma 3.1 — If the inequality (3.1) has eventually positive solution, then the equation (3.2) also has eventually positive solution.

***Theorem* 3.1** — *Suppose that conditions (i) – (iv), (1.3), (1.4) and (1.6) hold and assume that there exist nondecreasing functions* $\eta, \rho, \theta : [t_0, \infty) \to \mathbb{R}$ *such that* $g(t) < \eta(t) < t$ *and* $\sigma(t) > \rho(t) > \theta(t) > t$ *for* $t \geq t_0$. *If the first order advanced equation*

$$y'(t) - p(t)h\left(\int_{\rho(t)}^{\sigma(t)} a^{-1/\alpha}(s)ds\right) h\left([\rho(t) - \theta(t)]^{1/\alpha}\right) h\left(y^{1/\alpha}[\theta(t)]\right) \;=\; 0, \tag{3.3}$$

the first order delay equation

$$z'(t) + q(t)f\left(\int_{t_0}^{g(t)} a^{-1/\alpha}(s)ds\right) f\left([\eta(t) - g(t)]^{1/\alpha}\right) f\left(z^{1/\alpha}[\eta(t)]\right) \;=\; 0, \tag{3.4}$$

and the second order equation

$$w''(t) + q(t)f\Big(A[g(t)]\Big)f(w^{1/\alpha}[g(t)]) \;=\; 0, \tag{3.5}$$

where $A(t) = \int_t^{\infty} a^{-1/\alpha}(s)ds,$ *are oscillatory, then equation (1.2) is oscillatory*.

PROOF : Let $x(t)$ be a nonoscillatory solution of equation (1.2), say, $x(t) > 0,\ x[g(t)] > 0$ and $x[\sigma(t)] > 0$ for $t \geq t_0 \geq 0.$ Clearly, $(a(t)\,(x'\,(t))^{\alpha})'' \geq 0$ for $t \geq t_0$. There exists a $t_1 \geq t_0$ such that $(a(t)\,(x'\,(t))^{\alpha})'$ and $x'(t)$ are of fixed sign for $t \geq t_1$. Then as earlier there are four possibilities to consider:

(I) $(a(t)\,(x'\,(t))^{\alpha})' > 0$ and $x'(t) > 0$ for $t \geq t_1$;

(II) $(a(t)\,(x'\,(t))^{\alpha})' > 0$ and $x'(t) < 0$ for $t \geq t_1$;

(III) $(a(t)\,(x'\,(t))^{\alpha})' < 0$ and $x'(t) < 0$ for $t \geq t_1$; and

(IV) $(a(t)\,(x'\,(t))^{\alpha})' < 0$ and $x'(t) > 0$ for $t \geq t_1$.

Now, the case (II) cannot hold. In fact, if we let $y(t) = a(t)\,(x'\,(t))^{\alpha}$ for $t \geq t_1,$ then we see that $y''(t) > 0$ and $y'(t) > 0$ for $t \geq t_1$ and hence $\lim_{t\to\infty} y(t) = \infty$ which contradicts the negativity of $x'(t)$. Next, we shall consider:

Case (I): For $t \geq s \geq t_1,$ we have

$$a\,(t)\,(x'\,(t))^{\alpha} \geq a\,(s)\,(x'\,(s))^{\alpha}$$

and hence

$$x\,(t) \geq \left[a\,(s)\,(x'\,(s))^{\alpha}\right]^{1/\alpha} \int_s^t a^{-1/\alpha}(u)du.$$

Replacing t and s by $\sigma(t)$ and $\rho(t)$ respectively, we find

$$\begin{aligned} x[\sigma(t)] \ &\geq \ \left[a\,(\rho\,(t))\,(x'\,(\rho\,(t)))^{\alpha}\right]^{1/\alpha} \int_{\rho(t)}^{\sigma(t)} a^{-1/\alpha}(s)ds \\ &= \ y^{1/\alpha}\,[\rho\,(t)] \int_{\rho(t)}^{\sigma(t)} a^{-1/\alpha}(s)ds \quad \text{for} \quad t \geq t_2 \geq t_1, \end{aligned} \tag{3.6}$$

where $y(t) = a(t)\,(x'\,(t))^{\alpha}$. Using (3.6) and (1.4) in equation (1.2), we obtain

$$y''(t) \ \geq \ p(t)h(x[\sigma(t)]) \ \geq \ p(t)h\left(\int_{\rho(t)}^{\sigma(t)} a^{-1/\alpha}(s)ds\right) h\left(y^{1/\alpha}[\rho(t)]\right) \quad \text{for} \quad t \geq t_2. \tag{3.7}$$

For $t \geq s \geq t_2$, we have

$$y\,(t) - y\,(s) = \int_s^t y'\,(u)\,du$$

and since $y'(t)$ is increasing

$$y(t) \ \geq \ (t-s)y'(s)$$

or

$$y(t)^{1/\alpha} \ \geq \ (t-s)^{1/\alpha} \left(y'(s)\right)^{1/\alpha}.$$

Replacing t and s in the above inequality by $\rho(t)$ and $\theta(t)$ respectively, we get

$$y^{1/\alpha}[\rho(t)] \ \geq \ (\rho(t)-\theta(t))^{1/\alpha} \left(y'[\theta(t)]\right)^{1/\alpha} \quad \text{for} \quad t \geq t_3 \geq t_2. \tag{3.8}$$

Using (3.8) and (1.4) in (3.7) and letting $z(t) = y'(t)$ for $t \geq t_3$, we have

$$z'(t) \ \geq \ p(t)h\left(\int_{\rho(t)}^{\sigma(t)} a^{-1/\alpha}(s)ds\right) h\left([\rho(t)-\theta(t)]^{1/\alpha}\right) h\left(z^{1/\alpha}[\theta(t)]\right) \quad \text{for} \quad t \geq t_3.$$

By Lemma 2.1, we arrive at the desired contradiction.

Case (III): As in the proof of Theorem 2.1 – Case (III), we obtain (2.12). There exists a $t_2 \geq t_1$ such that

$$x[g(t)] \ \geq \ A[g(t)]w^{1/\alpha}[g(t)] \quad \text{for} \quad t \geq t_2, \tag{3.9}$$

where $w(t) = -a(t)\left(x'(t)\right)^{\alpha} > 0.$ Using (3.9) and (1.3) in equation (1.2), we have

$$w''(t) + q(t)f(A[g(t)])f\left(w^{1/\alpha}[g(t)]\right) \ \leq \ 0 \quad \text{for} \quad t \geq t_2.$$

Using Lemma 3.1, we see that the equation (3.5) has an eventually positive solution, which contradicts the hypothesis.

Case (IV) : For $t \geq t_1$, we have

$$x(t) \ = \ x(t_1) + \int_{t_1}^{t} x'(s)ds \ \geq \ \left(\int_{t_1}^{t} a^{-1/\alpha}(s)ds\right)\left(a(t)\left(x'(t)\right)^{\alpha}\right)^{1/\alpha}.$$

There exists a $t_2 \geq t_1$ such that

$$x[g(t)] \ \geq \ \left(\int_{t_1}^{g(t)} a^{-1/\alpha}(s)ds\right) y^{1/\alpha}[g(t)] \quad \text{for} \quad t \geq t_2, \tag{3.10}$$

where $y(t) = a(t)\left(x'(t)\right)^{\alpha}$ for $t \geq t_2$.

Using (3.10) and (1.3) in equation (1.2), we get

$$y''(t) \ \geq \ q(t)f\left(\int_{t_1}^{g(t)} a^{-1/\alpha}(s)ds\right) f\left(y^{1/\alpha}[g(t)]\right) \quad \text{for} \quad t \geq t_2. \tag{3.11}$$

Clearly, $y'(t)$ is negative and increasing for $t \geq t_2$. Therefore, for $t \geq s \geq t_2$, we have

$$y(t) - y(s) \ = \ \int_s^t y'(u)du \leq y'(t)(t-s),$$

or

$$y(s) \ \geq \ (t-s)(-y'(t)).$$

Replacing s and t in the above inequality by $g(t)$ and $\eta(t)$ respectively, we get

$$y[g(t)] \ \geq \ (\eta(t) - g(t))(-y'[\eta(t)]) \quad \text{for} \quad t \geq t_3 \geq t_2. \tag{3.12}$$

Using (3.12) and (1.3) in (3.11), and letting $z(t) = -y'(t) > 0$ for $t \geq t_3$, we have

$$z'(t) + q(t)f\left(\int_{t_1}^{g(t)} a^{-1/\alpha}(s)ds\right) f\left([\eta(t) - g(t)]^{1/\alpha}\right) f\left(z^{1/\alpha}[\eta(t)]\right) \ \leq \ 0 \quad \text{for} \quad t \geq t_3.$$

By Lemma 2.1, we arrive at the desired contradiction. This completes the proof. ∎

From Theorems A., B. and Theorem 3.1., the following result is immediate.

***Theorem* 3.2** — *Let conditions (i) – (iv), (1.3), (1.4) and (1.6) hold and suppose that there exist nondecreasing functions* $\eta, \rho, \theta : [t_0, \infty) \to \mathbb{R}$ *such that* $g(t) < \eta(t) < t$ *and* $\sigma(t) > \rho(t) > \theta(t) > t$ *for* $t \geq t_0$. *Equation (1.2) is oscillatory if*

$$\int^{\infty} q(s)f\Big(A[g(s)]\Big)ds \ = \ \infty, \tag{3.13}$$

and one of the following conditions holds:

(II$_1$) $\dfrac{f(u^{1/\alpha})}{u} \geq k$ *and* $\dfrac{h(u^{1/\alpha})}{u} \geq h$ for $u \neq 0$ *and some* $k, \ h > 0$

$$\limsup_{t\to\infty} h^3 \int_t^{\theta(t)} p(s)\left(\rho(s) - \theta(s)\right) \left(\int_{\rho(s)}^{\sigma(s)} a^{-1/\alpha}(\tau)d\tau\right)^{\alpha} ds \ > \ 1$$

and

$$\limsup_{t\to\infty} k^3 \int_{\eta(t)}^{t} q(s)\left(\eta(s) - g(s)\right) \left(\int_{t_0}^{g(s)} a^{-1/\alpha}(\tau)d\tau\right)^{\alpha} ds \ > \ 1 \quad \textit{for} \quad t_0 \geq 0.$$

(II$_2$) $\dfrac{f(u^{1/\alpha})}{u} \geq k$ *and* $\dfrac{h(u^{1/\alpha})}{u} \geq h$ for $u \neq 0$ *and some* $k, \ h > 0$

$$\liminf_{t\to\infty} h^3 \int_t^{\theta(t)} p(s)\left(\rho(s) - \theta(s)\right) \left(\int_{\rho(s)}^{\sigma(s)} a^{-1/\alpha}\left(\tau\right)d\tau\right)^{\alpha} ds \ > \ \frac{1}{e}$$

and

$$\liminf_{t\to\infty} k^3 \int_{\eta(t)}^{t} q(s)\left(\eta(s)-g(s)\right)\left(\int_{t_0}^{g(s)} a^{-1/\alpha}(\tau)d\tau\right)^{\alpha} ds \;>\; \frac{1}{e}.$$

(II$_3$) $\dfrac{u}{f(u^{1/\alpha})} \to 0$ as $u \to 0$ and $\dfrac{u}{h(u^{1/\alpha})} \to 0$ as $u \to \infty$,

$$\limsup_{t\to\infty} \int_{t}^{\theta(t)} p(s)h\left(\int_{\rho(s)}^{\sigma(s)} a^{-1/\alpha}(\tau)\,d\tau\right) h\left([\rho(s)-\theta(s)]^{1/\alpha}\right) ds \;>\; 0$$

and

$$\limsup_{t\to\infty} \int_{\eta(t)}^{t} q(s)f\left(\int_{t_0}^{g(s)} a^{-1/\alpha}(\tau)d\tau\right) f\left([\eta(s)-g(s)]^{1/\alpha}\right) ds \;>\; 0.$$

(II$_4$) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$ and $\displaystyle\int^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty$,

$$\int^{\infty} p(s)h\left(\int_{\rho(s)}^{\sigma(s)} a^{-1/\alpha}(\tau)\,d\tau\right) h\left([\rho(s)-\theta(s)]^{1/\alpha}\right) ds \;=\; \infty$$

and

$$\int^{\infty} q(s)f\left(\int_{t_0}^{g(s)} a^{-1/\alpha}(\tau)d\tau\right) f\left([\eta(s)-g(s)]^{1/\alpha}\right) ds \;=\; \infty\,.$$

When the condition (1.5) holds, we eliminate the case (III) and from Theorem 3.1, we obtain

***Theorem* 3.3** — *Suppose that conditions (i) – (iv), (1.3), (1.4) and (1.5) hold and assume that there exist nondecreasing functions* $\eta,\rho,\theta : [t_0,\infty) \to \mathbb{R}$ *such that* $g(t) < \eta(t) < t$ *and* $\sigma(t) > \rho(t) > \theta(t) > t$ *for* $t \geq t_0$. *If the first order advanced equation (3.3) and the first order delay equation (3.4) are oscillatory, then equation (1.2) is oscillatory*.

The following example is illustrative.

Example 3.1 — Consider the third order mixed equation

$$\left(t^6\left(x'(t)\right)^3\right)'' \;=\; \left(\frac{1}{\sqrt{t}}\right)^{2-\beta} x^{\beta}[\sqrt{t}] + t^{2\gamma/3}x^{\gamma}[2t], \quad t \geq 5. \tag{3.14}$$

Here, $a(t) = t^6$, $\alpha = 3$, $q(t) = (1/\sqrt{t})^{2-\beta}$ and $p(t) = t^{2\gamma/3}$, β and γ are the ratios of positive odd integers with $0 < \beta \leq 3 \leq \gamma$. Also $g(t) = \sqrt{t}$ and $\sigma(t) = 2t$. For $t \geq 5$, we let $\eta(t) = 2\sqrt{t}$, $\rho(t) = 5t/3$ and $\theta(t) = 4t/3$.

It is easy to check that all conditions of Theorem 3.2 for different values of β and γ are satisfied and hence equation (3.14) is oscillatory.

We note that there are many sufficient conditions for the oscillation of equation (3.5) rather than (3.13), see [1–3].

Next, from the proof of Theorem 3.1–Case (III), we obtain the second order inequality

$$w''(t) + q(t) f(A[g(t)]) f\left(w^{1/\alpha}[g(t)]\right) \;\leq\; 0 \quad \text{for} \quad t \geq t_2 \tag{3.15}$$

for $w(t) = -a(t)\left(x'(t)\right)^{\alpha}$. Since $w'(t) > 0$ and $w(t) > 0$ for $t \geq t_2$, there exist a $t_3 \geq t_2$ and a constant $k,\ 0 < k < 1$ such that

$$w[g(t)] \;\geq\; kg(t)w'[g(t)] \quad \text{for} \quad t \geq t_3. \tag{3.16}$$

Using (3.16) and (1.3) in (3.15), we obtain

$$v'(t) + \bar{c}\, q(t)\, f\Big(A[g(t)]\Big) f\left(g^{1/\alpha}(t)\right) f\left(v^{1/\alpha}[g(t)]\right) \;\leq\; 0 \quad \text{for} \quad t \geq t_3, \tag{3.17}$$

where $\bar{c} = f(k^{1/\alpha})$ and $v(t) = w'(t)$ for $t \geq t_3$.

Next, we can combine equations (3.4) and (3.17) in one, say,

$$y'(t) + Q(t) f\left(y^{1/\alpha}[\eta(t)]\right) \;=\; 0, \tag{3.18}$$

where

$$Q(t) \;=\; \min\left\{ q(t) f\left(\int_{t_0}^{g(t)} a^{-1/\alpha}(s)ds\right) f\left([\eta(t) - g(t)]^{1/\alpha}\right),\;\; c\, q(t) f\Big(A[g(t)]\Big) f\left(g^{1/\alpha}(t)\right)\right\}$$

for any constant $c,\ 0 < c < 1$ and all $t \geq t_0$. Thus, from Theorem 3.1 we get the following corollary:

Corollary 3.1 — Let the hypotheses of Theorem 3.1 hold and let equations (3.4) and (3.5) be replaced by equation (3.18). Then equation (1.2) is oscillatory.

In a similar way, we can combine equations (2.5) and (2.6) in one and restate Theorem 2.1. The details are left to the reader.

4. Remarks

1. Conditions (1.3) and (1.4) may be discarded if we let $f(x) = x^{\beta}$ and $h(x) = x^{\gamma}$, where β and γ are the ratios of positive odd integers.

2. We note that conditions (2.21) and (2.22) respectively can be replaced by

$$\limsup_{t\to\infty} k^2 \int_{g(t)}^{t} q(s) \left(\int_{T}^{g(s)} \left(\frac{u-T}{a(u)} \right)^{1/\alpha} du \right)^{\alpha} ds \;>\; 1$$

and

$$\liminf_{t\to\infty} k^2 \int_{g(t)}^{t} q(s) \left(\int_{T}^{g(s)} \left(\frac{u-T}{a(u)} \right)^{1/\alpha} du \right)^{\alpha} ds \;>\; \frac{1}{e}.$$

3. The results of Section 3 can be obtained for equations of the form

$$\left(a(t)\,(x'\,(t))^{\alpha}\right)'' \;=\; q(t)f(x[g(t)]),$$

where the argument $g(t)$ is of mixed type, say, $g(t) = t - \sin t$. The details are left to the reader.

4. The results of this paper are extendable to neutral equations of the form

$$\frac{d^2}{dt^2}\left(a(t) \left(\frac{d}{dt}[x(t)+c(t)x[\alpha(t)]] \right)^{\alpha} \right) + q(t)f(x[g(t)]) \;=\; 0$$

and

$$\frac{d^2}{dt^2}\left(a(t) \left(\frac{d}{dt}[x(t)+c(t)x[\alpha(t)]] \right)^{\alpha} \right) \;=\; q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]),$$

where $c(t),\ \alpha(t) \in C([t_0,\infty),\mathbb{R})$ and $\lim_{t\to\infty}\alpha(t) = \infty$. The details are left to the reader.

It will be interesting to derive different oscillation criteria for the oscillation of equations (1.1) and (1.2) via other techniques rater than those presented here.

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